

Metrics on diagram groups and uniform embeddings in a Hilbert space

G. N. Arzhantseva, V. S. Guba and M. V. Sapir*

Abstract. We give first examples of finitely generated groups having an intermediate, with values in $(0, 1)$, Hilbert space compression (which is a numerical parameter measuring the distortion required to embed a metric space into Hilbert space). These groups include certain diagram groups. In particular, we show that the Hilbert space compression of Richard Thompson's group F is equal to $1/2$, the Hilbert space compression of $\mathbb{Z} \wr \mathbb{Z}$ is between $1/2$ and $3/4$, and the Hilbert space compression of $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ is between 0 and $1/2$. In general, we find a relationship between the growth of H and the Hilbert space compression of $\mathbb{Z} \wr H$.

Mathematics Subject Classification (2000). 20F65, 20F69

Keywords. Richard Thompson's group F , diagram groups, Hilbert space compression, subgroup distortion.

1. Introduction

The study of uniform embeddings of metric spaces into Hilbert space was initiated by Gromov.

Definition 1.1. Let (Γ, d) be a metric space. Let \mathcal{H} be a separable Hilbert space. A map $f: \Gamma \rightarrow \mathcal{H}$ is said to be a *uniform embedding* [9] if there exist non-decreasing functions ρ_1, ρ_2 from $[0, \infty)$ to itself such that

- (1) $\rho_1(d(x, y)) \leq \|f(x) - f(y)\|_{\mathcal{H}} \leq \rho_2(d(x, y))$ for all $x, y \in \Gamma$;
- (2) $\lim_{r \rightarrow +\infty} \rho_i(r) = +\infty$ for $i = 1, 2$.

Gromov suggested that a finitely generated group G (viewed as a metric space with a word length metric) uniformly embeddable into Hilbert space should satisfy the Novikov Conjecture [10, page 67]. That was proved in [28], see also [26], [19].

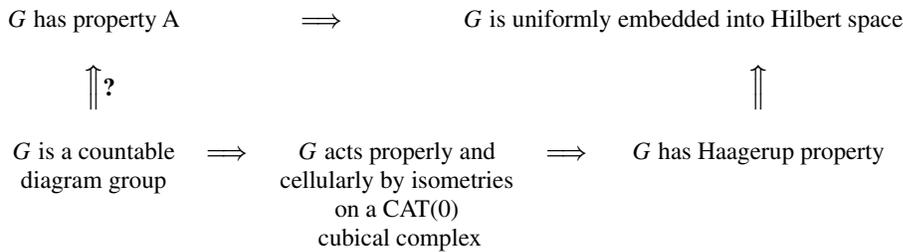
*This work has been supported by the Swiss National Science Foundation, No. PP002-68627. The first two authors also acknowledge the hospitality of the Centre de Recerca Matemàtica (Barcelona) during autumn 2004. The work of the third author has been supported in part by the NSF grant DMS 0245600.

Guoliang Yu introduced a Følner-type condition on finitely generated groups G , called property A, which is a weak form of amenability and which guarantees the existence of a uniform embedding of the metric space into Hilbert space [28]. That property is interesting in itself because it turned out to be equivalent to the exactness of G , and to the existence of an amenable action of G on a compact space (see [27] for a detailed discussion of property A).

Among “classical” groups for which property A has not been proved so far, is the R. Thompson group F , which is the group of all piecewise linear orientation preserving self-homeomorphisms of the unit interval with finitely many dyadic singularities and all slopes integer powers of 2.

It is not known whether F is amenable or not, so the question about a weak amenability is interesting.

It is worth noticing that, by a result of Niblo and Reeves, a group acting properly and cellularly by isometries on a CAT(0) cubical complex has the Haagerup property [24]. In particular, by a result of Farley [8], this holds for diagram groups over finite semigroup presentations. Moreover, this holds for all countable diagram groups because they can be embedded into a certain “universal” diagram group [15]. It follows immediately from the definition of the Haagerup property that such a group can be embedded uniformly into Hilbert space. However, the interaction between the Haagerup property and Guoliang Yu’s property A is rather complicated [5, Chapter 1.3], [18], [26]. Notice that amenable groups satisfy both the Haagerup property and property A.



Guentner and Kaminker [18] introduced a natural quasi-isometry invariant of a group that shows how close to a quasi-isometry can a uniform embedding of a group into a Hilbert space be.

Definition 1.2 (cf. [18, Definition 2.2]). The Hilbert space compression of a finitely generated discrete group G is the number $R(G)$, which is the supremum of all $\alpha \geq 0$ for which there exists a uniform embedding of G into a Hilbert space with $\rho_1(n) = Cn^\alpha$ with a constant $C > 0$ and linear ρ_2 (see Definition 1.1).

By [18], Hilbert space compression strictly greater than $1/2$ implies property A. Any group that is not uniformly embeddable into a Hilbert space (such groups exist by [11]) has Hilbert space compression 0. It is proved in [18] that free groups have Hilbert space compression 1 (although there are no quasi-isometric embeddings of a free group of rank > 1 into a Hilbert space [2], [22]¹). More generally, by Campbell and Niblo [4], any discrete group acting properly, co-compactly on a finite dimensional CAT(0) cubical complex has Hilbert space compression 1.

Note that till now there were no examples of groups with compression strictly between 0 and 1.

One of our main results is the following.

Theorem 1.3. *The Hilbert space compression of R. Thompson's group F equals $1/2$.*

In particular, this shows that the result by Guentner and Kaminker on groups with compression strictly greater than $1/2$ cannot be applied to F . However, one can extract a stronger fact from [18]. Suppose that a finitely generated group G has a uniform embedding into Hilbert space with linear $\rho_2(n)$ and $\rho_1 \gg \sqrt{n}$ (i.e. $\lim_{n \rightarrow \infty} \rho_1(n)/\sqrt{n} = \infty$). The proof of [18, Theorem 3.2] shows that property A holds for G in this case. For R. Thompson's group F we actually show that $\rho_1 \leq C\sqrt{n} \log n$ for some constant C .

Question 1.4. Is it true that there exists a uniform embedding of F into Hilbert space with linear ρ_2 and $\rho_1 \gg \sqrt{n}$?

If the answer is positive, then F has property A.

It is known [12], [14], [8] that geometric properties of F are better understood when F is considered as a diagram group. Recall that a diagram group is the fundamental group of the space of positive paths on a directed 2-complex [15] (an equivalent definition in terms of semigroup presentations from [12] is given below). For example, F is the diagram group of the dunce hat considered as a directed 2-complex with one vertex, one edge x and one directed cell $x = x^2$.

Our interest in uniform embeddings for diagram groups was stimulated in part by a connection with certain metric properties of diagram groups which have been studied independently before. Recall that elements of a diagram group G can be represented by diagrams, which are plane cellular complexes subdivided into a number of regions (cells). This allows one to introduce a canonical *diagram metric* dist_d on G such that the $\text{dist}_d(g_1, g_2)$ is the number of cells in the diagram with minimal number of cells representing the element $g_1^{-1}g_2$. This metric is left invariant. It is proper provided the corresponding directed 2-complex (semigroup presentation) is finite or when the diagram group is finitely generated. It is known (this is proved in Burillo [3] using

¹We thank D. Sonkin for bringing these papers to our attention.

a different terminology) that for the R. Thompson group F considered as a diagram group of the dunce hat, the diagram metric is bi-Lipschitz equivalent to the word metric. We say that a finitely generated diagram group satisfies *property B* if its diagram metric is bi-Lipschitz equivalent to the word metric.

Theorem 1.5. *The Hilbert space compression of a finitely generated diagram group with property B is at least $1/2$.*

Question 1.6. Do all finitely generated diagram groups satisfy property B?

We think that is a very interesting question.

In [12], [15], it is proved that there are (finitely presented) diagram groups containing all countable diagram groups as subgroups. Such diagram groups are called *universal*. One of these diagram groups (in this paper it is denoted by U) corresponds to the semigroup presentation $\langle x, a \mid x^2 = x^3, ax = a \rangle$.

Theorem 1.7. *The universal diagram group U satisfies property B.*

Notice that Theorem 1.7 does not imply (at least directly) a positive answer to Question 1.6. Indeed, U contains the direct product $G = \mathbb{F}_2 \times \mathbb{F}_2$ of two free groups of rank 2 (which is a countable diagram group). By a result of Mikhailova [23], the group G contains a non-recursively distorted subgroup H . Hence the word metric on a diagram group may not be equivalent to the metric induced by the embedding of H into U . Certainly Theorem 1.7 implies that all undistorted subgroups of U satisfy property B. Clearly the diagram metric on a finitely generated diagram group does not exceed a constant times the word metric. An easy argument shows that for any finitely generated diagram group, there exists a recursive function $f(n)$ such that the word metric does not exceed $f(\text{dist}_d)$.

Besides the R. Thompson group F , the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$ is another typical representative of the class of diagram groups. It corresponds to the presentation $\langle a, b, b_1, b_2, c \mid ab = a, bc = c, b = b_1, b_1 = b_2, b_2 = b \rangle$ [14]. It also satisfies property B. Hence Theorem 1.5 can be applied. The group $\mathbb{Z} \wr \mathbb{Z}$ is amenable, so it satisfies property A, but the next theorem shows that the compression of that group is not 1.

Theorem 1.8. *The Hilbert space compression of the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$ belongs to $[1/2, 3/4]$.*

Question 1.9. What is the Hilbert space compression of $\mathbb{Z} \wr \mathbb{Z}$?

We also prove the following general result.

Theorem 1.10. *Let H be a finitely generated group with a super-polynomial growth function. Then the Hilbert space compression of $\mathbb{Z} \wr H$ is at most $1/2$.*

This theorem immediately implies

Corollary 1.11. *The Hilbert space compression of the group $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ is at most $1/2$.*

Note that the group $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ is amenable and so it satisfies property A. Thus property A does not imply that Hilbert space compression is bigger than $1/2$. It is not known whether $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ is a diagram group (the most probable answer seems to be negative). Thus Theorem 1.5 does not apply and we do not know if the Hilbert space compression of this group is exactly $1/2$.

Question 1.12. What is the Hilbert space compression of the group $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$? Are there amenable groups with Hilbert space compression 0 or arbitrary close to 0?

Finally, we would like to mention that our proof of Theorem 1.5 yields actually a stronger result. Namely, a lower bound on the *equivariant* Hilbert space compression $R_G(G)$. This is the Hilbert space compression defined by restricting to G -equivariant uniform embeddings of G into Hilbert spaces equipped with actions of G by *affine isometries* [18, Section 5].

Theorem 1.13. *The equivariant Hilbert space compression of a finitely generated diagram group with property B is at least $1/2$.*

The estimate $1/2$ cannot be improved since $R_G(G) > 1/2$ implies amenability [18] and there are non-amenable diagram groups with property B². Moreover, one can see from the proof that the diagram metric of a finitely generated diagram group is a conditionally negative definite function on the group. Recall that a function $\psi : G \rightarrow \mathbb{R}$, satisfying $\psi(g) = \psi(g^{-1})$ is said to be conditionally negative definite if $\sum \psi(g_i g_j^{-1}) c_i c_j \leq 0$ for all $n \in \mathbb{N}$ and $g_1, \dots, g_n \in G, c_1, \dots, c_n \in \mathbb{R}$ such that $\sum c_i = 0$. The existence of a continuous conditionally negative definite function $\psi : G \rightarrow [0, \infty)$ with $\lim_{g \rightarrow \infty} \psi(g) = \infty$ is equivalent to the a-T-menability [1], hence to the Haagerup approximation property [5]. For more applications see [17] and references therein.

2. Preliminary information about diagram groups

2.1. Diagram groups. Let us briefly recall the concept of a diagram group and some terminology from this area. Details can be found in [12]. For the reasons of the

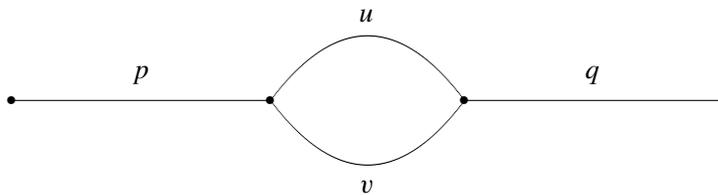
²We thank Y. de Cornulier for this remark.

present paper, it is enough to use the definition in terms of semigroup presentations given in [12] rather than an equivalent definition from [15].

Let X be an alphabet. We denote by X^* the set of all words over X and by X^+ the set of all nonempty words. The length of a word w is denoted by $|w|$.

Let $\mathcal{P} = \langle X \mid \mathcal{R} \rangle$ be a semigroup presentation. Here \mathcal{R} consists of ordered pairs of nonempty words over X . We always assume that if $(u, v) \in \mathcal{R}$, then $(v, u) \notin \mathcal{R}$.

A triple of the form $(p, u = v, q)$ is called an *atomic 2-path*, where $p, q \in X^*$, $(u, v) \in \mathcal{R} \cup \mathcal{R}^{-1}$. The following labelled plane graph is called an *atomic diagram* (over \mathcal{P}):



Here each segment labelled by a word w is subdivided into $|w|$ edges labelled by the letters of w . (Segments have a left-to-right orientation.)

Notice that each atomic diagram has the *top path* (in our case it is labelled by puq) and the *bottom path* (labelled by pvq). This atomic diagram is defined uniquely by a 2-path $(p, u = v, q)$ so we will often use this notation for the atomic diagram itself.

Suppose that we have a sequence $\Delta_1, \dots, \Delta_k$ of atomic diagrams. Let us consider the case when the bottom label of each Δ_i ($1 \leq i < k$) coincides with the top label of Δ_{i+1} . We can thus identify the bottom path of Δ_i with the top path of Δ_{i+1} ($1 \leq i < k$) to obtain the new labelled plane graph denoted by $\Delta := \Delta_1 \circ \dots \circ \Delta_k$ (the *concatenation of the above atomic diagrams*). One can naturally define the top path (which will be the top path of Δ_1) and the bottom path (the bottom path of Δ_k) of Δ .

For every word $w \in X^+$, we define the *trivial diagram* denoted by $\varepsilon(w)$, which is the interval labelled by w . Its top and bottom paths coincide with itself.

Definition 2.1. A *diagram* Δ over a semigroup presentation \mathcal{P} is either a trivial diagram, or a concatenation of atomic diagrams over \mathcal{P} . The diagrams are considered identical whenever they are isotopic as labelled plane graphs.

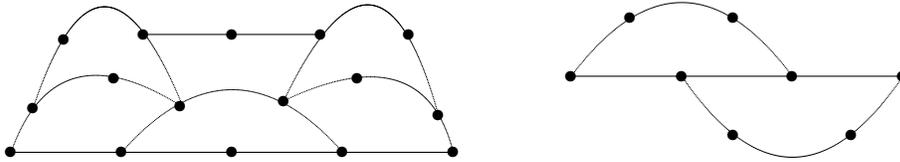
For a diagram Δ , its top and bottom paths will be denoted by $[\Delta]$ and $[\Delta]$, respectively. They start at the same vertex denoted by $\iota(\Delta)$ and end at the same vertex denoted by $\tau(\Delta)$. The diagram itself is situated “between” its top and bottom paths. If w' is the label of the top path of Δ and w'' is the label of its bottom path, then we say that Δ is a (w', w'') -diagram. If $w' = w''$, then we say that the diagram is *spherical*.

Given two diagrams Δ' and Δ'' , one can define their *concatenation* denoted by $\Delta' \circ \Delta''$, provided the labels of $\lfloor \Delta' \rfloor$ and $\lceil \Delta'' \rceil$ coincide.

For every word $w \in X^+$, the set of all (w, w) -diagrams over \mathcal{P} forms a monoid with $\varepsilon(w)$ as a unit.

For any diagram Δ , by Δ^{-1} we denote the mirror image of Δ with respect to the axis connecting $\iota(\Delta)$ and $\tau(\Delta)$. Clearly, the top (bottom) path of Δ becomes the bottom (top) path of Δ^{-1} .

Here is an example of two diagrams over $\langle x \mid x^3 = x^2 \rangle$:



The label of each edge of these diagrams is x .

Notice that each cell in a diagram can be considered as a diagram itself. If the cell corresponds to a relation $u = v$, we shall denote this diagram by $(u = v)$.

Let Δ be a diagram over \mathcal{P} . Suppose that it contains a pair of cells π', π'' such that the bottom path of π' coincides with the top path of π'' and the top label of π' equals the bottom label of π'' . In this case we say that the cells π' and π'' form a *dipole* in Δ . One can define the operation of removing the dipole. Namely, given a dipole formed by π', π'' , we first remove the common boundary of these cells and then glue $\lceil \pi' \rceil$ with $\lfloor \pi'' \rfloor$ (the paths we glue have the same label). We get a new diagram over \mathcal{P} that has fewer cells. Proceeding in such a way, we get to a diagram that has no dipoles, called a *reduced* diagram.

Kilibarda [20], [21] proved that the process of cancelling dipoles in diagrams is confluent, that is, the result does not depend on the order in which we remove the dipoles. Two diagrams are called *equivalent* if the corresponding reduced diagrams are the same.

For every word $w \in X^+$, the set of all reduced (w, w) -diagrams with operation “concatenation followed by removing all dipoles” is a group [12, Lemma 5.2], denoted by $\mathcal{D}(\mathcal{P}, w)$.

Definition 2.2. The group $\mathcal{D}(\mathcal{P}, w)$ is called a *diagram group* over a semigroup presentation \mathcal{P} with *base* w .

Example 1 ([12, Example 6.4]). The diagram group $\mathcal{D}(\mathcal{P}, x)$ over the semigroup presentation $\mathcal{P} = \langle x \mid x^2 = x \rangle$ with base x is R. Thompson’s group F that has the following group presentation:

$$\langle x_0, x_1, x_2, \dots \mid x_j x_i = x_i x_{j+1} \ (j > i) \rangle.$$

There is one more important binary operation on diagrams. Given two diagrams Δ_1, Δ_2 over \mathcal{P} , one can identify $\tau(\Delta_1)$ and $\iota(\Delta_2)$. This gives a new diagram over \mathcal{P} denoted by $\Delta_1 + \Delta_2$. This operation is associative but not commutative.

2.2. Universal diagram groups. In [15], Guba and Sapir found finitely presented diagram groups each of which contains every countable diagram group as a subgroup. Such diagram groups are called *universal*. For instance, the diagram group $U = \mathcal{D}(\mathcal{P}, a)$, where $\mathcal{P} = \langle x, a \mid x^3 = x^2, ax = a \rangle$ is universal. It has the following Thompson-like group presentation:

$$\langle x_0, x_1, x_2, \dots \mid x_j x_i = x_i x_{j+1} (j - i > 1) \rangle.$$

It is easy to see that U can be generated by x_0, x_1, x_2 . Indeed, $x_n = x_2^{x_0^{n-2}}$ for all $n \geq 3$ ($a^b = b^{-1}ab$ by definition). (In fact this group is finitely presented [15].)

We are going to describe explicitly the procedure of expressing an element of U as a word in generators x_0, x_1, x_2, \dots . Let $g \in U$ be represented by an (a, a) -diagram Δ over \mathcal{P} . One can decompose Δ into a product of atomic diagrams. It is easy to see that the atomic diagrams that can occur have the form $(1, a = ax, x^m)^\epsilon$, where $m \geq 0, \epsilon = \pm 1$ or the form $(ax^k, x^2 = x^3, x^m)^\epsilon$, where $k, m \geq 0, \epsilon = \pm 1$. In the first case we assign the identity to the atomic diagram and in the second case we assign to it the element x_m^ϵ . Multiplying these elements, we get an expression for the element $g \in U$, see [15, Section 6].

Given a diagram Ξ over $\langle x \mid x^3 = x^2 \rangle$, we can canonically assign a diagram Δ over $\langle x, a \mid x^3 = x^2, ax = a \rangle$ as follows. First we take the sum $\varepsilon(a) + \Xi$ and then add a number of cells of the form $a = ax$ on the top to make the top label equal to a . Then we add a number of cells of the form $ax = a$ on the bottom in order to make the bottom label also equal to a . The result is an (a, a) -diagram that represents an element of the group U .

On the above picture, where two diagrams over $\langle x \mid x^3 = x^2 \rangle$ are drawn, the described operation leads to the following elements of U , respectively: $x_3 x_1^{-1} x_0^{-1} x_3^{-2} x_1^{-1}$ and $x_1^{-1} x_0$. Notice that the way to decompose a diagram into the product of atomic factors is not unique in general. For the first case, we always chose the rightmost cell that can be included into the next atomic factor. This procedure will be described later in details.

3. Main results

Definition 3.1. Let M be a set and $p, q: M \rightarrow [0, \infty)$ be arbitrary functions. Then $p \preceq q$ if there exists a constant $c > 0$ such that $p(m) \leq cq(m)$ for every $m \in M$. These functions are equivalent, $p \sim q$, if $p \preceq q$ and $q \preceq p$.

Let G be a group generated by a set Σ . For any $g \in G$, we denote by $\ell_\Sigma(g)$ the length of the shortest word over $\Sigma^{\pm 1}$ that represents g . Notice that if G is finitely generated, then all functions of the form ℓ_Σ are equivalent for all finite sets of generators. Therefore, the function $g \mapsto \ell(g)$ is unique up to equivalence. The function ℓ on G defines the *word length metric* on G as follows: the (word) distance between $g_1, g_2 \in G$ is $\ell(g_1^{-1}g_2)$.

We will also omit the subscript Σ on ℓ if the generating set of G is clear.

Let $G = \mathcal{D}(\mathcal{P}, w)$ be a diagram group over a semigroup presentation \mathcal{P} with base w . Let $\psi: H \hookrightarrow G$ be an embedding of a finitely generated group H into G .

Definition 3.2. We say that $\psi: H \hookrightarrow G$ is a *B-embedding*, whenever $\#(h) \sim \ell(h)$ for all $h \in H$. Here $\#(h)$ is the number of cells in the reduced (w, w) -diagram representing $\psi(h)$ and ℓ is the length function on H .

An obvious argument implies that $\#(h) \leq \ell(h)$ in all cases. Indeed, let C be the maximum number of cells of diagrams that represent generators of H . Then any element that has length n in these generators can be represented by a diagram over \mathcal{P} with at most Cn cells.

However, not every embedding of the above form is a *B-embedding*.

Example 2. Let us consider the direct product $G = \mathbb{F}_2 \times \mathbb{F}_2$ of two free groups of rank 2. It is known that this is a diagram group, see, for example, [12, Section 8]. By a well-known result of Mikhailova [23], there is a finitely generated subgroup H in G with undecidable membership problem. This implies that no recursive function $f(n)$ can have the property $\ell(h) \leq f(\#(h))$ for all $h \in H$.

The most important case is when $G = \mathcal{D}(\mathcal{P}, w)$ is a finitely generated diagram group and $H = G$ with the identical embedding ψ .

Definition 3.3. A finitely generated diagram group $G = \mathcal{D}(\mathcal{P}, w)$ over a semigroup presentation \mathcal{P} with base w has *property B* whenever the identical embedding $G \hookrightarrow G$ is a *B-embedding*.

In other words, for all $g \in G$, one has $\#(g) \sim \ell(g)$, where $\#(g)$ is the number of cells in the reduced diagram representing g .

Notice that we cannot simply say that G has property B itself. This concept depends on the presentation \mathcal{P} and base w .

R. Thompson's group F has property B as a diagram group over $\langle x \mid x^2 = x \rangle$ with base x . This immediately follows from a result of Burillo [3].

Proof of Theorem 1.5. Let G be a finitely generated group that can be B-embedded into a diagram group. We need to show that the Hilbert space compression of G is at least $1/2$.

Given a diagram group $\mathcal{D}(\mathcal{P}, w)$, let us build the following geometric object associated with this group. Let us take all reduced diagrams over \mathcal{P} that have w as a top label. We identify all top paths of these diagrams. This gives a 1-path p labelled by w .

Suppose that there are two reduced (w, \cdot) -diagrams Δ_1, Δ_2 with decompositions of the form $\Delta_i \equiv \Delta'_i \circ \Delta''_i$, where $i = 1, 2$ and Δ'_1, Δ'_2 are isotopic. Then we identify Δ'_1 with Δ'_2 via this isotopy. We do that for all pairs of diagrams and all decompositions of them with the above property. The object we get as a result is a directed 2-complex \mathcal{T} , which turns out to be a rooted 2-tree in the sense of [15]. This directed 2-complex can be viewed as a semigroup presentation if we assign different labels to different edges and consider pairs of words written on the boundaries of the cells as relations. It is proved in [15] that for any path q in \mathcal{T} with the same endpoints as p there exists a unique reduced (p, q) -diagram over \mathcal{T} .

Let \mathbf{F} be the set of all geometric 2-cells of \mathcal{T} . By $\mathbb{R}^{\mathbf{F}}$ we denote the vector space over \mathbb{R} with \mathbf{F} as a basis. Clearly, this vector space is a subset in Hilbert space $\mathcal{H} = \ell_2(\mathbf{F})$.

Every element g of the diagram group $\mathcal{D}(\mathcal{P}, w)$ can be uniquely represented by a reduced (w, w) -diagram Δ . This diagram can be naturally embedded into \mathcal{T} . (The top of the diagram is identified with the path p under this embedding.) Let us assign to g a function ϕ_g from \mathbf{F} to \mathbb{R} , where $\phi_g(f) = 1$ if $f \in \mathbf{F}$ is contained in the image of Δ under the above embedding and $\phi_g(f) = 0$ otherwise.

So we have a mapping $\phi: \mathcal{D}(\mathcal{P}, w) \rightarrow \mathcal{H}$ from the diagram group $\mathcal{D}(\mathcal{P}, w)$ to \mathcal{H} defined by the rule $g \mapsto \phi_g$.

As we have already mentioned in the introduction, one can define a canonical *diagram metric* on $\mathcal{D}(\mathcal{P}, w)$ as follows: given two elements $g_1, g_2 \in \mathcal{D}(\mathcal{P}, w)$, one can define the *diagram distance*, denoted by $\text{dist}_d(g_1, g_2)$ between these elements as the number of cells in the reduced diagram over \mathcal{P} representing $g_1^{-1}g_2$.

Suppose that the diagram distance between two diagrams Δ_1, Δ_2 from $\mathcal{D}(\mathcal{P}, w)$ equals n . Let us consider the images of the diagrams Δ_i ($i = 1, 2$) in \mathcal{T} . They can be decomposed as $\Delta_i \equiv \Psi \circ \bar{\Delta}_i$ ($i = 1, 2$), where Ψ is the “greatest common divisor” of Δ_1 and Δ_2 . In this case we see that $\bar{\Delta}_1, \bar{\Delta}_2$ do not have common cells in \mathcal{T} . The total number of cells in $\bar{\Delta}_1$ and $\bar{\Delta}_2$ equals n . Since $\phi_{g_1} - \phi_{g_2}$, as an element of $\mathbb{R}^{\mathbf{F}}$ is a vector whose coordinates are 0, 1 or -1 , and the number of non-zero coordinates is n , we conclude that the norm of $\phi_{g_1} - \phi_{g_2}$ is \sqrt{n} .

Now given a finitely generated group G , which is B-embedded into a diagram group $\mathcal{D}(\mathcal{P}, w)$, we see that the word length metric in G is equivalent to the diagram metric induced on G as a subset in $\mathcal{D}(\mathcal{P}, w)$. Therefore, for some positive constants C_1, C_2 one has inequalities

$$C_1\sqrt{\text{dist}(g_1, g_2)} \leq \|\phi_{g_1} - \phi_{g_2}\| \leq C_2\sqrt{\text{dist}(g_1, g_2)} \leq C_2\text{dist}(g_1, g_2).$$

Hence the Hilbert space compression of G is at least $\frac{1}{2}$. □

Proof of Theorem 1.13. Let $G = \mathcal{D}(\mathcal{P}, w)$ be a diagram group with property B and

$$\phi: G \rightarrow \mathcal{H}, \quad g \mapsto \phi_g$$

be the mapping constructed in the previous proof. It suffices to show that this is a G -equivariant embedding. The group G acts on itself by left multiplications. Therefore, it remains to find an affine isometric action $G \rightarrow \text{Isom}(\mathcal{H})$, $g \mapsto g \cdot$, such that $\phi_{gh} = g \cdot \phi_h$ for $g, h \in G$.

Let $g \mapsto \pi_g$ be an orthogonal representation of G on \mathcal{H} such that ϕ satisfies the cocycle identity $\phi_{gh} = \pi_g(\phi_h) + \phi_g$ for $g, h \in G$. Then we define the desired action by $g \cdot \varphi = \pi_g(\varphi) + \phi_g$ for $\varphi \in \mathcal{H}$. \square

We will use the following result which is a generalization of the well-known parallelogram theorem to higher dimensions [7], see also [6]. Namely, let $E^n \subset \mathbb{R}^n$ be an n -dimensional hypercube. Suppose that we have a mapping of the set of vertices of E^n into a metric space M . In this case we will say that we have a *skew cube* in M . For every edge of E^n (there are exactly $2^{n-1}n$ of them), by an edge of the skew cube we will mean the distance in M between the images of the endpoints of the edge. Similarly, for each (long) diagonal of E^n (which connects opposite vertices of E^n) we consider the corresponding *diagonal* of the skew cube.

Lemma 3.4 (Skew Cube Inequality [7]). *For every skew cube in a Hilbert space, the sum of squares of its diagonals does not exceed the sum of squares of its edges.*

Proof of Theorem 1.3. We need to prove that the Hilbert space compression of R. Thompson’s group F equals $1/2$.

The fact that the compression is at least $1/2$, follows from Theorem 1.5.

For any $n \geq 0$, let us define 2^n elements of F that commute pairwise. All these elements will be reduced (x, x) -diagrams over $\mathcal{P} = \langle x \mid x^2 = x \rangle$. For $n = 0$, let Δ be the diagram that corresponds to the generator x_0 . Namely, if π is the diagram that consists of one cell of the form $x = x^2$, then Δ is $\pi \circ (\varepsilon(x) + \pi) \circ (\pi + \varepsilon(x))^{-1} \circ \pi^{-1}$ by definition. It has 4 cells.

Suppose that $n \geq 1$ and we have already constructed diagrams Δ_i ($1 \leq i \leq 2^{n-1}$) that commute pairwise. For every i we consider two (x^2, x^2) -diagrams: $\varepsilon(x) + \Delta_i$ and $\Delta_i + \varepsilon(x)$. We get 2^n spherical diagrams with base x^2 that obviously commute pairwise. It remains to conjugate them to obtain 2^n spherical diagrams with base x having the same property. Namely, we take $\pi \circ (\varepsilon(x) + \Delta_i) \circ \pi^{-1}$ and $\pi \circ (\Delta_i + \varepsilon(x)) \circ \pi^{-1}$.

Let us denote the elements of F obtained in this way by g_i ($1 \leq i \leq 2^n$). These elements define a 2^n -dimensional skew cube in F . It follows easily from the construction that each g_i has exactly $2n + 4$ cells as a diagram. So the word length of each g_i is $O(n)$. Now for any $\epsilon_i = \pm 1$ ($1 \leq i \leq 2^n$) we consider the product

of the form $g = g_1^{\epsilon_1} \dots g_{2^n}^{\epsilon_{2^n}}$. It is easy to see from definitions that the diagram that represents g has the form $\Gamma_n \circ (\Delta^{\epsilon_1} + \dots + \Delta^{\epsilon_{2^n}}) \circ \Gamma_n^{-1}$, where Γ_n is defined by induction in the following way: $\Gamma_0 = \varepsilon(x)$, $\Gamma_{k+1} = \pi \circ (\Gamma_k + \Gamma_k)$ ($k \geq 0$). In particular, the number of cells in Γ_n equals $2^n - 1$ and so g is represented by a diagram with exactly $2(2^n - 1) + 4 \cdot 2^n = 3 \cdot 2^{n+1} - 2$ cells. Since F satisfies property B , the word length of an element $g = g_1^{\epsilon_1} \dots g_{2^n}^{\epsilon_{2^n}}$ will be at least $C2^n$, where $C > 0$ is a constant that does not depend on the ϵ_i 's.

Now consider a uniform embedding of F into a Hilbert space \mathcal{H} with linear ρ_2 . In the image of our skew cube in F formed by g_1, \dots, g_{2^n} , each edge will be equal to $O(n)$. The Skew Cube Inequality implies that there exists a diagonal of the corresponding skew cube in \mathcal{H} that does not exceed $O(n) \cdot \sqrt{2^n} = O(n2^{\frac{n}{2}})$. This means that some points in F that were at distance $d \geq C2^n$ from each other will be mapped to points in \mathcal{H} at distance $O(n2^{\frac{n}{2}}) = O(\sqrt{d} \log_2 d)$. Therefore, the compression cannot exceed $1/2$. \square

In order to prove Theorem 1.7, we need the following simple construction, called the *rightmost decomposition* of a diagram. The idea of such a decomposition applied to the presentation $\langle x \mid x^2 = x \rangle$ was used in [13] to get a new normal form for the elements of R. Thompson's group F .

Definition 3.5 (Rightmost decomposition of a diagram). Let Δ be a diagram over a semigroup presentation \mathcal{P} . A *rightmost decomposition* of the diagram Δ into a product of atomic factors is defined as follows. We proceed by induction on the number of cells in Δ . If Δ has no cells, then the product has no factors. Let Δ have cells. Then it has at least one top cell (a cell whose top path is a subpath in the top path of Δ). Let us consider the rightmost top cell π of Δ . Let $[\Delta] = p[\pi]q$. Then Δ is a concatenation of an atomic diagram $\Pi = \varepsilon(p) + \pi + \varepsilon(q)$ and some diagram Δ' that has fewer cells. (Here Δ' is obtained from Δ by deleting the path $[\pi]$.) Taking the product of Π and the rightmost decomposition of Δ' , we obtain the rightmost decomposition of Δ .

Proof of Theorem 1.7. Let $\mathcal{P} = \langle x, a \mid x^3 = x^2, ax = a \rangle$. We need to prove that the universal diagram group $U = \mathcal{D}(\mathcal{P}, a)$ has property B.

Let Δ be an (a, a) -diagram over \mathcal{P} with N cells. It suffices to prove that the element $g \in U$ represented by Δ has length at most KN in generators x_0, x_1, x_2 , where $K > 0$ is a constant independent on g .

Let e_0 be the top edge of Δ . Suppose that there is an (a, ax) -cell whose top edge coincides with e_0 . Then we denote by e_1 the edge labelled by a on the bottom path of this cell. If e_1 is the top path of an (a, ax) -cell, then e_2 denotes the edge on the bottom path of this cell labelled by a . Proceeding in such a way, we finally obtain a sequence of edges e_0, \dots, e_k ($k \geq 0$).

Analogously, changing in the previous paragraph top by bottom, we define the sequence of edges f_0, \dots, f_m ($m \geq 0$), where f_0 is the bottom edge of Δ . A very easy geometric observation is that e_k must coincide with f_m . It is also easy to see that Δ is a concatenation of the form $\Delta = \Delta_1 \circ (\varepsilon(a) + \Delta') \circ \Delta_2$, where Δ_1 consists of (a, ax) -cells only, Δ_2 consists of (ax, a) -cells only and Δ' is an (x^k, x^m) -diagram over $\mathcal{P}' = \langle x \mid x^3 = x^2 \rangle$.

Let us apply the rightmost decomposition to $\varepsilon(a) + \Delta'$. Each factor is an atomic diagram of the form $(ax^s, x^2 = x^3, x^t)^\epsilon$, where $s, t \geq 0, \epsilon = \pm 1$. According to the description of U given in Subsection 2.2, this atomic diagram corresponds to x_j^ϵ . (Notice also that no generators correspond to atomic diagram with (a, ax) -cells and/or their inverses.)

Therefore, the rightmost decomposition of Δ' allows us to decompose g as a product (in U) of the form

$$g = x_{j_1}^{\epsilon_1} \dots x_{j_r}^{\epsilon_r} \quad \text{with } \epsilon_i = \pm 1, \tag{1}$$

where $r = N - k - m$ is the number of cells in Δ' . We need to establish some easy properties of the subscripts and the superscripts in (1).

Suppose that (1) contains a subword of the form $x_i^\epsilon x_j$. Then we claim that $i \leq j + 1$. Indeed, otherwise the cell that corresponds to x_j is located to the right of the cell corresponding to x_i^ϵ . (One can also see that $i > j + 1$ would imply that $x_i^\epsilon x_j$ equals $x_j x_{i+1}^\epsilon$ in U .)

Now suppose that (1) contains a subword of the form $x_i^\epsilon x_j^{-1}$. In this case we claim that $i \leq j + 2$. (Now $i > j + 2$ would also imply that the cell that corresponds to x_j^{-1} is located to the right of the one corresponding to x_i^ϵ . The element $x_i^\epsilon x_j^{-1}$ would be also equal to $x_j^{-1} x_{i-1}^\epsilon$ in U .)

For every $j \geq 0$ we replace each letter x_j in (1) by the word $u_j^{-1} v_j u_j$, where $u_j = 1, v_j = x_j$ if $j = 0, 1$ and $u_j = x_0^{j-2}, v_j = x_2$ if $j \geq 2$. We get the following word W in generators x_0, x_1, x_2 :

$$W = u_{j_1}^{-1} v_{j_1}^{\epsilon_1} u_{j_1} u_{j_2}^{-1} v_{j_2}^{\epsilon_2} \dots v_{j_r}^{\epsilon_r} u_{j_r}.$$

Since $x_j = u_j^{-1} v_j u_j$ in U , the word W represent the same element $g \in U$. Now let W_i denote the freely irreducible form of the word $u_{j_i} u_{j_{i+1}}^{-1}$ for all $1 \leq i < r$. It is also convenient to define the words $W_0 = u_{j_1}^{-1}$ and $W_r = u_{j_r}$. We are going to estimate the length of the word

$$\bar{W} = W_0 v_{j_1}^{\epsilon_1} W_1 \dots v_{j_r}^{\epsilon_r} W_r.$$

We know that $|W_0| = j_1 \leq k - 2$ (the first cell chosen in Δ' is at a distance j_1 from the rightmost point of Δ and the length of its top path is at least 2). Analogously,

$|W_r| = j_r \leq m - 2$. Also $|v_{j_1}| + \cdots + |v_{j_r}| = r$. So we estimate some part of the length of \bar{W} as follows:

$$|W_0| + |W_r| + |v_{j_1}| + \cdots + |v_{j_r}| \leq k + m + r - 4 < N. \quad (2)$$

It remains to estimate the sum $|W_1| + \cdots + |W_{r-1}|$. It is easy to see from the definitions that $|W_i| \leq |j_{i+1} - j_i|$ for all $1 \leq i < r$. Let I be the set of all $1 \leq i < r$ such that $j_i > j_{i+1}$. Then $|W_1| + \cdots + |W_{r-1}| \leq \sum_{1 \leq i < r} |j_{i+1} - j_i| = S_1 + S_2$, where $S_1 = \sum_{i \in I} (j_i - j_{i+1})$ and $S_2 = \sum_{i \notin I} (j_{i+1} - j_i)$.

For every $i \in I$, we know that $j_i - j_{i+1} \leq 2$. Hence, $S_1 \leq 2|I| \leq 2(r-1) < 2r$. On the other hand,

$$S_2 - S_1 = \sum_{1 \leq i < r} (j_{i+1} - j_i) = j_r - j_1 \leq j_r \leq m - 2 < k + m = N - r.$$

This gives $S_2 = S_1 + (S_2 - S_1) < N + r$ and so $S_1 + S_2 < N + 3r \leq 4N$. Using (2), we finally have

$$|\bar{W}| = (|W_0| + |W_r| + |v_{j_1}| + \cdots + |v_{j_r}|) + (|W_1| + \cdots + |W_{r-1}|) < 5N.$$

So our statement is true for $K = 5$.

Notice that the constant here is not optimal. \square

Let us recall the definition of the (restricted) wreath product of groups. Let G, H be two groups. If ϕ is a function from H to G , then the *support* of ϕ is the set $\text{supp}(\phi) = \{b \in H \mid \phi(b) \neq 1\}$. The restricted wreath product $G \wr H$ of the above groups will consist of all pairs of the form (b, ϕ) , where $b \in H$ and ϕ is a function from H to G with finite support. The group operation is defined in the standard way, that is, $(b_1, \phi_1)(b_2, \phi_2) = (b_1 b_2, \phi_1^{b_2} \phi_2)$, where $\phi^\beta(b) = \phi(\beta b)$ by definition ($\beta \in H$). For each of the groups G, H we have a canonical embedding into $G \wr H$. Namely, $a \mapsto (1, \phi_a)$, where $\phi_a(1) = a$, $\phi_a(b) = 1$ whenever $b \neq 1$. Also $b \mapsto (b, \mathbf{1})$, where $\mathbf{1}$ is the function from H to G that takes all elements to 1.

We will deal with restricted wreath products of finitely generated groups. If we fix some generating sets for G and for H , then the union of these sets (via canonical embeddings) will generate $G \wr H$. Hence we have three length functions: on G , on H , and on $G \wr H$. For simplicity, we will always freely speak about lengths in each of these three groups provided the generating sets for G and for H are clear. The same will concern Cayley graphs of these groups. The same symbol ℓ will be used to denote the length function for each of these groups. This will not lead to a confusion.

Working with lengths, we need the following fact proved in [25] (see Theorem 1.2).

Lemma 3.6. *Let G, H be finitely generated groups with fixed generating sets. Then the length of an element $(b, \phi) \in G \wr H$ is equal to the sum $\sum \ell(\phi(\beta)) + M$, where*

$\ell(a)$ is the length of $a \in G$ and M is the length of the shortest path in the Cayley graph of H starting at the identity, ending at b and visiting each element of $\text{supp}(\phi)$ at least once. The sum is taken over all $\beta \in \text{supp}(\phi)$.

Let $G = \mathcal{D}(\mathcal{P}, w)$ be a diagram group. We say that the base w is *protected* if w is a letter such that $w = uvv$ modulo \mathcal{P} implies that both words u, v are empty (for any u, v). For any diagram group $G = \mathcal{D}(\mathcal{P}, w)$, one can construct a new semigroup presentation \mathcal{P}' adding 3 new letters a, b, c to the alphabet and one new defining relation $a = bwc$. It is not hard to see that G will be isomorphic to the diagram group $\mathcal{D}(\mathcal{P}', a)$, where the base a is now protected.

It is shown in [14, Theorem 11] that, given a diagram group $G = \mathcal{D}(\mathcal{P}, y)$ with protected base y , one can construct a new semigroup presentation $\bar{\mathcal{P}}$ adding two new letters x, z to the alphabet and two new defining relations $x = xy, yz = z$ (see also Example 10 before the theorem we refer to). In this situation, the diagram group $\mathcal{D}(\bar{\mathcal{P}}, xz)$ will be isomorphic to the restricted wreath product $G \wr \mathbb{Z}$. (The concept of a protected base we gave slightly differs from the original one although the proof goes without any essential changes.) Now we are going to prove that if G has property B (as a diagram group over \mathcal{P}) then $G \wr \mathbb{Z}$ also has property B (as a diagram group over $\bar{\mathcal{P}}$).

Lemma 3.7. *In the above notation, if $G = \mathcal{D}(\mathcal{P}, y)$ has property B, then $G \wr \mathbb{Z} = \mathcal{D}(\bar{\mathcal{P}}, xz)$ also has property B.*

Proof. It suffices to prove that any diagram over $\bar{\mathcal{P}}$ with N cells has length $O(N)$ in the group $G \wr \mathbb{Z}$. Any reduced (xz, xz) -diagram over $\bar{\mathcal{P}}$ can be obtained as follows. Let us take finitely many (y, y) -diagrams over \mathcal{P} and let Ξ be their sum. Now let us add a few cells of the form $x = xy$ and a few cells of the form $z = yz$ to the diagram $\varepsilon(x) + \Xi + \varepsilon(z)$ on the top in order to make the top label equal to xz . Then we add a few cells of the form $xy = x$ and $yz = z$ on the bottom to make the bottom label also equal to xz . The result is an (xz, xz) -diagram Δ . Let us denote by o the vertex on the bottom path of Δ that splits this path into the product of two edges. Clearly, the vertex o is contained in Ξ and splits it into a sum. Let $\Xi = \Xi_k + \dots + \Xi_0 + \Xi_{-1} + \dots + \Xi_{-m}$, where each Ξ_j ($-m \leq j \leq k$) is a (y, y) -diagram and the vertex o subdivides Ξ into the sum of the two summands. All the Ξ_j that form the first summand satisfy $j \geq 0$, the ones that form the second summand satisfy $j < 0$. (Notice that each of the summands can be empty.)

Let $g_j \in G$ be the element of $\mathcal{D}(\mathcal{P}, y)$ represented by Ξ_j ($-m \leq j \leq k$). We also define $g_j = 1$ for any integer j not in $[-m, k]$. Then, according to [14, Example 10], the diagram Δ represents the element h of the group $G \wr \langle t \rangle$ of the following form: (t^d, ϕ) , where d, ϕ are defined as follows. The exponent d on t is the difference between the number of (z, yz) -cells and the number of (yz, z) -cells in Δ . The function ϕ from $\mathbb{Z} = \langle t \rangle$ is defined by the rule $\phi(t^j) = g_j$ for any integer j .

It remains to estimate the length of h . We will use Lemma 3.6. We see that the length of h equals

$$\sum_{j=-m}^k \ell(g_j) + M, \tag{3}$$

where M is the length of the shortest path in the Cayley graph of $\mathbb{Z} = \langle t \rangle$ that starts at the identity, ends at t^d and visits all those vertices t^j for which $g_j \neq 1$. It is clear that $M \leq 2(k + m + 1) + |d|$. Obviously, $|d|$ does not exceed the number of cells in Δ . It is also clear that $k + m + 1$ is the number of (x, xy) -cells plus the number of (z, yz) -cells in Δ . This implies $M = O(N)$.

Since $G = \mathcal{D}(\mathcal{P}, y)$ has property B, there exists a constant $C > 0$ such that the length of each $g \in G$ does not exceed $C \cdot \#(g)$, where $\#(g)$ denotes the number of cells in the reduced diagram over \mathcal{P} representing g . Therefore, the first summand in (3) does not exceed $C(\#(g_{-m}) + \dots + \#(g_0) + \dots + \#(g_k)) \leq CN$. Thus we finally have that the length of h in $G \wr \mathbb{Z}$ is $O(N)$. \square

Corollary 3.8. *Let $\mathcal{P} = \langle a, b, b_1, b_2, c \mid ab = a, bc = c, b = b_1, b_1 = b_2, b_2 = b \rangle$. Then the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$ has property B as a diagram group $\mathcal{D}(\mathcal{P}, ac)$.*

Indeed, \mathbb{Z} is the diagram group over $\langle b, b_1, b_2 \mid b = b_1, b_1 = b_2, b_2 = b \rangle$ with protected base b .

Let H be a finitely generated group with fixed generating set. By its *growth function* $\gamma(n)$ (with respect to this set of generators) we mean the number of elements in the ball of radius n around the identity in the Cayley graph of H .

Theorems 1.8 and 1.10 will follow from the next result.

Theorem 3.9. *Let H be a finitely generated group with fixed generating set. Suppose that its growth function satisfies the condition $\gamma(n) \geq n^k$ for some $k > 0$. Then the Hilbert space compression α of the group $\mathbb{Z} \wr H$ satisfies the following inequality:*

$$\alpha \leq \frac{1 + k/2}{1 + k}.$$

Proof. Let x be a generator of \mathbb{Z} in the wreath product $\mathbb{Z} \wr H$. Every element in $\mathbb{Z} \wr H$ is a pair (b, ϕ) where $b \in H$, ϕ is a function $g \mapsto x^{m_g}$ from H to \mathbb{Z} with finite support.

Let B_n be the ball of radius n around 1 in the Cayley graph of H . Consider the set

$$X_n = \{ w_b = b^{-1}x^{2n+1-\ell(b)}b \mid b \in B_n \},$$

where $\ell(b)$ is the length of b . Obviously, w_b has the form $(1, \phi_b)$ in $\mathbb{Z} \wr H$, where the support of ϕ_b is $\{ b \}$ and $\phi_b(b) = x^{2n+1-\ell(b)}$. Notice that $\ell(x^r) = |r|$ in $\langle x \rangle$ for all

$r \in \mathbb{Z}$. Hence by Lemma 3.6, the length of w_b in $\mathbb{Z} \wr H$ equals $2n + 1 - \ell(b) + M$, where $M = 2\ell(b)$. Therefore, $\ell(w_b) = 2n + 1 + \ell(b)$ is always between $2n + 1$ and $3n + 1$. Clearly, all elements of X_n pairwise commute. The number of them is exactly $\gamma(n)$.

It is easy to see that for every choice of $\epsilon_b \in \{1, -1\}$, $b \in B_n$, the length of the element $w = \prod_{b \in B_n} w_b^{\epsilon_b}$ is greater than $n\gamma(n)$. Indeed, w has the form $(1, \phi)$ in $\mathbb{Z} \wr H$, where $\text{supp}(\phi) = B_n$ and $\phi(b) = x^{2n+1-\ell(b)}$ for all $b \in B_n$. Hence, by Lemma 3.6, the length of w is at least

$$\sum_{b \in B_n} \ell(\phi(b)) = \sum_{b \in B_n} (2n + 1 - \ell(b)) > n\gamma(n).$$

Consider the skew cube spanned by X_n in $\mathbb{Z} \wr H$ and its image in \mathcal{H} under the uniform embedding f . The sides of the skew cube in \mathcal{H} do not exceed Cn for some constant C since our group is finitely generated. So the sum of squares of sides does not exceed $C^2 2^{\gamma(n)-1} \gamma(n) n^2$. On the other hand, the number of diagonals is $2^{\gamma(n)-1}$. Hence by the Skew Cube Inequality, there exists a diagonal of the skew cube in \mathcal{H} which does not exceed $Cn\sqrt{\gamma(n)}$. Thus there exist two points in the Cayley graph of G at distance $d \geq n\gamma(n)$ whose images under f are at distance at most $Cn\sqrt{\gamma(n)}$. If $\rho_1(d) = Kd^\alpha$ for some constant $K > 0$ in the definition of the uniform embedding, then $Cn\sqrt{\gamma(n)} \geq K(n\gamma(n))^\alpha$. Hence

$$n^{1-\alpha} \geq (\gamma(n))^{\alpha-1/2} \geq n^{k(\alpha-1/2)}.$$

This implies $1 - \alpha \geq k(\alpha - 1/2)$ and so $\alpha \leq (1 + k/2)/(1 + k)$. □

The next results are Theorem 1.8, Theorem 1.10, and Corollary 1.11 announced in the Introduction.

Theorem 3.10. *The Hilbert space compression of the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$ belongs to $[1/2; 3/4]$.*

Proof. The lower bound follows from Theorem 1.5 and the fact that $\mathbb{Z} \wr \mathbb{Z}$ satisfies property B as a diagram group $\mathcal{D}(\mathcal{P}, ac)$ over $\mathcal{P} = \langle a, b, b_1, b_2, c \mid ab = a, bc = c, b = b_1, b_1 = b_2, b_2 = b \rangle$ (Corollary 3.8).

The upper bound follows from Theorem 3.9 with $G = \mathbb{Z}$. Indeed, the growth function of \mathbb{Z} is linear, that is, $\gamma(n) \geq n^k$ for $k = 1$. Hence $\alpha \leq (1 + k/2)/(1 + k) = 3/4$. □

Recall that a finitely generated group has a *super-polynomial* growth whenever its growth function $\gamma(n)$ exceeds n^k for all $k \geq 1$.

Theorem 3.11. *Let H be a finitely generated group with super-polynomial growth. Then the Hilbert space compression of $\mathbb{Z} \wr H$ is at most $1/2$.*

Proof. Indeed, $\gamma(n) \geq n^k$ implies $\alpha \leq (1 + k/2)/(1 + k)$. The right-hand side of this inequality approaches $1/2$ as $k \rightarrow \infty$. \square

Corollary 3.12. *The Hilbert space compression of the group $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ is at most $1/2$.*

References

- [1] M. E. B. Bekka, P.-A. Cherix, A. Valette, Proper affine isometric actions of amenable groups. In *Novikov conjectures, index theorems and rigidity* (Oberwolfach, 1993), Vol. 2, London Math. Soc. Lecture Note Ser. 227, Cambridge University Press, Cambridge 1995, 1–4. [Zbl 0959.43001](#) [MR 1388307](#)
- [2] J. Bourgain, The metrical interpretation of superreflexivity in Banach spaces. *Israel J. Math.* **56** (2) (1986), 222–230. [Zbl 0643.46013](#) [MR 0880292](#)
- [3] J. Burillo, Quasi-isometrically embedded subgroups of Thompson’s group F . *J. Algebra* **212** (1) (1999), 65–78. [Zbl 0951.20028](#) [MR 1670622](#)
- [4] S. J. Campbell and G. A. Niblo, Hilbert space compression and exactness of discrete groups. *J. Funct. Anal.* **222** (2) (2005), 292–305. [Zbl 02175271](#) [MR 2132393](#)
- [5] P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg and A. Valette, *Groups with the Haagerup property. Gromov’s a -T-menability*. Progr. Math. 197, Birkhäuser Verlag, Basel 2001. [Zbl 1030.43002](#) [MR 1852148](#)
- [6] M. Deza and H. Maehara, A few applications of negative-type inequalities. *Graphs Combin.* **10** (3) (1994), 255–262. [Zbl 0813.05024](#) [MR 1304381](#)
- [7] P. Enflo, On the nonexistence of uniform homeomorphisms between L_p -spaces. *Ark. Mat.* **8** (1969), 103–105. [Zbl 0196.14002](#) [MR 0271719](#)
- [8] D. Farley, Finiteness and CAT(0) properties of diagram groups. *Topology* **42** (5) (2003), 1065–1082. [Zbl 1044.20023](#) [MR 1978047](#)
- [9] M. Gromov, Asymptotic invariants of infinite groups. In *Geometric group theory* (Sussex, 1991), Vol. 2, London Math. Soc. Lecture Note Ser. 182, Cambridge University Press, Cambridge 1993, 1–295. [Zbl 0841.20039](#) [MR 1253544](#)
- [10] M. Gromov, Problems (4) and (5). In *Novikov conjectures, index theorems and rigidity* (S. Ferry, A. Ranicki and J. Rosenberg, eds.), Vol. 1, London Math. Soc. Lecture Note Ser. 226, Cambridge University Press, Cambridge 1995, 67. [Zbl 0829.00027](#) [MR 1388294](#)
- [11] M. Gromov, Random walk in random groups. *Geom. Funct. Anal.* **13** (1) (2003), 73–146. [Zbl 01971826](#) [MR 1978492](#)
- [12] V. S. Guba and M. V. Sapir, Diagram groups. *Mem. Amer. Math. Soc.* **130** (620) (1997). [Zbl 0930.20033](#) [MR 1396957](#)
- [13] V. S. Guba and M. V. Sapir, The Dehn function and a regular set of normal forms for R. Thompson’s group F . *J. Austral. Math. Soc. (Ser. A)* **62** (1997), 315–328. [Zbl 0899.20013](#) [MR 1443551](#)
- [14] V. S. Guba, M. V. Sapir, On subgroups of the R. Thompson group F and other diagram groups. *Mat. Sb.* **190** (8) (1999), 3–60; English transl. *Sb. Math.* **190** (7–8) (1999), 1077–1130. [Zbl 01406862](#) [MR 1725439](#)

- [15] V. S. Guba and M. V. Sapir, Diagram groups and directed 2-complexes: homotopy and homology. *J. Pure Appl. Algebra* **205** (1) (2006), 1–47. [Zbl 05004392](#) [MR 193190](#)
- [16] V. S. Guba and M. V. Sapir, Diagram groups are totally orderable. *J. Pure Appl. Algebra* **205** (1) (2006), 48–73. [Zbl 05004393](#) [MR 2193191](#)
- [17] E. Guentner and J. Kaminker, Exactness and the Novikov conjecture. *Topology* **41** (2) (2002), 411–418; Addendum to this paper: *Topology* **41** (2) (2002), 419–420. [Zbl 0992.58002](#) [MR 1876896](#); [MR 1876897](#)
- [18] E. Guentner and J. Kaminker, Exactness and uniform embeddability of discrete groups. *J. London Math. Soc.* (2) **70** (3) (2004), 703–718. [Zbl 1082.46049](#) [MR 2160829](#)
- [19] N. Higson, Bivariant K -theory and the Novikov conjecture. *Geom. Funct. Anal.* **10** (3) (2000), 563–581. [Zbl 0962.46052](#) [MR 1779613](#)
- [20] V. Kilibarda. On the algebra of semigroup diagrams. PhD Thesis, Univ. of Nebraska, Lincoln 1994.
- [21] V. Kilibarda, On the algebra of semigroup diagrams. *Internat. J. Algebra Comput.* **7** (1997), 313–338. [Zbl 0879.20029](#) [MR 1448329](#)
- [22] N. Linial and M. Saks, The Euclidean distortion of complete binary trees. *Discrete Comput. Geom.* **29** (1) (2003), 19–21. [Zbl 1008.05040](#) [MR 1946791](#)
- [23] K. A. Mikhailova, The membership problem for direct products of groups. *Dokl. Akad. Nauk SSSR* **119** (1958), 1103–1105 (in Russian). [Zbl 0084.25302](#) [MR 0100018](#)
- [24] G. A. Niblo, L. D. Reeves, Groups acting on CAT(0) cube complexes. *Geom. Topol.* **1** (1997), 7. [Zbl 0887.20016](#) [MR 1432323](#)
- [25] W. Parry, Growth series of some wreath products. *Trans. Amer. Math. Soc.* **331** (2) (1992), 751–759. [Zbl 0793.20034](#) [MR 1062874](#)
- [26] G. Skandalis, J. L. Tu, and G. Yu, The coarse Baum-Connes conjecture and groupoids. *Topology* **41** (4) (2002), 807–834. [Zbl 1033.19003](#) [MR 1905840](#)
- [27] J.-L. Tu, Remarks on Yu’s property A for discrete metric spaces and groups. *Bull. Soc. Math. France* **129** (2001), 115–139. [Zbl 1036.58021](#) [MR 1871980](#)
- [28] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.* **139** (1) (2000), 201–240. [Zbl 0956.19004](#) [MR 1728880](#)

Received April 20, 2005

G. N. Arzhantseva, Section de Mathématiques, Université de Genève, CP 64, 1211 Genève 4, Switzerland

E-mail: Goulvara.Arjantseva@math.unige.ch

V. S. Guba, Department of Mathematics, Vologda State University, 6 S. Orlov St., Vologda, 160600, Russia

E-mail: guba@uni-vologda.ac.ru

M. V. Sapir, Department of Mathematics, Vanderbilt University, U.S.A.

E-mail: m.sapir@vanderbilt.edu