

Transcendental submanifolds of $\mathbb{R}P^n$

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Abstract. In this paper we give examples of closed smooth submanifolds of $\mathbb{R}P^n$ which are isotopic to nonsingular projective subvarieties of $\mathbb{R}P^n$ but they can not be isotopic to the real parts of nonsingular complex projective subvarieties of $\mathbb{C}P^n$.

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0. Introduction

Let $j: \mathbb{R}^n \hookrightarrow \mathbb{R}P^n$ be the canonical imbedding as a chart. Real algebraic sets in \mathbb{R}^n are not in general real algebraic sets in $\mathbb{R}P^n$. The Zariski closure of the image of an algebraic set (under j) usually has extra components at infinity. An algebraic subset of \mathbb{R}^n which remains an algebraic set in $\mathbb{R}P^n$ is called a *projectively closed algebraic set* ([AK1]). Not every algebraic set is projectively closed. In general, isotoping a submanifold of the projective space $\mathbb{R}P^n$ to an algebraic subset is a much harder problem than the corresponding problem in the affine case \mathbb{R}^n . In this paper we produce a transcendental submanifold of $\mathbb{R}P^n$ in the sense of [AK5]. That is, we find a smooth submanifold of $\mathbb{R}P^n$ which is isotopic to a nonsingular projective algebraic subset, but which can not be isotoped to the real part of any complex nonsingular algebraic subset of $\mathbb{C}P^n$. This results generalizes the affine examples of [AK5] to the projective case. We want to thank MSRI for giving us the opportunity to work together.

1. Preliminaries

By a *closed* (sub)manifold we mean a compact (sub)manifold without boundary.

Let V be a real (or complex) algebraic set defined over \mathbb{R} , and let $R = \mathbb{Z}_2$ (or $R = \mathbb{Z}$). Then we can define algebraic homology groups $H_*^A(V; R)$ to be the subgroup of $H_*(V; R)$ generated by the compact real (or complex) algebraic subsets of V

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(cf. [AK1]). We define $H_A^*(V; R)$ to be the Poincaré duals of the groups $H_*^A(V; R)$ when defined. By the resolution of singularities theorem ([H]), $H_*^A(V; R)$ is also the subgroup generated by the classes $g_*([S])$ where $g: S \rightarrow V$ is an entire rational function, S is a compact nonsingular real (or complex) algebraic set and $[S]$ is the fundamental class of S . Therefore even when V is real, we can define $H_*^A(V; \mathbb{Z})$ to be the subgroup generated by $g_*([S])$ where $g: S \rightarrow V$ is an entire rational function from an oriented compact nonsingular real algebraic set and $[S]$ is the fundamental class of S .

Now let $V \subset \mathbb{R}P^n$ be a nonsingular projective real algebraic set of dimension v , and suppose its complexification $V_{\mathbb{C}} \subset \mathbb{C}P^n$ is nonsingular. Let $j: V \hookrightarrow V_{\mathbb{C}}$ denote the inclusion. Define $\bar{H}_{2k}^A(V_{\mathbb{C}}; \mathbb{Z})$ to be the subgroup of $H_{2k}^A(V_{\mathbb{C}}; \mathbb{Z})$ generated by irreducible complex algebraic subsets defined over \mathbb{R} with k -dimensional real parts. In other words it is generated by the complexification of k -dimensional real algebraic subsets of V in $V_{\mathbb{C}}$. Again by the resolution theorem, $\bar{H}_{2k}^A(V_{\mathbb{C}}; \mathbb{Z})$ is generated by the classes $g_*([L_{\mathbb{C}}])$, where $L_{\mathbb{C}}$ is an irreducible nonsingular complex projective algebraic set defined over \mathbb{R} with nonempty real part and $g: L_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$ is a regular map defined over \mathbb{R} . Let $\bar{H}_A^{2k}(V_{\mathbb{C}}; \mathbb{Z})$ denote the Poincaré dual of $\bar{H}_{2v-2k}^A(V_{\mathbb{C}}; \mathbb{Z})$. Define

$$\bar{H}_{\mathbb{C}\text{-alg}}^{2k}(V; \mathbb{Z}) = j^* \bar{H}_A^{2k}(V_{\mathbb{C}}; \mathbb{Z}).$$

Let $\bar{H}_{\mathbb{C}\text{-alg}}^{2k}(V; \mathbb{Z}_2)$ to be the mod 2 reduction of $\bar{H}_{\mathbb{C}\text{-alg}}^{2k}(V; \mathbb{Z})$ (under the obvious coefficient homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}_2$). Define the natural subgroup of $H_A^{2k}(V; \mathbb{Z}_2)$

$$H_A^k(V; \mathbb{Z}_2)^2 = \{ \alpha^2 \mid \alpha \in H_A^k(V; \mathbb{Z}_2) \}.$$

Recall that Theorem A (b) of [AK5] relates these groups to each other:

Theorem 1. *For all k the following holds: $\bar{H}_{\mathbb{C}\text{-alg}}^{2k}(V; \mathbb{Z}_2) = H_A^k(V; \mathbb{Z}_2)^2$.*

Let $M \subset V$ be a closed smooth submanifold of a nonsingular algebraic set V . The problem of whether M is isotopic to a nonsingular algebraic subset of V is an old one. If we allow stabilization (replacing V by $V \times \mathbb{R}^k$ for sufficiently large k) the problem becomes solvable ([N] and [T] for the affine case, [K] for the projective case, and [AK2] for the general case where there are obstructions).

If we don't allow stabilization the problem becomes much harder, in which the complexification of V begins to play an important role. In [AK3] it was shown that every closed smooth $M \subset \mathbb{R}^n$ can be isotoped to a nonsingular (topological) component of an algebraic subset of \mathbb{R}^n . More generally in [AK4] it was shown that any immersed submanifold of \mathbb{R}^n can be isotoped to a nonsingular algebraic subset of \mathbb{R}^n if and only if M is cobordant through immersions to an algebraic subset of \mathbb{R}^n . Here we need a very special case of these theorems, the one which allows us to isotop some submanifolds of $\mathbb{R}P^n$ (the examples in the next section) to projectively closed

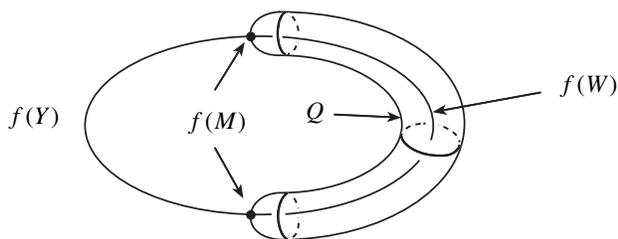
algebraic subsets. Obviously the following lemmas hold in more general contexts, but to make the examples of the main theorem in the next section transparent, we chose to state them in this special form, which is enough to prove the theorem.

Lemma 2. *Every closed codimension one submanifold of \mathbb{R}^k can be C^∞ approximated by a nonsingular projectively closed algebraic subset.*

Proof. This for example is proven in [K] (also Theorem 2.8.2 of [AK1]), but can also be seen from Seifert’s original proof [S] by noting that the highest degree terms of the polynomial he constructs are a constant times $|x|^{2n}$ (clearly the zeros of such polynomials are projectively closed algebraic sets). \square

Lemma 3. *Let $M^m \subset Y^{m+1}$ be closed smooth manifolds with M separating Y . Let $f: Y \rightarrow \mathbb{R}^k$ be a smooth imbedding. Then $f(M)$ is isotopic to a nonsingular projectively closed subvariety of \mathbb{R}^k . In particular, by viewing $f(M)$ as a submanifold of $\mathbb{R}P^k$ via the natural inclusion $\mathbb{R}^k \subset \mathbb{R}P^k$, $f(M)$ is isotopic to a nonsingular projective algebraic subvariety of $\mathbb{R}P^k$.*

Proof. By [AK3] we can isotop $f(Y)$ to a nonsingular topological component Y' of a real algebraic subvariety of \mathbb{R}^k . Let $W \subset Y$ be one of the codimension zero components of $Y - M$, and let $Q^{k-1} \subset \mathbb{R}^k$ be the boundary of a small tubular neighborhood of $f(W)$ in \mathbb{R}^k . By Lemma 2 above we can isotop Q to a projectively closed nonsingular algebraic subvariety Q' of \mathbb{R}^k which is C^1 close to Q . Then $V := Y' \cap Q'$ gives the desired variety. V is isotopic to $f(M)$ by transversality. V is projectively closed since Q' is projectively closed. \square



2. Transcendental submanifolds

In [AK5] we constructed smooth submanifolds $M \subset \mathbb{R}^n$ which are isotopic to nonsingular algebraic subsets of \mathbb{R}^n , but not isotopic to the real parts of nonsingular complex algebraic subsets of $\mathbb{C}P^n$. So, this means that either every algebraic model of M in \mathbb{R}^n develops extra components at infinity when its Zariski closure is taken in $\mathbb{R}P^n$ (i.e. it

is not projectively closed), or M admits nonsingular algebraic models in $\mathbb{R}P^n$ but the complexifications all such models necessarily contain singular points in $\mathbb{C}P^n$. The following theorem eliminates the first possibility, hence it gives genuine topological obstructions to moving smooth submanifolds of $\mathbb{R}P^n$ to nonsingular algebraic sets in the strong sense (i.e. no singularities in complexification).

Theorem 4. *There are closed smooth submanifolds $M \subset \mathbb{R}P^n$ which can be approximated (via a small isotopy) by nonsingular subvarieties of $\mathbb{R}P^n$, but they can not be isotoped to the real parts of nonsingular complex algebraic subvarieties of $\mathbb{C}P^n$ defined over \mathbb{R} .*

Proof. By [MM] for any s there is an m such that we have an imbedding $\mathbb{R}P^m \subset \mathbb{R}^{2m-s}$. Let $M = \mathbb{R}P^m \times S^1 \subset \mathbb{R}^{2m-s} \times \mathbb{R}^3 = \mathbb{R}^{2m+3-s} \subset \mathbb{R}P^n$, where $n = 2m + 3 - s$. Let $Y = \mathbb{R}P^m \times S^2$ in $\mathbb{R}P^n$, so by Lemma 2 above M can be isotoped to a nonsingular projectively closed algebraic subset V^v of $\mathbb{R}P^n$ where $v = m + 1$.

We claim that when $s \geq 3$ when m is even, and $s \geq 5$ when $m = 4k + 1$, V can not be the real part of a nonsingular complex algebraic subset $V_{\mathbb{C}}$ of $\mathbb{C}P^n$ (defined over \mathbb{R}). Suppose such a $V_{\mathbb{C}}$ exists. By a version of the Lefschetz hyperplane theorem due to Larsen [Hr], for $i \leq 2v - n = s - 1$, the restriction induces an isomorphism

$$H^i(\mathbb{C}P^n; \mathbb{Z}) \xrightarrow{\cong} H^i(V_{\mathbb{C}}; \mathbb{Z}).$$

In particular when $i \leq s - 1$, the group $\bar{H}_{\mathbb{C}\text{-alg}}^i(V; \mathbb{Z})$ lies in the image of $H^i(\mathbb{R}P^n; \mathbb{Z})$ under restriction. But since V lies in a chart \mathbb{R}^n of $\mathbb{R}P^n$, from the diagram

$$\begin{array}{ccc} H^i(V_{\mathbb{C}}; \mathbb{Z}_2) & \xrightarrow{j^*} & H^i(V; \mathbb{Z}_2) \\ \cong \uparrow & & \uparrow \\ H^i(\mathbb{C}P^n; \mathbb{Z}_2) & \xrightarrow{j^*} & H^i(\mathbb{R}P^n; \mathbb{Z}_2) \end{array}$$

we conclude that $\bar{H}_{\mathbb{C}\text{-alg}}^i(V; \mathbb{Z}_2) = 0$ for $0 < i \leq s - 1$. On the other hand by [AK1] the Stiefel–Whitney classes of V are represented by real algebraic subsets since V is a nonsingular real algebraic set. Hence, when m is even $w_1(V) = \alpha \times 1$ is algebraic, also when $m = 4k + 1$ then $w_2(V) = \alpha^2 \times 1$ is algebraic, where α is the generator of $H^1(\mathbb{R}P^m; \mathbb{Z}_2)$. So when m is even and $s \geq 3$, or when $m = 4k + 1$ and $s \geq 5$ we get a contradiction to Theorem 1 above, for example when m is even

$$0 \neq \alpha^2 \times 1 = w_1^2(M) \in H_A^1(V; \mathbb{Z}_2)^2 = \bar{H}_{\mathbb{C}\text{-alg}}^2(V; \mathbb{Z}_2) = 0.$$

We should point out that the above mentioned theorem of [MM] actually implies that there are imbeddings $\mathbb{R}P^m \subset \mathbb{R}^{2m-s}$ for the pairs (m, s) with m even and $s \geq 3$, or $m = 4k + 1$ and $s \geq 5$. This is what is used in the proof. \square

Remark 5. Recall that on any nonsingular real algebraic variety structure V of a smooth manifold M , the Stiefel–Whitney classes, and mod 2 reductions of Pontryagin classes of M are algebraic in V ([AK1], [AK5]). So the examples in the above theorem generalize in many directions. For example we can take M^v to be any closed smooth manifold which admits a separating imbedding into a closed manifold Y^{v+1} , such that $Y^{v+1} \subset \mathbb{R}^{2v-q}$ with $2i \leq q$, and $\alpha \in H^i(M; \mathbb{Z}_2)$ such that $\alpha^2 \neq 0$ and α lies in the subring of the cohomology group generated by Stiefel–Whitney and Pontryagin classes.

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