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On $(-P \cdot P)$-constant deformations of Gorenstein surface singularities

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Abstract. Let $\pi\colon X \to T$ be a small deformation of a normal Gorenstein surface singularity $X_0 = \pi^{-1}(0)$ over the complex number field $\mathbb{C}$. Suppose that T is a neighborhood of the origin of $\mathbb{C}$ and that X_0 is not log-canonical. We show that if a topological invariant $-P_t \cdot P_t$ of $X_t = \pi^{-1}(t)$ is constant, then, after a suitable finite base change, π admits a simultaneous resolution $f\colon M \to X$ which induces a locally trivial deformation of each maximal string of rational curves at an end of the exceptional set of $M_0 \to X_0$; in particular, if X_0 has a star-shaped resolution graph, then π admits a weak simultaneous resolution (in other words, π is an equisingular deformation).

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1. Introduction

We continue the study of a family of Gorenstein surface singularities preserving a certain topological invariant ([15]). Let (X_0, x_0) be a normal complex Gorenstein surface singularity and $\pi\colon X \to T$ a flat deformation of (X_0, x_0), where T is a reduced complex space. Let $f\colon M \to X$ be a proper modification with the exceptional set E. Then $f\colon M \to X$ is called a *very weak* simultaneous resolution if $\pi \circ f$ is flat and $f_t\colon M_t \to X_t$ is a resolution of X_t for all $t \in T$. Laufer proved [11, Theorem 4.3] that the constancy of a *topological* invariant $-K \cdot K$ in the deformation π implies the existence of a simultaneous canonical model (which is also called a simultaneous RDP resolution); then he obtained the following

Theorem 1.1 (Laufer [11, Theorem 5.7])**.** *π admits a very weak simultaneous resolution after a finite base change if and only if $-K_t \cdot K_t$ is constant, where K_t is the canonical divisor on the minimal resolution space of $X_t = \pi^{-1}(t)$.*

However, the structure of the exceptional divisor in a very weak simultaneous resolution can vary greatly. Let us recall a strong kind of simultaneous resolution;

$f\colon M \to X$ is called a *weak* simultaneous resolution if it is a very weak simultaneous resolution and the morphism $E \to T$ induced by $\pi \circ f$ is a locally trivial deformation. If a weak simultaneous resolution of π exists, then π is called an equisingular deformation [20]. It is shown [11, Theorem 6.4] that π admits a weak simultaneous resolution if and only if each singularity (X_t, x_t) is homeomorphic to (X_0, x_0). But, at present, there is no statement about the existence of weak simultaneous resolutions similar to Theorem 1.1.

In this paper, we deal with deformations of Gorenstein surface singularities preserving the topological invariant $-P \cdot P$, where P denotes the nef-part of the Zariski decomposition of the log-canonical divisor on a good resolution [21]. We shall show that such a family has a simultaneous resolution with some nice properties; it is a weak simultaneous resolution in a special case. Assume that T is a sufficiently small neighborhood of the origin of the complex number field $\mathbb{C}$ and the (X_0, x_0) is not a log-canonical singularity. In [14], we obtained that if the topological invariant $-P_t \cdot P_t$ is constant, then π admits a simultaneous log-canonical model; it is a log-version of Laufer's result mentioned before Theorem 1.1. In [15], we proved that the constancy of $-P_t \cdot P_t$ implies not only the log-version above, but also the existence of a simultaneous resolution $f\colon M \to X$, after a finite base change, such that each $f_t\colon M_t \to X_t$ is a resolution with the exceptional divisor having only normal crossings, and f_t is minimal among resolutions with such properties. Our new result in this paper gives a geometric characterization of $(-P \cdot P)$-constant deformations that clarifies what structure of the exceptional set is preserved. We prove the following

Theorem 1.2. *Assume that $-P_t \cdot P_t$ is constant. Then, after a finite base change, there exists a section $\gamma\colon T \to X$ of π such that each $\gamma(t)$ is a non-log-canonical singularity and a simultaneous resolution $f\colon M \to X$ which satisfy the following conditions*:

(1) *for each $t \in T$, $f_t\colon M_t \to X_t$ is a resolution with the exceptional divisor having only normal crossings, and f_t is minimal among resolutions with such properties*;
(2) *if E denotes the reduced divisor such that $f(E) = \gamma(T)$, then the restriction E_t of E is the reduced divisor supported on $f^{-1}(\gamma(t))$*;
(3) *there exists a reduced divisor $S \le E$ such that S_t is the sum of all maximal strings of rational curves at the ends of E_t for each $t \in T$ and that $\pi \circ f|_S\colon S \to T$ is a locally trivial deformation.*

Any singular point on $X_t \setminus \{\gamma(t)\}$ is a rational double point of type A_n.

Corollary 1.3. *Assume that $-P_t \cdot P_t$ is constant and that the resolution graph of (X_0, x_0) is star-shaped. Then each X_t has only one singular point x_t and π is an equisingular deformation.*

In case where X_t has only a singularity x_t, an outline of the proof of Theorem 1.2 is as follows. Let $f\colon M \to X$ be a resolution which satisfies the condition (1) of Theorem 1.2 and $g\colon Y \to X$ the simultaneous log-canonical model (the existence of them follows from [14] and [15], respectively). Denote by E and F the exceptional divisor of f and g, respectively. First, we shall show that there exists a morphism $h\colon M \to Y$ such that $f = g \circ h$. Let $P = h^*(K_Y + F)$ and $N = K_M + E - P$. Next, we verify that the restriction $P_t + N_t$ is the Zariski decomposition of the log-canonical divisor on M_t. Then it follows that $S := \mathrm{Supp}(N)$ satisfies the condition (3) of Theorem 1.2.

In [3], Ishii proved that for a small deformation of any normal surface singularity, the constancy of the invariant $-K \cdot K$ implies the existence of the simultaneous canonical model of the deformation. We hope that Theorem 1.2 may be generalized to the non-Gorenstein case.

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Notation and terminology

We denote by $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Q}$, the set of integers, the set of positive integers and the set of rational numbers, respectively. Let X be a normal variety. For a $\mathbb{Q}$-divisor $D = \sum d_i D_i$ on X, where D_i are distinct prime divisors, we write $D_{red} = \sum_{d_i \neq 0} D_i$. We say that a resolution $f\colon M \to X$ of X is *semigood* (resp. *good*) if the exceptional set of f is a divisor having only normal crossings (resp. simple normal crossings). Let $g\colon Y \to X$ be a partial resolution and E the reduced exceptional divisor of g. Then g is called a *canonical model* of X if Y has only canonical singularities and K_Y is g-ample; it is called a *log-canonical model* of X if the pair (Y, E) has only log-canonical singularities and $K_Y + E$ is g-ample.

2. Preliminaries

In this section, we review some results on surface singularities needed later. A *minimal semigood* (resp. *minimal good*) resolution of a normal surface singularity is the smallest resolution among all semigood (resp. good) resolutions. The minimal semigood resolution is obtained from the minimal good resolution by contracting each (-1)-curve intersecting one component twice. The *weighted dual graph* of a normal surface singularity is that of the exceptional divisor on the minimal good resolution of the singularity.

Let (X, x) be a normal surface singularity and $f\colon (M, A) \to (X, x)$ the minimal semigood resolution with the exceptional divisor A. Let K be a canonical divisor on M and $A = \bigcup_{i=1}^t A_i$ the decomposition into irreducible components. We call a divisor (resp. $\mathbb{Q}$-divisor) on M supported in A a *cycle* (resp. $\mathbb{Q}$-*cycle*). For any divisors D and E on M, the intersection number $D \cdot E$ is defined as $\nu(D) \cdot \nu(E)$, where $\nu(D)$ denotes a $\mathbb{Q}$-cycle determined by $(\nu(D) - D) \cdot A_i = 0$ for $1 \leq i \leq t$. Let $P + N$ be the Zariski decomposition of $K + A$: N is an effective $\mathbb{Q}$-cycle such that $P = K + A - N$ is f-nef and $P \cdot A_i = 0$ for all $A_i \leq N_{red}$ (see [17, Theorem A.1]). The intersection number $-P \cdot P$ is a topological invariant of the singularity (X, x), and its fundamental properties are stated in [21].

Definition 2.1. Let $S = \sum_{i=1}^{n} A_i$ be a chain of nonsingular rational curves. We call S a string at an end of A if $A_i \cdot A_{i+1} = 1$ for $1 \leq i \leq n-1$, and these account for all intersections in A among the A_i's, except that A_n intersects exactly one other curve. Let $S^* = \sum_{i=1}^{n} a_i A_i$ be a $\mathbb{Q}$-cycle such that $S^* \cdot A_1 = -1$ and $S^* \cdot A_i = 0$ $(i > 1)$. Note that $a_i > 0$ for $i = 1, \ldots, n$.

Lemma 2.2. *In the situation above, we have the inequalities*

$$a_{n-j+1} \leq j a_{n-j}/(j+1), \quad j = 1, \ldots, n-1.$$

Hence $a_1 > a_2 > \cdots > a_n$.

Proof. Let $-b_i = A_i \cdot A_i$. Then $b_i \geq 2$. By the definition of S^*, we have $a_{k-1} - b_k a_k + a_{k+1} = 0$ for $1 \leq k \leq n$, where $a_0 = 1$ and $a_{n+1} = 0$. It is clear that $a_n \leq a_{n-1}/2$. Now use induction on j. □

Proposition 2.3 (Wahl [21, Proposition 2.3, (2.7)])**.** *Suppose (X, x) is not a quotient, simple elliptic, or cusp singularity. Let $\{S_1, \ldots, S_p\}$ be the set of all maximal strings at the ends of A. Then $N = \sum_{i=1}^{p} S_i^*$.*

Lemma 2.4 (see [13, Lemma 1.8])**.** *If (X, x) is not a rational double point, then $[N] = 0$, where $[N]$ denotes the integral part of N.*

The m-th L^2-plurigenus of (X, x) is expressed as

$$\delta_m(X, x) = \dim_{\mathbb{C}} \mathcal{O}_X(mK_X)/f_* \mathcal{O}_M(mK + (m-1)A)$$

(see [22, pp. 67–68]). $\delta_1(X, x)$ is equal to the geometric genus $p_g(X, x)$.

Theorem 2.5 (see [13])**.** *There exists a bounded function $v(m)$ such that*

$$\delta_{m+1}(X, x) = -(P \cdot P)m^2/2 - (K \cdot P)m/2 + p_g(X, x) + v(m)$$

for $m \geq 0$. If (X, x) is a Gorenstein singularity with $p_g(X, x) \geq 1$, then the function $v(m)$ is determined by the weighted dual graph of the maximal strings at the ends of A.

Assume that (X, x) is not a log-canonical singularity, or equivalently that $\nu(P) \neq 0$ (see [21, Remark 2.4], [6, §9]). Let $g\colon Y \to X$ be the log-canonical model and F the exceptional divisor of g. Then we obtain a morphism $h\colon M \to Y$, which is the minimal resolution of the singularities of Y, and $P \sim_{\mathbb{Q}} h^*(K_Y + F)$ (see

[15, §3]). Let C be a reduced cycle which is the sum of the components A_i such that $P \cdot A_i = 0$. Then C is exactly the exceptional divisor for h, and contains no (-1)-curves. Let C_0 be the sum of the components $A_i \leq C$ such that $A_i \cdot A_i = -2$.

Definition 2.6. Let $\bar{X}$ be a normal surface obtained by contracting the cycle C_0 on M. Then $\bar{X}$ has only rational double points. We call the natural morphism $\bar{X} \to X$ an RDP good resolution of the singularity (X, x).

Lemma 2.7. *The natural morphism $h' \colon \bar{X} \to Y$ is the canonical model of Y.*

Proof. Since a rational double point is a canonical singularity, it suffices to show that $K_{\bar{X}}$ is h'-ample. Let $\varphi \colon M \to \bar{X}$ be the contraction. Then for any irreducible curve $\ell \subset \varphi(C)$, we have $K_{\bar{X}} \cdot \ell = K \cdot \varphi_*^{-1}\ell > 0$, where $\varphi_*^{-1}\ell$ denotes the strict transform of ℓ. Hence $K_{\bar{X}}$ is h'-ample. □

The following theorem gives another construction of the RDP good resolution.

Theorem 2.8 (see [15, Theorem 3.2])**.** *Let r be a positive integer such that rN is a cycle, and let $f' \colon (M', A') \to (X, x)$ be any semigood resolution. Then there exists a positive integer $\beta(X, x)$ determined by the weighted dual graph of (X, x) such that for any $m \geq \beta(X, x)$, the blowing-up of X with respect to the sheaf $f'_*\mathcal{O}_{M'}(K_{M'} + mr(K_{M'} + A'))$ is the RDP good resolution of (X, x).*

3. Simultaneous resolution

Let (X_0, x_0) be a normal Gorenstein surface singularity and $\pi \colon X \to T$ a deformation of $X_0 = \pi^{-1}(0)$, where T is an open neighborhood of the origin of $\mathbb{C}$. Then each X_t is normal and Gorenstein. We assume that (X_0, x_0) is not log-canonical. The aim of this section is to show that a simultaneous RDP good resolution of π is obtained as the canonical model of a simultaneous log-canonical model of π.

For any morphism $h \colon W \to X$, we denote by W_t the fiber $(\pi \circ h)^{-1}(t)$ and by h_t the restriction $h|_{W_t} \colon W_t \to X_t$.

Definition 3.1 (cf. Laufer [11, V])**.** Let $f \colon M \to X$ be a resolution of the singularities of X and E the exceptional set of f. We call $f \colon M \to X$ a weak simultaneous resolution if each f_t is a resolution of X_t and $\pi \circ f|_E \colon E \to T$ is a locally trivial deformation of the exceptional divisor of M_0.

We assume that T is sufficiently small so that $\pi|_{X \setminus X_0} : X \setminus X_0 \to T \setminus \{0\}$ admits a weak simultaneous resolution. We note that if π admits a weak simultaneous resolution along a section $\gamma \colon T \to X$ of π, then the weighted dual graph of $(X_t, \gamma(t))$ is the same as that of (X_0, x_0) (see [11, VI]).

Let us review some results on simultaneous partial resolutions studied in [14] and [15]. Let $g\colon Y \to X$ be the log-canonical model of X and F the reduced exceptional divisor of g.

Definition 3.2 (cf. [14, Definition 4.1 and Lemma 4.2])**.** We call the morphism g a simultaneous log-canonical model of π if for any $t \in T$ the restriction $g_t\colon Y_t \to X_t$ is the log-canonical model of X_t and F_t is a reduced divisor supported on the exceptional set of g_t.

Let $f(t)\colon \tilde{X}_t \to X_t$ be the minimal semigood resolution, A_t the exceptional divisor and K_t the canonical divisor on $\tilde{X}_t$. Let $A_{t,p}$ be the connected component of A_t which blows down to a singular point $p \in X_t$. Let $P_{t,p} + N_{t,p}$ be the Zariski decomposition of $K_t + A_{t,p}$, where $N_{t,p}$ is a $\mathbb{Q}$-divisor supported in $A_{t,p}$. We put $N_t := \sum_p N_{t,p}$ and $P_t \cdot P_t := \sum_p P_{t,p} \cdot P_{t,p}$.

Theorem 3.3 (see [14, Theorem 4.11])**.** *The following conditions are equivalent*:

(1) *g is the simultaneous log-canonical model of π*;

(2) *$-P_t \cdot P_t$ is constant.*

The next lemma follows from Theorem 3.3, [14, Remark 4.3], [15, Lemma 4.2] and [5, Proposition 2.2].

Lemma 3.4. *Suppose that $-P_t \cdot P_t$ is constant. Then there exists a section $\gamma\colon T \to X$ of π such that $(X_t, \gamma(t))$ is a non-log-canonical singularity and any singularity on $X_t \setminus \{\gamma(t)\}$ is a rational double point for each $t \in T$ (note that $g(F) = \gamma(T)$).*

The idea for the proof of the next lemma is due to Tomari [19].

Lemma 3.5. *Suppose that $-P_t \cdot P_t$ is constant. Let $\alpha\colon W \to Y$ be a morphism such that $g \circ \alpha$ is a semigood resolution of X, and let B be the exceptional set of $g \circ \alpha$. Then $\alpha_* \mathcal{O}_W(m(K_W + B) - B) = \mathcal{O}_Y(m(K_Y + F) - F)$ for any $m \in \mathbb{N}$.*

Proof. Let $L^W = K_W + B$ and $L^Y = K_Y + F$. Since X is Gorenstein and L^Y is g-ample, there exists a $\mathbb{Q}$-Cartier effective divisor F' supported on F such that $-F' \sim_{\mathbb{Q}} L^Y$. It is clear that $\alpha_* \mathcal{O}_W(mL^W - B) \subset \mathcal{O}_Y(mL^Y - F)$. To prove the converse, we may assume that Y is Stein. So it suffices to show the following

$$H^0(W, \mathcal{O}_W(mL^W - B)) \supset \alpha^* H^0(Y, \mathcal{O}_Y(mL^Y - F)).$$

Let $\omega \in H^0(Y, \mathcal{O}_Y(mL^Y - F))$. Then $\operatorname{div}(\omega) + mL^Y - F \geq 0$. Let n be a positive integer such that $nF \geq F'$. Then

$$\operatorname{div}(\omega) + mL^Y - (1/n)F' \geq \operatorname{div}(\omega) + mL^Y - F \geq 0.$$

Note that the left hand side is a $\mathbb{Q}$-Cartier divisor. Since L^Y is log-canonical, there exists an exceptional effective divisor Δ such that $L^W = \alpha^* L^Y + \Delta$. By

Lemma 3.4, we see that $Y \setminus F$ has only canonical singularities (see [16, Theorem 2.6]). Thus $\mathrm{Supp}(\Delta + \alpha^* F) = B$. It follows from the inequality above that

$$\mathrm{div}(\alpha^*\omega) + mL^W \geq m\Delta + (1/n)\alpha^* F'.$$

Since $\mathrm{Supp}(m\Delta + (1/n)\alpha^* F') = B$ and the left hand side is an integral divisor, we obtain that $\mathrm{div}(\alpha^*\omega) + mL^W \geq B$, i.e., $\alpha^*\omega \in H^0(W, \mathcal{O}_W(mL^W - B))$. □

Let $f\colon M \to X$ be a semigood resolution and E the exceptional divisor of f. Since $\pi|_{X \setminus X_0}$ admits a weak simultaneous resolution, there exists a positive integer r such that rN_t is a cycle for any $t \in T$. Assume that $r(K_Y + F)$ is a Cartier divisor. Let $\psi_m\colon X_m \to X$ be the blowing-up of X with respect to the sheaf $f_*\mathcal{O}_M(K_M + mr(K_M + E))$ for $m \geq 0$. Note that these sheaves are independent of the choice of the semigood resolution.

In the following, an RDP good resolution of X_t means a partial resolution which is the RDP good resolution of a non-log-canonical singularity (X_t, x_t) and an isomorphism over $X_t \setminus \{x_t\}$.

Theorem 3.6 (see the proof of [15, Theorem 4.2])**.** *Suppose that $-P_t \cdot P_t$ is constant. Let γ be as in Lemma 3.4 and $\beta(X)$ the maximum of $\{\beta(X_t, \gamma(t)) \mid t \in T\}$ (see Theorem 2.8). Then for any $m \geq \beta(X)$, there exists a neighborhood T_m of $0 \in T$ such that each $(\psi_m)_t\colon (X_m)_t \to X_t$ is the RDP good resolution for $t \in T_m$.*

To simplify the notation, we write T (resp. π) instead of T_m (resp. $\pi|_{\pi^{-1}(T_m)}$).

Proposition 3.7. *Suppose that $-P_t \cdot P_t$ is constant. Then the natural rational map $\varphi_m\colon X_m \to Y$ is a morphism for $m >> 0$. If $m \geq \beta(X)$ and φ_m is a morphism, then φ_m is the canonical model of Y.*

Proof. Assume that $m \geq \beta(X)$. Let A' be the exceptional set of $(\psi_m)_0\colon (X_m)_0 \to X_0$. Then φ_m is a morphism on $X_m \setminus A'$, since $\pi|_{X \setminus X_0}$ admits a weak simultaneous resolution. There exists an effective divisor Z on Y such that $K_Y \sim -Z$ and $\mathrm{Supp}(Z) = F$. Let $g'\colon Y' \to Y$ be the normalization of the blowing-up of Y with respect to the sheaf of ideals $\mathcal{O}_Y(-Z)$. We take a semigood resolution $f_m\colon M_m \to X$ of X such that the following diagram of morphisms is commutative:

$$\begin{array}{ccccc} M_m & & \xrightarrow{\tilde{\psi}_m} & & X_m \\ {\scriptstyle h_m}\downarrow & & & & \downarrow{\scriptstyle \psi_m} \\ Y' & \xrightarrow[g']{} & Y & \xrightarrow[g]{} & X \end{array}$$

where $f_m = \psi_m \circ \tilde{\psi}_m$. Let G' be a Cartier divisor on Y' such that $\mathcal{O}_{Y'}(G') = {g'}^*\mathcal{O}_Y(-Z)/\mathrm{torsion}$ and $G_m = h_m^* G'$. Let E_m be the exceptional divisor of f_m.

We put $L_m^M = mr(K_{M_m} + E_m)$, $L_m^Y = mr(K_Y + F)$ and $P_m = (g' \circ h_m)^* L_m^Y$. Let D_m be a Cartier divisor on M_m such that

$$\mathcal{O}_{M_m}(D_m) = f_m{}^* f_{m*} \mathcal{O}_{M_m}(K_{M_m} + L_m^M)/\text{torsion}.$$

Then D_m and P_m are f_m-nef.

Now let us show the claim: $D_m \sim G_m + P_m$ for $m >> 0$. Since L_1^Y is a g-ample Cartier divisor, the natural homomorphism

$$g^* g_* \mathcal{O}_Y(K_Y + L_m^Y) \to \mathcal{O}_Y(K_Y + L_m^Y)$$

is surjective for $m >> 0$. Then we have the surjection

$$(g' \circ h_m)^* g^* g_* \mathcal{O}_Y(K_Y + L_m^Y) \to \mathcal{O}_{M_m}(G_m + P_m).$$

By Lemma 3.5, the left hand side is equal to $f_m{}^* f_{m*} \mathcal{O}_{M_m}(K_{M_m} + L_m^M)$. Hence we have $\mathcal{O}_{M_m}(D_m) \cong \mathcal{O}_{M_m}(G_m + P_m)$.

To show that φ_m is a morphism, it suffices to prove that if $D_m \cdot \ell = 0$ for an irreducible curve $\ell \subset \tilde{\psi}_m^{-1}(A')$, then $P_m \cdot \ell = 0$. Let Λ be the set of irreducible curves on Y_0' which are $g \circ g'$-exceptional but not g'-exceptional. Since g' is isomorphic over the non-singular locus of Y, each curve in Λ is the strict transform of an irreducible component of F_0. We take m such that $D_m \sim G_m + P_m$ and $-m < \min\{G' \cdot \ell' \mid \ell' \in \Lambda\}$. Suppose that $D_m \cdot \ell = 0$ and $P_m \cdot \ell > 0$ for a curve $\ell \subset \tilde{\psi}_m^{-1}(A')$. Then $h_m(\ell) \in \Lambda$. Let d be the degree of the finite morphism $\ell \to h_m(\ell)$. Since L_1^Y is Cartier, $P_m \cdot \ell \geq dm$. Then we have $dG' \cdot h_m(\ell) = G_m \cdot \ell \leq -dm$: however it contradicts the choice of m.

Assume that φ_m is a morphism on X_m. By Lemma 2.7, the divisor $K_{X_m}|_{(X_m)_t}$ is $(\varphi_m)_t$-ample for any $t \in T$. Hence K_{X_m} is φ_m-ample. By Theorem 3.6 and [16, Theorem 2.6], X_m has only canonical singularities. Hence φ_m is the canonical model of Y. □

4. The main result

Let (X_0, x_0) be a normal Gorenstein surface singularity and $\pi\colon X \to T$ a deformation of $X_0 = \pi^{-1}(0)$. We always assume that T is sufficiently small; so $\pi|_{X \setminus X_0}$ admits a weak simultaneous resolution. We shall prove that the constancy of $-P_t \cdot P_t$ implies the existence of a simultaneous resolution $f\colon M \to X$ and a section $\gamma\colon T \to X$ which satisfy the following

Condition 4.1. Let E denote the reduced exceptional divisor on M such that $f(E) = \gamma(T)$.

(1) For each $t \in T$, $f_t\colon M_t \to X_t$ is the minimal semigood resolution and E_t is the reduced divisor supported on $f_t^{-1}(\gamma(t))$.

(2) There exists a divisor $S \leq E$ such that S_t is the sum of all maximal strings at the ends of E_t for each $t \in T$ and that $\pi \circ f|_S\colon S \to T$ is a locally trivial deformation.

Example 4.2. Let (X_0, x_0) be a minimally elliptic singularity which has the following weighted dual graph (we denote it by $A_n(w_1, w_2, w_3, w_4)$):

($n + 4$ vertices)

By using [4, Corollary 3.9], for any positive integer $k < n$, we can construct a deformation $\pi\colon X \to T$ of X_0, a section $\gamma\colon T \to X$ and a simultaneous resolution $f\colon M \to X$ which satisfy Condition 4.1 such that the weighted dual graph of $(X_t, \gamma(t))$ is $A_k(w_1, w_2, w_3, w_4)$ for $t \neq 0$.

In general, some rational double points of type A_q arise on X_t. There is a concrete example. According to Table 1 in [8, V], the weighted dual graph of the singularity $(\{z^2 - (y + x^3)(y^2 + x^{n+5}) = 0\}, o) \subset (\mathbb{C}^3, o)$ is $A_n(w_1, w_2, w_3, w_4)$. Assume that $n - k \geq 2$. Let us consider a family $X_t = \{z^2 - (y + x^3)(y^2 + x^{k+5}(x - t)^{n-k}) = 0\}$. If $t \neq 0$, then the points $(0, 0, 0)$ and $(t, 0, 0)$ are singularities of X_t; the singularity $(0, 0, 0)$ is an equisingular deformation of $(\{z^2 - (y + x^3)(y^2 + x^{k+5}) = 0\}, o)$, and $(t, 0, 0)$ is a rational double point of type A_{n-k-1}.

Theorem 4.3. *Assume that $-P_t \cdot P_t$ is constant. Then, after a finite base change, there exists a section $\gamma\colon T \to X$ such that each $(X_t, \gamma(t))$ is a non-log-canonical singularity and a simultaneous resolution which satisfy the conditions in Condition 4.1; furthermore $X_t \setminus \{\gamma(t)\}$ has only rational double points of type A_n.*

Proof. By Theorem 3.6, there exists a simultaneous RDP good resolution of π. It follows from [1] that there exists a finite base change $T' \to T$ and a resolution $f'\colon M' \to X' = X \times_T T'$ such that each $f'_t\colon M'_t \to X'_t$, $t \in T'$, is the minimal semi-good resolution; M' is obtained by resolving the singularities of the simultaneous RDP good resolution of $X' \to T'$ simultaneously. To simplify, we write $f\colon M \to X$ (resp. T) instead of $f'\colon M' \to X'$ (resp. T'). By Theorem 3.3, there exists the simultaneous log-canonical model $g\colon Y \to X$. By Proposition 3.7, we may assume that there exists a morphism $h\colon M \to Y$ such that $f = g \circ h$. Let $\gamma\colon T \to X$ be the section in Lemma 3.4. We will show that $f\colon M \to X$ and $\gamma\colon X \to T$ satisfy the conditions in Condition 4.1.

Let F (resp. E) be the reduced exceptional divisor on Y (resp. on M over $\gamma(T)$). We define the $\mathbb{Q}$-divisors $\mathcal{P}$ and $\mathcal{N}'$ on M by $\mathcal{P} = h^*(K_Y + F)$ and $\mathcal{N}' = K_M + E - \mathcal{P}$, respectively. Since $K_Y + F$ is log-canonical, $\mathcal{N}'$ is an effective exceptional divisor. Let $\mathcal{N}' = \sum n_i E^i$, where $\{E^i\}$ is the set of the exceptional prime divisors on M. Let $\mathcal{N} = \sum_{E^i \subset E} n_i E^i$. For each $t \in T$, we put $K_t = (K_M)_t$; in fact, $(K_M)_t$ is a canonical divisor on M_t. Now suppose $t \in T \setminus \{0\}$. Since $\pi|_{X \setminus X_0}$ admits a weak simultaneous resolution, E_t is the reduced exceptional divisor on M_t and $\mathcal{P}_t + \mathcal{N}_t$ is the Zariski decomposition of $K_t + E_t$ (by using the notation in the previous section, we can write $\mathcal{P}_t = P_{t,\gamma(t)}$ and $\mathcal{N}_t = N_{t,\gamma(t)}$). Let A be the exceptional set on M_0 and $P+N$ the Zariski decomposition of K_0+A. Then $P = h_0^*((K_Y+F)|_{Y_0}) = \mathcal{P}_0$. Since $K_0+E_0 = \mathcal{P}_0+\mathcal{N}_0$, we have $\mathcal{N}_0 - N = E_0 - A$. These divisors are effective since $[\mathcal{N}_0] = E_0 - A$ by Lemma 2.4. Thus $(\mathcal{N}_0)_{red} \geq N_{red}$ and $(E_0)_{red} = A$. If $\mathcal{N} = 0$, then $N = 0$ and $E_0 = A$; hence the conditions in Condition 4.1 are satisfied. Assume that $\mathcal{N} \neq 0$ and let $S = \mathcal{N}_{red}$.

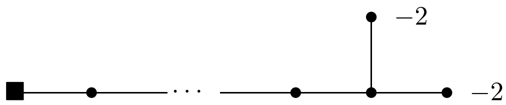
Let C be the cycle supported in A defined in Preliminaries and $C = \bigcup_{j=1}^n C^j$ the decomposition into connected components. Since $P \cdot \mathcal{N}_0 = 0$, we have $(\mathcal{N}_0)_{red} \leq C$. Let $H = A - C$. Each C^j is one of the following three types (see [6, Theorem 9.6]):

(1) Type A: C^j is a maximal string at an end of A.

(2) Type Ã: C^j has the following dual graph

where symbols • and ■ represent a component of C^j and H, respectively.

(3) Type D: C^j has the following dual graph



We write $S = \sum S^i$, where $\{S^i\}$ is a set of reduced divisors such that $\{(S^i)_t\}$ is the set of all maximal strings at the ends of E_t. Let S_t^i denote $(S^i)_t$. Note that $S_t^i \cdot S_t^j = 0$ if $i \neq j$. By [10, Lemma 3.1, Theorem 3.17], S_0^i is connected and reduced for any i. Hence each S_0^i is contained in an unique C^j. Let $A = \cup A_i$ be the decomposition into irreducible components.

Suppose that C^1 is a cycle of type Ã. Let $\sigma = \{i \,|\, S_0^i \leq C^1\}$. Assume that $\sigma \neq \emptyset$. Let A_k be a component at an end of $(\sum_{i\in\sigma} S_0^i)_{red}$. Assume that $A_k \leq S_0^i - \sum_{j\neq i} S_0^j$. Then the coefficient of A_k in S_0 is 1. Since A_k is not a component of N and $[\mathcal{N}] = 0$ by Lemma 2.4, it follows from Proposition 2.3 that the coefficient of A_k in $\mathcal{N}_0 - N$ is a positive number less than 1; however it contradicts that $\mathcal{N}_0 - N = E_0 - A$. If $A_k \subset S_0^i \cap S_0^j$, then $S_0^i \cdot S_0^j < 0$. Hence $\sigma = \emptyset$.

Next suppose that C^1 is a cycle of type D and that A_1 and A_2 are the maximal strings at ends of A in C^1. Let $C' = C^1 - A_1 - A_2$ and

$$\tau = \{i \,|\, S_0^i \text{ and } C' \text{ have a common component}\}.$$

Suppose that $\tau \neq \emptyset$ and A_k is the component of $\sum_{i\in\tau} S_0^i$ nearest to H. Assume that $A_k \subset S_0^i \cap S_0^j$ with $i \neq j$. Then the condition $S_t^i \cdot S_t^j = 0$ implies that any component of $S_0^i + S_0^j$ is a (-2)-curve. Thus there exists an open set in M containing $S^i \cup S^j$ which is a simultaneous resolution space of a deformation of a rational double point (see [11, p.12]); however S_0^i and S_0^j can have no common component by virtue of [10, Theorem 3.9] or [7, §4.3]. Hence $\tau = \emptyset$.

Now we obtain that $(\mathcal{N}_0)_{red} = N_{red}$. By arguments similar to above, we see that S_0 is a disjoint union of S_0^j's. Since $[\mathcal{N}] = 0$, we have $[\mathcal{N}_0] = 0$. It follows from $\mathcal{N}_0 - N = E_0 - A \geq 0$ that $\mathcal{N}_0 = N$ and $E_0 = A$. So (1) in Condition 4.1

follows. Let $S = \bigcup_{i=1}^{a} E^i$ be the decomposition into irreducible components. By Lemma 2.2, each $(E^i)_0$ is irreducible. Hence (2) in Condition 4.1 holds.

Next we will show a rational double point $p \in X_t \setminus \{\gamma(t)\}$ is of type A_n. Let D be a reduced exceptional divisor on M such that $D_t = f_t^{-1}(p)$. Then D_0 is reduced, connected and contained in C. By the minimality of the semigood resolution, any component of D_0 is a (-2)-curve. Let D_0' be the sum of the components $A_i \leq D_0$ such that $(D_0 - A_i) \cdot A_i = 2$. Note that if $A_i \leq D_0$ and $D_0 \cdot A_i = 0$ then $A_i \leq D_0'$. Since $A_i \cdot D_0 = 0$ for any $A_i \subset S_0^j$, we have $S_0^j \leq D_0'$ or $\mathrm{Supp}(S_0^j) \cap \mathrm{Supp}(D_0) = \emptyset$. Since S_0^j is a maximal string at an end of A, the first case does not occur. Hence D_0 is a chain and so is D_t. □

We use the notation of the proof of Theorem 4.3 in the following two remarks.

Remark 4.4. The converse of the theorem is true. In fact, the following conditions are equivalent:

(1) π admits a section and a simultaneous resolution as in Theorem 4.3 after a finite base change;

(2) $\delta_m(X_t) = \sum_{p \in \mathrm{Sing}(X_t)} \delta_m(X_t, p)$ is constant for any $m \in \mathbb{N}$;

(3) $-P_t \cdot P_t$ is constant.

We show a sketch of the proof. Suppose that (1) holds. Then we see that $\mathcal{P}_t \cdot \mathcal{P}_t$ and $K_t \cdot \mathcal{P}_t$ are constant. The existence of the simultaneous resolution implies that $p_g(X_t, \gamma(t))$ is constant too (see [11, Theorem 5.3]). Hence $\delta_m(X_t, \gamma(t))$ is constant by Theorem 2.5. Now (2) follows from the fact that $\delta_m = 0$ for any quotient singularity and $m \in \mathbb{N}$ ([22, Theorem 1.5]).

Remark 4.5. A component A_i is called a node unless it is a nonsingular rational curve with at most two intersections with other curves. Suppose that $-P_t \cdot P_t$ is constant. From the proof of the theorem, we see that X_t $(t \neq 0)$ has only one singular point $\gamma(t)$ if any chain in A connecting two nodes contains no (-2)-curves.

Corollary 4.6. *Suppose that $-P_t \cdot P_t$ is constant and that the weighted dual graph of (X_0, x_0) is a star-shaped graph. Then π admits a weak simultaneous resolution.*

Proof. If the weighted dual graph of (X_0, x_0) is a star-shaped graph, then X_t has only one singular point by Remark 4.5 and a simultaneous resolution with the conditions in Condition 4.1 is just a weak simultaneous resolution. Thus we need no finite base changes. □

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