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Analogies between group actions on 3-manifolds and number fields

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Abstract. Let a cyclic group G act either on a number field $\mathbb{L}$ or on a 3-manifold M. Let $s_{\mathbb{L}}$ be the number of ramified primes in the extension $\mathbb{L}^G \subset \mathbb{L}$ and s_M be the number of components of the branching set of the branched covering $M \to M/G$. In this paper, we prove several formulas relating $s_{\mathbb{L}}$ and s_M to the induced G-action on $Cl(\mathbb{L})$ and $H_1(M)$, respectively. We observe that the formulas for 3-manifolds and number fields are almost identical, and therefore, they provide new evidence for the correspondence between 3-manifolds and number fields postulated in arithmetic topology.

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1. Introduction and statement of the main results

Denote the ring of algebraic integers in a number field $\mathbb{K}$ by $\mathcal{O}_{\mathbb{K}}$. Mazur's calculations of the étale cohomology groups of rings of algebraic integers, [Ma2], show that for number fields $\mathbb{K}$ the groups $H^n_{et}(\operatorname{Spec}\mathcal{O}_{\mathbb{K}}, \mathbf{G}_m)$ vanish (up to 2-torsion) for $n > 3$, and that they are equal to $\mathbb{Q}/\mathbb{Z}$ for $n = 3$. Since, furthermore, the groups $H^*_{et}(\mathcal{O}_{\mathbb{K}}, \mathbf{G}_m)$ satisfy Artin–Verdier duality which is reminiscent of 3-dimensional Poincaré duality, B. Mazur and D. Mumford suggested a surprising analogy between the spaces $\operatorname{Spec}\mathcal{O}_{\mathbb{K}}$ and 3-dimensional manifolds. The points in $\operatorname{Spec}\mathcal{O}_{\mathbb{K}}$ (which are immersions of spectra of residue fields, $\operatorname{Spec}\mathcal{O}_{\mathbb{K}}/\mathfrak{p}$, into $\operatorname{Spec}\mathcal{O}_{\mathbb{K}}$) can be viewed as 1-dimensional objects and hence compared to knots in a 3-manifold. Note that the fundamental group of a circle is $\mathbb{Z}$ and the absolute Galois group of a finite field is the profinite completion of $\mathbb{Z}$. Further analogies between number fields and 3-manifolds were described by B. Mazur in [Ma1], and later, by A. Reznikov, M. Morishita and others in [Ma1, Mo1, Mo2, Ra, R1, R3, Wl], making a foundation for "arithmetic topology." At the heart of it lies a "dictionary"(which we call the MKR dictionary after Mazur, Kapranov, and Reznikov) matching the corresponding terms from 3-dimensional topology and number theory, [R2, Si]. Despite its limitations and inconsistencies, the dictionary can be used for translating statements from 3-dimensional topology into number theory, and vice versa, often with a surprising accuracy.

In this section we state the main results of this paper, which were motivated by the following two problems:

Let C_p be a cyclic group of prime order p.

Problem T. *Let C_p act on a smooth, closed, connected, oriented 3-manifold M by orientation preserving diffeomorphisms. In this situation the projection $\pi : M \to M/C_p$ is a branched covering. Given the induced C_p-action on the torsion and free parts of $H_1(M,\mathbb{Z})$, denoted by $H_{tor}(M)$ and $H_{free}(M) = H_1(M,\mathbb{Z})/H_{tor}(M)$, find the best lower and upper estimates on the number, s, of components of the branching set.*

Upon translation into number theory, Problem T assumes the following form:

Problem N. *Let C_p act on a number field $\mathbb{L}$ and let $\mathbb{K} = \mathbb{L}^{C_p}$. Given the induced C_p-action on the ideal class group, $Cl(\mathbb{L})$, and on the group of units, $\mathcal{O}^*_{\mathbb{L}}$, find the best lower and upper estimates on the number of ramified primes in the extension $\mathbb{K} \subset \mathbb{L}$. We will denote by s_0 and s the numbers of finite ramified primes and all ramified primes (including the infinite ones), respectively. (For $p \neq 2$, $s_0 = s$.)*

Our particular interest in these two problems stems from the fact that none of them has an elementary solution, and furthermore, our solutions are based on methods which are beyond the current scope of arithmetic topology. Therefore, we hope that the consideration of these two problems will provide a new insight into arithmetic topology.

According to the MKR dictionary ([R2]) $H_{tor}(M)$ and $H_{free}(M)$ correspond to $Cl(\mathbb{L})$ and $\mathcal{O}^*_{\mathbb{L}}/torsion$, respectively. For that reason, we will denote the latter two groups by $H_{tor}(\mathbb{L})$ and $H_{free}(\mathbb{L})$.

Let $\mathbb{F}_p$ denote the field of p elements. We will prove the following estimates for s.

Theorem 1.1. (1) *Under the assumptions of Problem T, and the additional assumption $H_{free}(M/C_p) = 0$, we have*

$$s \leq 1 + \dim_{\mathbb{F}_p} H^2(C_p, H_{tor}(M)) + \dim_{\mathbb{F}_p} H^1(C_p, H_{free}(M)).$$

(2) *Under the assumptions of Problem N*

$$s_0 \leq 1 + \dim_{\mathbb{F}_p} H^2(C_p, H_{tor}(\mathbb{L})) + \dim_{\mathbb{F}_p} H^1(C_p, H_{free}(\mathbb{L})).$$

Recall that if H is an abelian group with a C_p-action, $C_p =< \tau | \tau^p = 1 >$, then $H^i(C_p, H) = \mathrm{Ker}N/\mathrm{Im}S$ for odd $i > 0$ and $H^i(C_p, H) = \mathrm{Ker}S/\mathrm{Im}N$, for even $i > 0$, where N and S are homomorphisms from H to H given by the multiplication by $1 + \tau + \cdots + \tau^{p-1}$ and by $\tau - 1$, respectively. Furthermore, the method of Herbrand quotient ([Ko, Prop. 2.53]), implies that

$$H^1(C_p, H) \simeq H^2(C_p, H) \tag{1}$$

for finite H.

We do not have a satisfying explanation for the coincidence of the formulas of Theorem 1.1. Their proofs (given in Sections 3 and 4) are based on two very different methods. Examples 5.2 and 5.1 show that the extra assumption of Theorem 1.1(1) is necessary and its inequality cannot be improved. Similarly, the inequality of (2) cannot be improved – see comments following Theorem 1.3.

Here is an upper bound for s, which does not need the additional assumption of Theorem 1.1(1).

Theorem 1.2. *Under the assumptions of Problem T,*

$$s \le 1 + \dim_{\mathbb{F}_p} H^2(C_p, H_1(M)) + \dim_{\mathbb{F}_p} H^1(C_p, H_{free}(M)).$$

Searching for lower estimates for s one encounters the following problem: A G-action on a 3-manifold M induces G-actions on $H_1(M)$, $H_{tor}(M)$, and $H_{free}(M)$. However, $H_1(M,\mathbb{Z})$ and $H_{free}(M) \oplus H_{tor}(M)$ do not need to be isomorphic as G-modules! (See Example 5.2.) Since this problem does not occur in number theory, the next result requires that $H_1(M,\mathbb{Z}) = H_{free}(M) \oplus H_{tor}(M)$ as G-modules.

Theorem 1.3. (1) *If conditions of Problem T are satisfied, $s > 0$, and $H_1(M,\mathbb{Z}) = H_{free}(M) \oplus H_{tor}(M)$ as C_p-modules then*

$$s \ge 1 + \dim_{\mathbb{F}_p} H^2(C_p, H_{tor}(M)).$$

(2) *If conditions of Problem N are satisfied, $s > 0$, and $Cl(\mathbb{K})$ has no elements of order p then*

$$s \ge 1 + \dim_{\mathbb{F}_p} H^2(C_p, H_{tor}(\mathbb{L})).$$

To illustrate these results we consider quadratic number fields. Let $\mathbb{L} = \mathbb{Q}(\sqrt{d})$ where $d > 0$ is a square-free integer. $\mathrm{Gal}(\mathbb{L}/\mathbb{Q}) = C_2$ acts on $Cl(\mathbb{L})$ by the involution $I \to I^{-1}$. Hence, by (1), $H^1(C_2, Cl(\mathbb{L})) = H^2(C_2, Cl(\mathbb{L})) = Cl(\mathbb{L})^{C_2}$. A simple calculation yields $H^1(C_2, H_{free}(\mathbb{L})) = \mathbb{F}_2$ and, therefore, Theorems 1.1(2) and 1.3(2) imply the following estimates (similar to a formula of Gauss):

$$1 \le s - \dim_{\mathbb{F}_2} Cl(\mathbb{L})^{C_2} \le 2.$$

These estimates cannot be improved. In fact $s - \dim_{\mathbb{F}_2} Cl(\mathbb{L})^{C_2}$ is either 1 or 2 depending if the norm of fundamental unit in $\mathcal{O}_{\mathbb{L}}^*$ is -1 or 1 ($d = 10$ represents the first case, $d = 15$ the second).

The topological examples are more laborious and, therefore, they are postponed to Section 5. Example 5.2 shows that the extra assumption in Theorem 1.3(1) (saying that $H_1(M,\mathbb{Z}) = H_{free}(M) \oplus H_{tor}(M)$ as C_p-modules) is necessary and that the inequality of Theorem 1.3(1) is sharp.

The situation with cyclic extensions of $\mathbb{Q}$ of dimensions greater than 2 is much simpler. The following generalization of the Gauss formula follows from [La, Lemma 13.4.1].

Theorem 1.4. *If $\mathbb{L}$ is a Galois extension of $\mathbb{Q}$ of prime degree $p \neq 2$ then*

$$H_{tor}(\mathbb{L})^{C_p} = (\mathbb{Z}/p)^{s-1}.$$

The topological counterpart of the above formula is the following restatement of a result due to Reznikov, [R3, Thm. 15.2.5]. It follows from our Theorems 1.3(1) and 1.1(1), see the proof in Section 3.

Corollary 1.5. *If M is a rational homology sphere, $H_{tor}(M/C_p) = 0$, and $s \neq 0$ then*

$$H_{tor}(M)^{C_p} = (\mathbb{Z}/p)^{s-1}.$$

Example 5.1 shows that the assumption $H_{free}(M) = 0$ is necessary.

Despite the fact that all above formulas are pairwise identical we do not know any uniform proof of them. In fact, the statements concerning group actions on manifolds are proved by calculations on spectral sequences associated with equivariant cohomology, and the statements concerning Problem N follow from results of P. E. Conner and J. Hurrelbrink, [CH], which are proved by local methods (ideles) and class field theory. This suggests that the results of class field theory can be interpreted in terms of 3-dimensional topology. (This suggestion was formulated before, for example by B. Mazur and A. Reznikov.) Analogously, one might expect that equivariant cohomology can be formulated and used in the framework of number theory.

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2. Preliminaries

2.1. Transfer and norm maps

For any G-action on M consider the map $\pi_* : H_1(M,\mathbb{Z}) \to H_1(M/G,\mathbb{Z})$ induced by $\pi : M \to M/G$ and the transfer map $\pi_\sharp : H_1(M/G,\mathbb{Z}) \to H_1(M,\mathbb{Z})$ defined as follows: For any $x \in H_1(M/G,\mathbb{Z})$ represented by a closed curve c disjoint from the branching set, $\pi_\sharp(x) = [\pi^{-1}(c)] \in H_1(M,\mathbb{Z})$.

The induced maps

$$\pi_* : H_{free}(M,\mathbb{Z}) \to H_{free}(M/G,\mathbb{Z}),\ \pi_* : H_{tor}(M,\mathbb{Z}) \to H_{tor}(M/G,\mathbb{Z}),$$

$$\pi_\sharp : H_{free}(M/G) \to H_{free}(M),\ \ \pi_\sharp : H_{tor}(M/G) \to H_{tor}(M),$$

satisfy the following conditions:
(1) $\pi_* \pi_\sharp$ is the multiplication by $|G|$.
(2) $\pi_\sharp \pi_*(x) = \sum_{g \in G} g \cdot x$.
In particular we have

$$\operatorname{rank} H_{free}(M/C_p) \leq \operatorname{rank} H_{free}(M). \tag{2}$$

Given a Galois extension $\mathbb{K} \subset \mathbb{L}$ with $\mathrm{Gal}(\mathbb{L}/\mathbb{K}) = G$, we have the norm map $\pi_* : \mathcal{O}_\mathbb{L} \to \mathcal{O}_\mathbb{K}$, $\pi_*(x) = \prod_{g \in G} gx$ and the embedding $\pi_\sharp : \mathcal{O}_\mathbb{K} \hookrightarrow \mathcal{O}_\mathbb{L}$. The induced maps $\pi_* : H_{free}(\mathbb{L}) \to H_{free}(\mathbb{K})$ and $\pi_\sharp : H_{free}(\mathbb{K}) \to H_{free}(\mathbb{L})$ satisfy properties identical to those above:
(1) $\pi_* \pi_\sharp(x) = x^{|G|}$.
(2) $\pi_\sharp \pi_*(x) = \prod_{g \in G} gx$.
Furthermore, we can define $\pi_\sharp$ and π_* for ideals: $\pi_\sharp(I) = I \cdot \mathcal{O}_\mathbb{L}$ for $I \lhd \mathcal{O}_\mathbb{K}$, and $\pi_*(J) = \prod_{g \in G} gJ$ for $J \lhd \mathcal{O}_\mathbb{L}$. The induced maps

$$\pi_\sharp : Cl(\mathbb{K}) \to Cl(\mathbb{L}), \quad \pi_* : Cl(\mathbb{L}) \to Cl(\mathbb{K}),$$

once again satisfy the conditions
(1) $\pi_* \pi_\sharp(I) = I^{|G|}$,
(2) $\pi_\sharp \pi_*(I) = \prod_{g \in G} gI$.

2.2. A rough classification of $\mathbb{Z}[C_p]$-modules

Let $\mathbb{Z}_{(p)}$ denote the ring of integers localized at (p), p prime.

Proposition 2.1. *If V is a free abelian group and an indecomposable $\mathbb{Z}[C_p]$-module then $V \otimes \mathbb{Z}_{(p)}$ is isomorphic to one of the following $\mathbb{Z}_{(p)}[C_p]$-modules:*
(F) *the free module $\mathbb{Z}_{(p)}[C_p]$,*
(T) *the trivial module, $\mathbb{Z}_{(p)}$, or*
(AI) *the augmentation ideal,* $\operatorname{Ker} \varepsilon$, *for* $\varepsilon : \mathbb{Z}_{(p)}[C_p] \to \mathbb{Z}_{(p)}$, $\varepsilon(g) = 1$.

The proof follows from the classification of indecomposable C_p-representations over $\mathbb{Z}$, due to Diederichsen and Reiner, [CR, Theorem §74.3]. □

For any C_p-module V the cohomology groups $H^i(C_p, V)$ for $i > 0$ depend on the parity of i only and

$$H^{2n}(C_p, V) = \hat{H}^0(C_p, V), \quad H^{2n-1}(C_p, V) = \hat{H}^1(C_p, V), \quad \text{for } n > 0,$$

are linear spaces over the field of p-elements, $\mathbb{F}_p$. Here $\hat{H}^*(C_p, V)$ denotes the Tate cohomology groups.

The type of a $\mathbb{Z}[C_p]$-module V (as described in Proposition 2.1) can be determined by the Tate cohomology of C_p with coefficients in V. If V is a free abelian group and an indecomposable $\mathbb{Z}[C_p]$-module then V is of type

(F) iff $\hat{H}^i(C_p, V) = 0$ for $i = 0, 1$,
(T) iff $\hat{H}^0(C_p, V) = \mathbb{Z}/p$, $\hat{H}^1(C_p, V) = 0$,
(AI) iff $\hat{H}^0(C_p, V) = 0$, $\hat{H}^1(C_p, V) = \mathbb{Z}/p$.

Any C_p-module V gives rise to two other C_p-modules, V^* and $V^\#$, defined as follows: $V^* = \mathrm{Hom}(V, \mathbb{Z})$, and $g \in C_p$ acts on $\mathrm{Hom}(V, \mathbb{Z})$ by sending $f(\cdot)$ to $f(g^{-1}\cdot)$. The second module, $V^\#$, is equal to V as an abelian group and the action of $g \in C_p$ on $v \in V^\#$ is given by $g^{-1}v$.

Lemma 2.2. *For any C_p-module V,*
(1) $\hat{H}^*(C_p, V^\#) = \hat{H}^*(C_p, V)$
(2) *if V is a free abelian group then* $\hat{H}^*(C_p, V^*) = \hat{H}^*(C_p, V)$.

Proof. (1) As before, let τ be a generator of C_p and let $N, S : V \to V$ be given by $\cdot \sum_{i=0}^{p-1} \tau^i$ and by $\cdot(\tau - 1)$ respectively. $\hat{H}^n(C_p, V)$ is equal to $\mathrm{Ker}\, N/\mathrm{Im}\, S$ or $\mathrm{Ker}\, S/\mathrm{Im}\, N$, depending if n is odd or even. Similarly, $\hat{H}^n(C_p, V^\#)$ is equal to $\mathrm{Ker}\, N^\#/\mathrm{Im}\, S^\#$ or $\mathrm{Ker}\, S^\#/\mathrm{Im}\, N^\#$, depending if n is odd or even, where $N^\#, S^\# : V \to V$ are given by $\cdot \sum_{i=0}^{p-1} \tau^{-i}$ and by $\cdot(\tau^{-1} - 1)$ respectively. Now (1) follows from the fact that $N^\# = N$, $\mathrm{Ker}\, S^\# = \mathrm{Ker}\, S$, and $\mathrm{Im}\, S^\# = \mathrm{Im}\, S$.

(2) Let $V = \mathbb{Z}^d$. Since $\hat{H}^*(C_p, V)$ is a p-group, $Tor(\hat{H}^*(C_p, V), \mathbb{Z}_{(p)}) = 0$ (see [We, Calculation 3.1.1]) and by the universal coefficient theorem,

$$\hat{H}^i(C_p, V \otimes \mathbb{Z}_{(p)}) = \hat{H}^i(C_p, V) \otimes \mathbb{Z}_{(p)} = \hat{H}^i(C_p, V).$$

Therefore, it is sufficient to show that $V \otimes \mathbb{Z}_{(p)}$ and $\mathrm{Hom}(V, \mathbb{Z}) \otimes \mathbb{Z}_{(p)}$ are isomorphic as C_p-modules, and for that it is enough to consider indecomposable C_p-modules V only. Such modules are classified in Proposition 2.1. We leave it to the reader to check that in each of the three possible cases we get

$$V \otimes \mathbb{Z}_{(p)} \simeq \mathrm{Hom}(V \otimes \mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \simeq \mathrm{Hom}(V, \mathbb{Z}) \otimes \mathbb{Z}_{(p)}$$

as needed. □

3. Cyclic group actions on manifolds

In this section we will prove the results concerning cyclic group actions on 3-manifolds which were announced in the introduction. As references for the homological algebra (group cohomology and spectral sequences) we recommend [Bro, We].

3.1. Equivariant cohomology

Let (X, A) be a relative C_p–CW-complex. In other words, let X be a CW-complex with a C_p-action, A be a subcomplex, and let the C_p-action on X preserve A and be compatible with the CW-structure on X. Additionally, we require that if $g \in C_p$ maps a cell $\sigma \subset X$ into itself then g acts trivially on σ; compare [AP]. For any smooth C_p-action on a manifold M, (M, M^{C_p}) is a relative C_p–CW-complex, [Il].

Denote the cochain complex for the cellular cohomology of (X, A) with coefficients in $\mathbb{Z}$ by $(C^*(X, A), \delta)$. Let (P^*, δ') be a complete projective resolution of $\mathbb{Z}$ considered as a $\mathbb{Z}[C_p]$-module with the trivial C_p-action, cmp. [Bro, Ch. VI.3]. Then $D^{**}(X, A) = \mathrm{Hom}_{\mathbb{Z}[C_p]}(P^*, C^*(X, A))$ is a double complex with two differentials (a) $\delta_v = \delta : D^{kl}(X, A) \to D^{k,l+1}(X, A)$ and (b) $\delta_h : D^{kl}(X, A) \to D^{k+1,l}(X, A)$ dual to δ'. The cohomology of the associated total complex,

$$D^s(X, A) = \bigoplus_{k+l=s} D^{kl}(X, A),$$

$$d(\alpha) = d_h(\alpha) + (-1)^k d_v(\alpha), \text{ for } \alpha \in D^{kl}(X, A)$$

is the Tate equivariant cohomology.

Consider the "first" spectral sequence $(E_*^{**}(X, A), d_*)$ associated with $(D^{**}(X, A), \delta_h, \delta_v)$ – that is the spectral sequence induced by the vertical filtration of $(D^*(X, A), d)$. Its first term is $E_1^{kl}(X, A) = H^l(X, A; \mathbb{Z})$, for $k, l \in \mathbb{Z}$.

The next two results describing the properties of $(E_*^{**}(X, A), d_*)$ belong to the folk knowledge.

Lemma 3.1. *If X is a connected C_p–CW-complex and $X^{C_p} \neq \emptyset$, then all differentials $d_r^{*,r-1} : E_r^{*,r-1}(X) \to E_r^{*0}(X)$ are 0 for $r \geq 2$. (In other words, the 0th row of $E_2^{**}(X)$ survives to infinity.)*

Proof. Let $\{*\}$ denote the one point space and let x_0 be a zero cell in X^{C_p}. The C_p-equivariant maps: $* \to x_0 : \{*\} \to X$ and $X \to \{*\}$ induce maps

$$(E_r^{**}(*), d_r) \to (E_r^{**}(X), d_r) \to (E_r^{**}(*), d_r),$$

whose composition is the identity. $E_2^{i0}(X) = H^i(C_p, H^0(X, \mathbb{Z}))$ is either 0 or $\mathbb{F}_p$. Hence, if $d_r^{k,r-1} : E_r^{k,r-1}(X) \to E_r^{k+r,0}(X)$ is not 0 for some k, r then $E_{r+1}^{k+r,0}(X) = 0$ and $E_r^{k+r,0}(X) = \mathbb{F}_p$. Hence, $k + r$ is even and, hence, $E_{r+1}^{k+r,0}(*) = \hat{H}^{k+r}(C_p, H^0(*)) = \mathbb{F}_p$. Therefore

$$E_{r+1}^{**}(*) \to E_{r+1}^{**}(X) \to E_{r+1}^{**}(*)$$

cannot be the identity. □

Lemma 3.2. *If (M, B) is a C_p–CW-complex such that M is a connected, closed, oriented n-manifold and B is an n-dimensional ball then all differentials $d_r^{*,n} : E_r^{*,n}(M) \to E_r^{*,n-r+1}(M)$ are 0 for $r \geq 2$. (In other words, the n-th row of $E_2^{**}(M)$ survives to infinity.)*

Proof (due to T. Skjelbred). The sequence of cochain complexes

$$C^*(M, M \setminus \mathrm{int}\, B) \to C^*(M) \to C^*(M \setminus \mathrm{int}\, B)$$

induced by the embedding $M \setminus \operatorname{int} B \hookrightarrow M$ yields a sequence of spectral sequences

$$E_r^{**}(M, M \setminus \operatorname{int} B) \xrightarrow{\alpha_r} E_r^{**}(M) \to E_r^{**}(M \setminus \operatorname{int} B)$$

which is exact for $r = 1$. Since $E_1^{*n}(M \setminus \operatorname{int} B) = H^n(M \setminus \operatorname{int} B) = 0$, the map $\alpha_1^{*n} : E_1^{*n}(M, M \setminus \operatorname{int} B) \to E_1^{*n}(M)$ is onto.

Since

$$E_1^{*k}(M, M \setminus \operatorname{int} B) = H^k(M, M \setminus \operatorname{int} B) = H^k(B^n, \partial B^n) = H_{n-k}(B^n),$$

all non zero elements of $E_r^{**}(M, M \setminus \operatorname{int} B)$ for $r \geq 1$ lie in the nth row. Since α_r commutes with differentials and all differentials in $E_r^{**}(M, M \setminus \operatorname{int} B)$ are 0 for $r \geq 2$, $d_r^{*,n} : E_r^{*,n}(M) \to E_r^{*,n-r+1}(M)$ is zero for $r \geq 2$. □

3.2. Cyclic group actions on 3-manifolds

Let M be a smooth, connected, closed, oriented 3-manifold with a smooth orientation preserving C_p-action with a fixed point.

By the result of Illman [Il] recalled above, M can be given a structure of a relative C_p–CW-complex. Furthermore, since $M^{C_p} \neq \emptyset$, one may assume that there exists a C_p–CW-subcomplex B of M homeomorphic to a 3-ball.

In this situation, the second term of $E_*^{**}(M)$ is:

$$\begin{array}{ccccc}
\dots & \mathbb{F}_p & 0 & \mathbb{F}_p & \dots \\
\dots & \hat{H}^0(C_p, H^2(M)) & \hat{H}^1(C_p, H^2(M)) & \hat{H}^0(C_p, H^2(M)) & \dots \\
\dots & \hat{H}^0(C_p, H^1(M)) & \hat{H}^1(C_p, H^1(M)) & \hat{H}^0(C_p, H^1(M)) & \dots \\
\dots & \mathbb{F}_p & 0 & \mathbb{F}_p & \dots
\end{array}$$

By Poincaré duality $H^2(M, \mathbb{Z}) = H_1(M, \mathbb{Z})^{\#}$, $H^1(M, \mathbb{Z}) = H_{free}(M)^*$ as C_p-modules, and by Lemma 2.2,

$$\begin{aligned} \hat{H}^n(C_p, H^2(M, \mathbb{Z})) &\simeq \hat{H}^n(C_p, H_1(M, \mathbb{Z})), \\ \hat{H}^n(C_p, H^1(M, \mathbb{Z})) &\simeq \hat{H}^n(C_p, H_{free}(M, \mathbb{Z})). \end{aligned} \tag{3}$$

Let

$$\Psi^n : \hat{H}^n(C_p, H_1(M, \mathbb{Z})) \to \hat{H}^n(C_p, H_{free}(M))$$

denote the homomorphism given by the composition of $d_2^{n2} : E_2^{n,2} \to E_2^{n+2,1}$ with the isomorphisms (3). One needs to be aware that Ψ^n is not the map induced by the natural homomorphism $H_1(M, \mathbb{Z}) \to H_{free}(M)$.

Theorem 3.3. *If $M^{C_p} \neq \emptyset$ (p prime) then M^{C_p} is a union of s circles, where*

$$s = 1 + \dim_{\mathbb{F}_p} \operatorname{Ker} \Psi^n + \dim_{\mathbb{F}_p} \operatorname{Coim} \Psi^{n-1},$$

for any n.

Before giving the proof we note that this result immediately implies Theorem 1.2. It also implies Theorems 1.1(1) and 1.3(1).

Proof of Theorem 1.1(1). If $H_{free}(M/C_p) = 0$ then by property (2) of the transfer map (cf. Subsection 2.1) $H_{free}(M)$ is annihilated by $\sum_{g\in C_p} g$. By Proposition 2.1, $H_{free}(M) \otimes \mathbb{Z}_{(p)}$ is a sum of three types of indecomposable $\mathbb{Z}_{(p)}[C_p]$-modules. Since $\operatorname{Ker}\varepsilon$ is the only indecomposable $\mathbb{Z}_{(p)}[C_p]$-module which is annihilated by $\sum_{g\in C_p} g$, $H_{free}(M) \otimes \mathbb{Z}_{(p)}$ is a sum of modules of this type. We compute that $H^2(C_p, \operatorname{Ker}\varepsilon) = 0$. Since $\otimes\mathbb{Z}_{(p)}$ is an exact functor in the category of C_p-modules, $H^2(C_p, H_{free}(M)) = H^2(C_p, H_{free}(M)\otimes\mathbb{Z}_{(p)}) = 0$. Now, the short exact sequence

$$0 \to H_{tor}(M) \to H_1(M) \to H_{free}(M) \to 0$$

implies that

$$\dim_{\mathbb{F}_p} H^2(C_p, H_1(M)) \leq \dim_{\mathbb{F}_p} H^2(C_p, H_{tor}(M)).$$

Therefore, Theorem 1.1(1) follows from Theorem 1.2. □

Proof of Theorem 1.3(1). If $H_1(M, \mathbb{Z}) = H_{free}(M) \oplus H_{tor}(M)$ as C_p-modules, the dimension of $\operatorname{Ker}\Psi^n$ is bounded below by

$$\dim_{\mathbb{F}_p} \hat{H}^n(C_p, H_1(M)) - \dim_{\mathbb{F}_p} \hat{H}^n(C_p, H_{free}(M)) \\ = \dim_{\mathbb{F}_p} \hat{H}^n(C_p, H_{tor}(M)).$$

□

Proof of Corollary 1.5. By inequality (2) in Section 2.1, $H_{free}(M) = 0$ implies $H_{free}(M/C_p) = 0$. Hence by Theorems 1.1(1) and 1.3(1),

$$s = 1 + \dim_{\mathbb{F}_p} H^2(C_p, H_{tor}(M)).$$

Since $H_{tor}(M/C_p) = 0$, $\pi_\sharp\pi_*(x) = \sum_{g\in C_p} g\cdot x = 0$ for all $x \in H_{tor}(M)$. Hence $H^2(C_p, H_{tor}(M)) = H_{tor}(M)^{C_p}$. □

We complete this section by proving Theorem 3.3.

By Lemmas 3.1 and 3.2 all differentials in the spectral sequence $(E_r^{*,*}(M), d_r)$, for $r \geq 2$, are 0 except possibly $d_2^{k2} : E_2^{k2}(M) \to E_2^{k+2,1}(M)$. Therefore,

$$\sum_{k+l=n} E_\infty^{kl}(M) = \mathbb{F}_p + \operatorname{Ker} d_2^{n-2,2} + \operatorname{Coim} d_2^{n-3,2} \simeq \mathbb{F}_p + \operatorname{Ker}\Psi^{n-2} + \operatorname{Coim}\Psi^{n-3}.$$

Since $(E_*^{**}(M), d_*)$ converges to $\hat{H}^*_{C_p}(M, \mathbb{Z})$, $\hat{H}^n_{C_p}(M, \mathbb{Z})$ and $\sum_{k+l=n} E_\infty^{kl}(M)$ have equal numbers of elements. By the Localization Theorem, [Bro, Ch. VII Prop. 10.1], $\hat{H}^n_{C_p}(M, \mathbb{Z}) = \hat{H}^n_{C_p}(M^{C_p}, \mathbb{Z})$. Since M^{C_p} is composed of circles, Theorem 3.3 follows from the following lemma.

Lemma 3.4. $\hat{H}^n_{C_p}(M^{C_p}, \mathbb{Z}) = \mathbb{F}_p^s$, *for any* n.

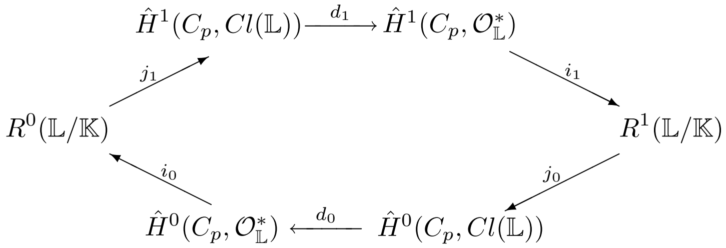
Proof. Since the C_p-action on M^{C_p} is trivial, the total complex of $(D^{**}(M^{C_p}), \delta, \delta')$ is isomorphic to $(P^*) \otimes (C^*(M^{C_p}, \mathbb{Z}), \delta^*)$ where (P^*) is the cochain complex $\xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot p} \ldots$ Hence, by Künneth formula, $\hat{H}^n_{C_p}(M^{C_p}, \mathbb{Z}) = H^n(D^{**}(M^{C_p}))$ is equal to

$$\bigoplus_{k+l=n} H^k(P^*) \otimes H^l(M^{C_p}, \mathbb{Z}) + \bigoplus_{k+l=n+1} Tor_{\mathbb{Z}}(H^k(P^*), H^l(M^{C_p}, \mathbb{Z})).$$

Since $H^k(P^*)$ is either $\mathbb{F}_p$ or 0 depending if k even or odd, the proof follows. □

4. Cyclic group actions on number fields

This section is devoted to proofs of Theorems 1.1(2) and 1.3(2). Our original proof of these results was based on the class field theory and the properties of idele group, cf. [Si]. However, as suggested to us by the referee, the proofs presented here are based on the work of Conner and Hurrelbrink, [CH]. Given a Galois extension $\mathbb{L}/\mathbb{K}$ with $\mathrm{Gal}(\mathbb{L}/\mathbb{K}) = C_p$ they have defined a hexagon



and proved that it is exact.[1]

4.1. Proof of Theorem 1.1(2)

Let μ_* denote the torsion part of $\mathcal{O}^*_{\mathbb{L}}$. Since μ_* is cyclic, $\dim_{\mathbb{F}_p} \hat{H}^1(C_p, \mu_*) \leq 1$ and the exact sequence

$$\hat{H}^1(C_p, \mu_*) \to \hat{H}^1(C_p, \mathcal{O}^*_{\mathbb{L}}) \to \hat{H}^1(C_p, \mathcal{O}^*_{\mathbb{L}}/\mu_*)$$

implies

$$\dim_{\mathbb{F}_p} \hat{H}^1(C_p, \mathcal{O}^*_{\mathbb{L}}) \leq 1 + \dim_{\mathbb{F}_p} \hat{H}^1(C_p, H_{free}(\mathbb{L})). \quad (4)$$

(Recall that $H_{free}(\mathbb{L}) = \mathcal{O}^*_{\mathbb{L}}/\mu_*$.)

By [CH, Thm. 5.1], the p-rank of $R_1(\mathbb{L}/\mathbb{K})$ is s_0 (the number of finite ramified primes) and, therefore, Theorem 1.1(2) follows from (4) and the exact hexagon of Conner and Hurrelbrink.

[1] Note that the Tate cohomology groups are denoted by $H^*(C_p, \cdot)$ in [CH].

4.2. Proof of Theorem 1.3(2)

By [CH, Lemma 9.1] the assumptions of Theorem 1.3(2) imply that j_1 is onto. Now the theorem follows from [CH, Thm. 4.2] stating that the p-rank of $R^0(\mathbb{L}/\mathbb{K})$ is $s-1$.

5. Examples

We end this paper with two examples of group actions on 3-manifolds illustrating various anomalies related to cyclic group actions on 3-manifolds. The first example shows that the inequality of Theorem 1.1(1) is sharp and that the assumption $H_{free}(M)=0$ of Corollary 1.5 is necessary.

Example 5.1. Consider a C_p-action on the lens space $L(p,1)$ with two circles of fixed points. ($L(p,1)$ is obtained from two handlebodies identified along their boundaries. C_p acts on each of these solid tori by rotation along their cores. These actions give rise to the desired C_p-action on $L(p,1)$.) Let U be an open ball in $L(p,1)$ such that $U\cap gU=\emptyset$ for all $g\neq e$ in C_p.

Consider now a C_p-action on S^3 with one circle of fixed points. Let V be an open ball in S^3 such that $V\cap gV=\emptyset$ for all $g\neq e$ in C_p, and let $\Psi:\partial V\to\partial U$ be any homeomorphism. By removing the balls gU from $L(p,1)$ and the balls gV from S^3 and by identifying the boundary spheres of $L(p,1)\setminus\bigcup_g gU$ and $S^3\setminus\bigcup_g gV$ by the homeomorphisms $g\Psi g^{-1}:g\partial V\to g\partial U$, we obtain a closed 3-manifold M with a C_p-action. Since $L(p,1)/C_p=S^3$ and $S^3/C_p=S^3$, we have $M/C_p=L(p,1)/C_p\#S^3/C_p=S^3$. Simple calculations show that $H_1(M)=\mathbb{F}_p\oplus\mathbb{Z}[\xi]/(1+\xi+\cdots+\xi^{p-1})$, where C_p acts on $\mathbb{Z}[\xi]/(1+\xi+\cdots+\xi^{p-1})$ by $\cdot\xi$ and it acts trivially on $\mathbb{F}_p$. Hence $H^2(C_p,H_{tor}(M))=\mathbb{F}_p$ and $H^1(C_p,H_{free}(M))=\mathbb{F}_p$. In this example $s=3$.

The second example shows that the assumption $H_{free}(M/C_p)=0$ in Theorem 1.1(1) is necessary. Furthermore, it shows that the assumption in Theorem 1.3(1) saying that $H_1(M)=H_{tor}(M)\oplus H_{free}(M)$ as C_p-modules is also necessary. (Take $n=1$, $k=p$ below.)

Example 5.2. (Based on an idea of J. Hempel.) Let ϕ be a Dehn twist on the torus T and let M_k be the 3-manifold obtained from $T\times[0,1]$ by identifying $T\times\{0\}$ with $T\times\{1\}$ via ϕ^k. Assume that ϕ fixes neighborhoods of points $p_1,\ldots,p_n$ and consider knots $\{p_i\}\times S^1$ with the framing $v_i\times S^1$, where v_i is an arbitrary tangent vector to T at p_i. By performing the surgery along these framed knots we obtain a new manifold $M_{k,n}$. If p divides k then the map $(x,t)\to(\phi(x)^{k/p},t+1/p \bmod 1)$ defines a C_p-action on M_k which survives the surgery. We have $H_1(M_k,\mathbb{Z})=\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}/k$ and $H_1(M_{k,n},\mathbb{Z})=\mathbb{Z}^n\oplus\mathbb{Z}/k$, for $n\geq 1$. Furthermore, $H_1(M_{p,n},\mathbb{Z})$

decomposes as a C_p-module into a sum $\mathbb{Z}^{n-1} \oplus \mathbb{Z} \oplus \mathbb{Z}/p$, with the trivial action on the first summand and the action

$$\mathbb{Z} \oplus \mathbb{Z}/p \to \mathbb{Z} \oplus \mathbb{Z}/p, \quad (x, y) \to (x, x + y)$$

on the second summand. For this action

$$H^2(C_p, H_{tor}(M_{p,n})) = \mathbb{F}_p, \qquad H^1(C_p, H_{free}(M_{p,n})) = 0,$$

and $s = n$. Note also that $M_{p,n}/C_p = M_{1,n}$.

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