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On the triple points of singular maps

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Abstract. The number of triple points (mod 2) of a self-transverse immersion of a closed $2n$-manifold M into $3n$-space are known to equal one of the Stiefel–Whitney numbers of M. This result is generalized to the case of generic (i.e. stable) maps with singularities. Besides triple points and Stiefel–Whitney numbers, a certain linking number of the manifold of singular values with the rest of the image is involved in the generalized equation which corrects an erroneous formula in [9].

If n is even and the closed manifold is oriented then the equations mentioned above make sense over the integers. Together, the integer- and mod 2 generalized equations imply that a certain Stiefel–Whitney number of closed oriented $4k$-manifolds vanishes. This Stiefel–Whitney number is in fact the first in a family which vanish on such manifolds.

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1. Introduction

In his classical paper [10] of 1946, Whitney showed that the number of double points of a self-transverse immersion of an n-manifold into $2n$-space is related to the Euler number of its normal bundle. Since then many results of a similar nature have been found. This paper deals with a generalization of one of these results, the Herbert–Ronga formula [5] which expresses the number of triple points of a self-transverse immersion of a closed $2n$-manifold into $3n$-space in terms of one of its characteristic numbers. More precisely, the Herbert–Ronga formula is extended to singular generic (i.e. stable) maps of $2n$-manifolds into $3n$-space. (In this paper all manifolds and maps are assumed to be C^∞-smooth, unless otherwise explicitly stated.) To state the formula, some notation is needed:

Let M be a closed $2n$-manifold and let $f\colon M \to \mathbb{R}^{3n}$ be a generic map. If $\Delta(f) \subset \mathbb{R}^{3n}$ denotes the set of double points of f then $\Delta(f)$ is an immersed n-dimensional submanifold with boundary. The self-intersection points of $\Delta(f)$ are the triple points of f. The boundary of $\Delta(f)$ is $\Sigma(f)$, the set of singular values of f.

Define $t_2(f) \in \mathbb{Z}_2$ as the mod 2-number of triple points of f. Let $\Sigma'(f)$ denote the $(n-1)$-dimensional submanifold of $\mathbb{R}^{3n}$ which is obtained by shifting $\Sigma(f)$ slightly along its outward normal vector field in $\Delta(f)$. Then $\Sigma'(f) \cap f(M) = \emptyset$. Define $l_2(f) \in \mathbb{Z}_2$ as the mod 2-linking number of the cycles $f(M)$ and $\Sigma'(f)$ in $\mathbb{R}^{3n}$. If $i_1 + \cdots + i_m = 2n$ then let $\bar{w}_{i_1} \ldots \bar{w}_{i_m}[M] \in \mathbb{Z}_2$ denote the product of the normal Stiefel–Whitney classes of M in dimensions $i_1, \ldots, i_m$ evaluated on the fundamental homology class of M.

Theorem 1. *Let M be a closed manifold of dimension $2n$ and let $f\colon M \to \mathbb{R}^{3n}$ be a generic map. Then*

$$t_2(f) + l_2(f) = \bar{w}_n^2[M] + \bar{w}_{n+1}\bar{w}_{n-1}[M] \tag{1}$$

Theorem 1 is proved in Section 2. It corrects the erroneous theorem on the second page of [9], in which the second term in the right hand side of Equation (1) is missing.

For closed oriented $4k$-manifolds Equation (1) can be lifted to an integer equation: If $n = 2k$ is even and M is oriented then there is an induced orientation on $\Delta(f)$ as well as on the triple points of f. Define $t(f) \in \mathbb{Z}$ as the algebraic number of triple points of f. The orientation of $\Delta(f)$ induces an orientation of its boundary $\Sigma(f)$ which in turn induces an orientation of $\Sigma'(f)$. Define $l(f) \in \mathbb{Z}$ as the linking number of the oriented cycles $f(M)$ and $\Sigma'(f)$ in $\mathbb{R}^{6k}$. Let $\bar{p}_k[M^{4k}]$ denote the k^{th} normal Pontryagin number of M. The following theorem is Lemma 4 in [1].

Theorem 2. *Let M be a closed oriented manifold of dimension $4k$ and let $f\colon M \to \mathbb{R}^{6k}$ be a generic map. Then*

$$3t(f) - 3l(f) = \bar{p}_k[M]. \tag{2}$$

Equation (2) turned out to be very useful: It is used in the derivation of a geometric formula for Smale invariants of immersions of spheres, see [1] and [2], and in the study of geometric features of the regular homotopy classification of immersions of 3-manifolds in 5-space, see [7].

If M is a closed oriented $4k$-manifold then the mod 2-reduction of $\bar{p}_k[M]$ equals $\bar{w}_{2k}^2[M]$. Hence Theorems 1 and 2 together imply that

$$\bar{w}_{2k+1}\bar{w}_{2k-1}[M] = 0 \tag{3}$$

for any closed oriented $4k$-manifold M. In fact, $\bar{w}_{2k+1}\bar{w}_{2k-1}[M]$ is the first in a sequence of Stiefel–Whitney numbers which vanish on closed oriented $4k$-manifolds. More precisely,

Theorem 3. (Stong). *If M is an oriented $4k$-manifold and $(2k_1 + 1) + \cdots + (2k_r + 1) = 4k$ then*

$$\bar{w}_{2k_1+1} \ldots \bar{w}_{2k_r+1}[M] = 0.$$

This theorem was communicated by R. Stong to the second author together with a proof of the first case (3). A proof of Theorem 3 is presented in Section 3.

2. Proof of Theorem 1

Fix a generic map $f\colon M \to \mathbb{R}^{3n}$ of a closed $2n$-manifold. Let $\tilde\Sigma \subset M$ denote the $(n-1)$-dimensional submanifold of singular points of f and let $\Sigma = f(\tilde\Sigma)$. Then f maps $\tilde\Sigma$ diffeomorphically to Σ.

Let $\tilde\Delta \subset M$ denote the closure of the preimages of multiple points of f. Then $\tilde\Delta$ is an immersed closed n-dimensional manifold with transverse double points at the preimages of triple points of f. Let $\tilde\Delta_{\rm res}$ denote the resolution of $\tilde\Delta$ and let $\tilde\iota\colon \tilde\Delta_{\rm res} \to M$ denote the natural immersion with image $\tilde\Delta \subset M$.

There is a natural involution $T\colon \tilde\Delta_{\rm res} \to \tilde\Delta_{\rm res}$ such that $f \circ \tilde\iota \circ T = f \circ \tilde\iota$. Since no triple point of f is singular we have a natural embedding $\tilde\Sigma \subset \tilde\Delta_{\rm res}$ and $\tilde\Sigma$ is the fix point set of T.

Let $\nu(\tilde\iota)$ denote the normal bundle of the immersion $\tilde\iota$ and let ν denote its restriction to $\tilde\Sigma$. Since ν is an n-dimensional vector bundle over an $(n-1)$-manifold there exists a non-zero section. Let $\tilde s$ be such a section.

A standard transversality argument allows us to extend $\tilde s$ to a section $\tilde S$ of $\nu(\tilde\iota)$ which is transverse to the 0-section and which satisfies the following two conditions:

- If x is a double point of $\tilde\iota$ then $\tilde S(x) \neq 0$.
- If $\tilde S(x) = 0$ then $\tilde S(T(x)) \neq 0$.

Let $\Delta \subset \mathbb{R}^{3n}$ denote the closure of the double points of f. Then Δ is an immersed submanifold with boundary Σ and Δ has triple points at the triple points of f. Let $\Delta_{\rm res}$ denote the resolution of Δ and let $\iota\colon \Delta_{\rm res} \to \mathbb{R}^{3n}$ denote the natural immersion with image Δ. Let $\nu(\iota)$ denote the normal bundle of the immersion ι. Note that there is a natural map $\Pi\colon \tilde\Delta_{\rm res} \to \Delta_{\rm res}$ which is a double cover of $\Delta_{\rm res} - \Sigma$ when restricted to $\tilde\Delta_{\rm res} - \tilde\Sigma$, and which maps $\tilde\Sigma$ diffeomorphically onto Σ.

Define the section S of $\nu(\iota)$ as follows:

$$S(y) = \begin{cases} df(\tilde S(y_1)) + df(\tilde S(y_2)) & \text{if } y \in \Delta_{\rm res} - \Sigma, \text{ where } y_1 \neq y_2,\ \Pi(y_1) = \Pi(y_2) = y, \\ 2df(\tilde S(y_1)) & \text{if } y \in \Sigma, \text{ where } \Pi(y_1) = y. \end{cases}$$

Let $C(\Sigma) \subset \Delta_{\rm res}$ be a small open collar on the boundary Σ of $\Delta_{\rm res}$. Let Δ'' denote the image of the immersion $y \mapsto \iota(y) + \epsilon S(y)$, $y \in \Delta_{\rm res} - C(\Sigma)$ for some small $\epsilon > 0$. Then, if ϵ and the collar $C(\Sigma)$ are small enough, Δ'' is a chain with boundary $\partial\Delta'' = \Sigma''$ satisfying $\Sigma'' \cap f(M) = \emptyset$. If lk_2 denotes the mod 2-linking number, $\bullet$ denotes the mod 2-intersection number, and $\sharp(F)$ denotes the mod 2-number of elements in the finite set F, then

$$\mathrm{lk}_2(\Sigma'', f(M)) = \Delta'' \bullet f(M) = \sharp(\tilde S^{-1}(0)) + t_2(f), \tag{4}$$

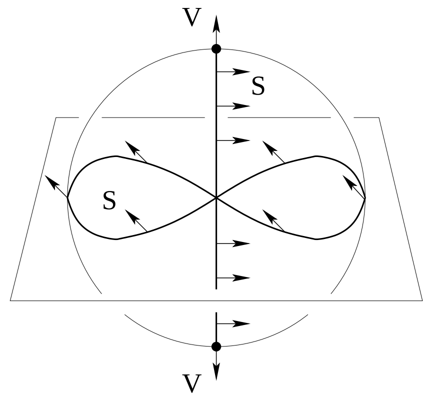

Figure 1. A piece of $f(M)$ (represented by a 2-sphere and a piece of a plane) with the double point set Δ (fat lines), its normal field S, and singularity set Σ (dots) with its outward normal field V in Δ.

since near each zero z of $\tilde{S}$ there is a unique intersection point of Δ'' and $f(M)$ near $f(z)$, and near each triple point of f there are exactly three such intersection points.

The homology class of the cycle $\tilde{\Delta}$ in M is Poincaré dual to n^{th} normal Stiefel–Whitney class $\bar{w}_n$ of M, see [6]. Thus

$$\bar{w}_n^2[M] = \tilde{\Delta} \bullet \tilde{\Delta} = \sharp(\tilde{S}^{-1}(0)), \tag{5}$$

since the image of a slight shift of the immersion $\tilde{\imath}$ along $\tilde{S}$ intersects $\tilde{\Delta}$ near each zero of $\tilde{S}$ and in *two* points near each double point of $\tilde{\imath}$.

Equations (4) and (5) imply

$$\mathrm{lk}_2(\Sigma'', f(M)) = \bar{w}_n^2[M] + t_2(f). \tag{6}$$

Recall that $\Sigma' \subset \mathbb{R}^{3n}$ is the submanifold which results when Σ is shifted slightly along its unit outward normal vector field V in Δ, and that $\Sigma' \cap f(M) = \emptyset$. We compare the linking numbers $\mathrm{lk}_2(\Sigma'', f(M))$ and $\mathrm{lk}_2(\Sigma', f(M))$:

Let $\tilde{\Sigma}_0 \subset M$ be the submanifold which results when $\tilde{\Sigma}$ is shifted a small distance along $\tilde{S}$. Let $\Sigma_0 = f(\tilde{\Sigma}_0)$ and for $p \in \Sigma$, let $p_0 = f(\tilde{p}_0)$ where $\tilde{p}_0$ is the point in $\tilde{\Sigma}_0$ corresponding to $\tilde{p} \in \tilde{\Sigma}$ with $f(\tilde{p}) = p$.

For small $\epsilon > 0$ and $p \in \Sigma$ let $l_p(\epsilon)$ be the segment of the straight line through $p + \epsilon V(p)$ and p_0 of length 2ϵ and centered at p_0. For $\epsilon > 0$ and the shifting of $\tilde{\Sigma}$ in M small enough,

$$\Gamma = \bigcup_{p \in \Sigma} l_p(\epsilon)$$

is a submanifold of $\mathbb{R}^{3n}$. If the collar $C(\Sigma)$ is chosen small enough and if the shifting distance along S is small enough then the boundary $\partial\Gamma$ of Γ is isotopic to $\Sigma' \cup \Sigma''$ in $\mathbb{R}^{3n} - f(M)$. Thus

$$\mathrm{lk}_2(\Sigma', f(M)) = \mathrm{lk}_2(\Sigma_0, f(M)) + \Gamma \bullet f(M) = \mathrm{lk}_2(\Sigma'', f(M)) + \Gamma \bullet f(M). \quad (7)$$

We compute $\Gamma \bullet f(M)$: The intersection $\Gamma \cap f(M)$ is a clean intersection. That is, $\Gamma \cap f(M) = \Sigma_0$ is a manifold and the tangent bundle

$$T\Sigma_0 = Tf(M) \cap T\Gamma \subset T\mathbb{R}^{3n}, \quad (8)$$

where all bundles in the left hand side are restricted to Σ_0.

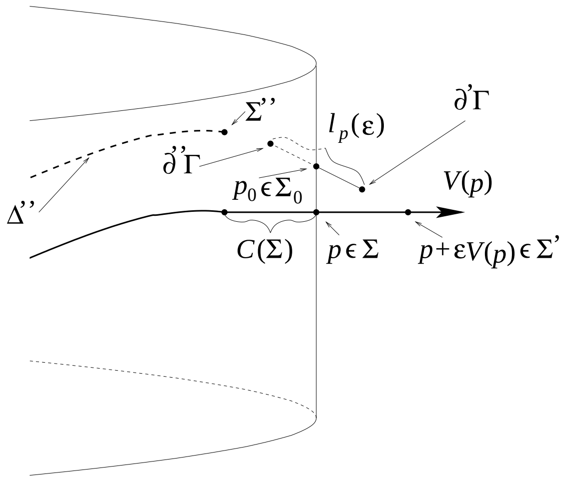

Figure 2. The normal space of Σ in $\mathbb{R}^{3n}$ at $p \in \Sigma$. In the figure the boundary of Γ is the union of $\partial'\Gamma$, isotopic to Σ' in $\mathbb{R}^{3n} - f(M)$, and $\partial''\Gamma$ isotopic to Σ'' in $\mathbb{R}^{3n} - f(M)$.

As in [4], we find

$$\Gamma \bullet f(M) = w_{n-1}(\xi),$$

where ξ is the so called excess bundle over Σ_0:

$$\xi = T\mathbb{R}^{3n}/(T\Gamma + Tf(M)),$$

where all bundles are restricted to Σ_0.

To finish the proof it remains to calculate $w_{n-1}(\xi)$. Note that

$$T\Gamma|\Sigma_0 = T\Sigma_0 \oplus \epsilon^1,$$

where ϵ^1 is the trivial line bundle directed along the intervals $l_p(\epsilon)$. Thus, by (8),

$$\xi \oplus Tf(M)|\Sigma_0 \oplus \epsilon^1 = T\mathbb{R}^{3n}|\Sigma_0. \quad (9)$$

The bundle $Tf(M)|\Sigma_0$ is identified with $TM|\tilde{\Sigma}_0$ by the differential of f. Hence if $i_0\colon \tilde{\Sigma}_0 \to M$ denotes the inclusion then $w(\xi) = i_0^*\bar{w}(M)$. Therefore, if F_V denotes the fundamental homology class of the manifold V and PD denotes the Poincaré duality operator,

$$\begin{aligned}\langle w_{n-1}(\xi), F_{\Sigma_0}\rangle &= \left\langle i_0^*\bar{w}(M), F_{\tilde{\Sigma}_0}\right\rangle = \left\langle \bar{w}(M), i_{0*}(F_{\tilde{\Sigma}_0})\right\rangle = \langle \bar{w}(M), \mathrm{PD}\,\bar{w}_{n+1}(M)\rangle \\ &= \langle \bar{w}(M)\cup \bar{w}_{n+1}(M), F_M\rangle = \bar{w}_{n-1}\bar{w}_{n+1}[M]. \qquad (10)\end{aligned}$$

Here, the third equality follows from the well-known formula $\mathrm{PD}\,\bar{w}_{n+1}(M) = i_*F_{\tilde{\Sigma}}$, where $i\colon \tilde{\Sigma} \to M$ denotes the inclusion, together with $i_*F_{\tilde{\Sigma}} = i_{0*}F_{\tilde{\Sigma}_0}$. Equations (6), (7), and (10) prove the theorem. □

3. Proof of Theorem 3

Let $\mathfrak{N}_*$, Ω_*, and Ω_*^U denote the cobordism ring, the oriented cobordism ring, and the complex cobordism ring, respectively. Note that there are natural forgetting homomorphisms

$$\Omega_*^U \longrightarrow \Omega_* \longrightarrow \mathfrak{N}_*.$$

For a manifold M, let $[M]$ denote its cobordism class.

Using some facts from cobordism theory which can all be found in Chapter 4 of Stong's book [8], we show that it is enough to prove the theorem for oriented $4k$-manifolds M such that either

(a) $[M] \in \Omega_{4k}$ maps to a square $[N \times N] \in \mathfrak{N}_{4k}$, or

(b) $[M]$ is a torsion element of Ω_{4k} (in fact, $[M]$ torsion implies $2\cdot[M] = 0$):

Let $\mathrm{Tors}(\Omega_*)$ denote the torsion subgroup of Ω_*. The homomorphism $\Omega_*^U \to \Omega_*$ induces an epimorphism

$$\Omega_*^U \longrightarrow \Omega_*/\,\mathrm{Tors}(\Omega_*).$$

and the image $\Omega_*^U \to \mathfrak{N}_*$ consists of squares of elements in $\mathfrak{N}_*$.

Hence, if M is any oriented $4k$-manifold then there exists some oriented $4k$-manifold V such that $[V]$ is torsion in Ω_{4k} and $[M]+[V] = [N\times N]$ in $\mathfrak{N}_{4k}$. This implies that the theorem follows once it is proved for manifolds satisfying (a) or (b) above.

First consider (a): let $M = N\times N$. Then $\bar{w}(M) = \bar{w}(N)\times\bar{w}(N)$ and hence

$$\bar{w}_{2k+1}(M) = \sum_{i+j=2k+1} \bar{w}_i(N)\times\bar{w}_j(N).$$

Thus

$$\begin{aligned}&\langle \bar{w}_{2k_1+1}(M)\dots\bar{w}_{2k_r+1}(M), F_M\rangle = \\ &= \sum \langle \bar{w}_{i_1}(N)\dots\bar{w}_{i_r}(N), F_N\rangle \cdot \langle \bar{w}_{j_1}(N)\dots\bar{w}_{j_r}(N), F_N\rangle\,. \qquad (11)\end{aligned}$$

Since $i_s + j_s$ is odd for all i_s, j_s there is a fixed point free involution T acting on the set of the terms in the sum in (11) such that $i_1 + \cdots + i_r = 2k = j_1 + \cdots + j_r$:

$$T\colon \langle \bar{w}_{i_1}(N) \dots \bar{w}_{i_r}(N), F_N \rangle \cdot \langle \bar{w}_{j_1}(N) \dots \bar{w}_{j_r}(N), F_N \rangle$$
$$\mapsto \langle \bar{w}_{j_1}(N) \dots \bar{w}_{j_r}(N), F_N \rangle \cdot \langle \bar{w}_{i_1}(N) \dots \bar{w}_{i_r}(N), F_N \rangle .$$

Thus the terms in the left hand side of (11) which does not vanish for dimensional reasons cancel in pairs and hence $\bar{w}_{2k_1+1} \dots \bar{w}_{2k_r+1}[M] = 0$.

Next consider (b): let $u\colon \mathfrak{N}_{4k} \to \mathbb{Z}_2$ denote the homomorphism induced by the product of odd-dimensional normal Stiefel–Whitney classes $\bar{w}_{2k_1+1} \dots \bar{w}_{2k_r+1}$, $\sum 2k_j + 1 = 4k$. Odd-dimensional Stiefel–Whitney classes are mod 2-reductions of twisted integer classes, see [3], p. 140. Hence, a product of an even number of such classes is an integer class so the map

$$\Omega_{4k} \xrightarrow{\ \pi\ } \mathfrak{N}_{4k} \xrightarrow{\ u\ } \mathbb{Z}_2$$

lifts to a homomorphism

$$\Omega_{4k} \xrightarrow{\ U\ } \mathbb{Z}.$$

Thus U and therefore $u \circ \pi$ is zero on any torsion element of Ω_{4k}. □

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