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Universal octonary diagonal forms over some real quadratic fields

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Abstract. In this paper, we will prove there are infinitely many integers n such that $n^2 - 1$ is square-free and $\mathbb{Q}(\sqrt{n^2-1})$ admits universal octonary diagonal quadratic forms.

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1. Introduction

A universal integral form over totally real number field K is a positive definite quadratic form over the ring of integers of K which represents all the totally positive integers of K. For example, the sum of four squares is universal integral over $\mathbb{Q}$. In 1917, Ramanujan [8] found there are exactly 54 universal positive diagonal integral quadratic forms over $\mathbb{Q}$. More concretely, he showed there are 54 diagonal quaternary quadratic forms $f(x, y, z, w) = ax^2 + by^2 + cz^2 + dw^2$ such that $a, b, c, d \in \mathbb{Z}^+$ and the equation $f = n$ is solvable for all $n \in \mathbb{Z}^+$. In 1947, M. Willerding [10] proved there are exactly 178 classic universal integral forms. More concretely, she showed there are 178 quaternary quadratic forms $f(x, y, z, w)$ up to equivalence such that f is positive definite integral quadratic form, the coefficients of cross terms of f are always even and the equation $f = n$ is solvable for all $n \in \mathbb{Z}^+$. On the other hand, the study of positive universal quadratic integral forms over totally real number fields was initiated by F. Götzky [3]. In 1928, he proved that the sum of four squares is universal over $\mathbb{Q}(\sqrt{5})$. H. Maass [6] improved this result. In 1941, he proved the sum of three squares is positive universal over $\mathbb{Q}(\sqrt{5})$. Four years later, Siegel [9] proved $\mathbb{Q}(\sqrt{5})$ is the only totally real number field, other than $\mathbb{Q}$, over which every (totally) positive integer is a sum of squares. In other words, he showed if a totally real number field K is different from $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{5})$, there is a totally positive algebraic integer α of K which cannot be represented by the sum of any number of squares. For example, if $K = \mathbb{Q}(\sqrt{2})$, $\alpha = 2 + \sqrt{2}$. In 1996, W. K. Chan, M.-H. Kim and S. Raghavan [1] classified all (totally) positive universal integral ternary lattices over real quadratic fields. Only $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$ admit universal integral ternary lattices and total number of universal integral ternary lattices over real quadratic fields is 11. Recently, the author [5] proved there are only finitely many real quadratic fields which admit universal integral septenary diagonal forms. The content of this paper is to prove if n^2-1 is square-free, there are universal octonary diagonal forms over $\mathbb{Q}(\sqrt{n^2-1})$. So we can prove there are infinitely many real quadratic fields which admit universal integral octonary diagonal forms. Obviously 8 is the minimal rank with this property.

2. Main Theorem

Throughout this chapter, we let $m = n^2 - 1$ be a positive square free integer, $K = \mathbb{Q}(\sqrt{m})$ and $\mathcal{O}_K$ be the ring of algebraic integers of K. Note that $\epsilon = n + \sqrt{m}$ is the fundamental unit of $\mathcal{O}_K$ and is totally positive.

Theorem 1. *The octonary diagonal form $x_1^2+x_2^2+x_3^2+x_4^2+\epsilon x_5^2+\epsilon x_6^2+\epsilon x_7^2+\epsilon x_8^2$ is universal over $\mathcal{O}_K$.*

This Theorem is a consequence of following Lemmas.

Lemma 1. *Let $1 \le b < 2n$. $\alpha = a + b\sqrt{m}$ is totally positive algebraic integer in K if and only if $nb \le a$.*

Proof. As $nb + b\sqrt{m} = b(n+\sqrt{m})$ is totally positive, the necessity is trivial. For the sufficiency, it suffices to prove $nb - 1 - (b\sqrt{m}) < 0$. This follows from

$$(nb-1)^2 - (b\sqrt{m})^2 = n^2b^2 - 2nb + 1 - b^2(n^2-1)$$

$$= (b-n)^2 - n^2 + 1 \le (n-1)^2 - n^2 + 1 < 0.$$

□

Lemma 2. *If $\alpha \in \mathcal{O}_K^+$, α belongs to*

$$S = \{a_0\epsilon^k + a_1\epsilon^{k+1} + \ldots + a_l\epsilon^{k+l} \mid k, l \in \mathbb{Z},\ a_0, a_1, \ldots, a_l \in \mathbb{N}\}.$$

Proof. Suppose $\alpha = a + b\sqrt{m} \notin S$. We may assume that $b > 0$ and $\mathrm{tr}_{K/\mathbb{Q}}(\alpha) \le \mathrm{tr}_{K/\mathbb{Q}}(\beta)$ for all elements $\beta \notin S$. Then, by Lemma 1, we have $b \ge 2n$. Since

$$bn - 1 + b\sqrt{m} = \epsilon^2 + (b-2n)\epsilon \in S,$$

we also have $a \le bn-1$. Then,

$$\alpha\epsilon^{-1} = (a+b\sqrt{m})(n-\sqrt{m}) = an - bm + (bn-a)\sqrt{m}.$$

So

$$\begin{aligned}\mathrm{tr}_{K/\mathbb{Q}}(\alpha\epsilon^{-1}) &= 2(an-bm) \le 2(n(bn-1) - b(n^2-1)) \\ &= 2(b-n) < 2a = \mathrm{tr}_{K/\mathbb{Q}}(\alpha).\end{aligned}$$

So $\alpha\epsilon^{-1} \in S$. Thus $\alpha \in S$. Contradiction. □

Lemma 3. *For $l \ge 2$, $\epsilon^l = -1 + b_1\epsilon + b_2\epsilon^2 + \ldots + b_{l-1}\epsilon^{l-1}$ where $b_1 \ge 2n-1$ and $b_2, \ldots, b_{l-1} \ge 2n-2$.*

Proof. We use induction on l. As $\epsilon^2 = 2n\epsilon - 1$, the assertion holds for $l = 2$. If this Lemma is true for $l = s \ge 2$,

$$\epsilon^{s+1} = \epsilon\epsilon^s = \epsilon(-1 + b_1\epsilon + b_2\epsilon^2 + \ldots + b_{s-1}\epsilon^{s-1})$$

$$= -\epsilon + \epsilon^2 + (b_1 - 1)\epsilon^2 + b_2\epsilon^2 + \ldots + b_{s-1}\epsilon^s$$

$$= -1 + (2n-1)\epsilon + (b_1-1)\epsilon^2 + b_2\epsilon^2 + \ldots + b_{s-1}\epsilon^s.$$

This proves the Lemma. □

Lemma 4. *If $\alpha \in \mathcal{O}_K^+$, $\alpha = p\epsilon^k + q\epsilon^{k+1}$ for some $p, q \in \mathbb{N}$ and $k \in \mathbb{Z}$.*

Proof. By Lemma 2, $\alpha = a_k\epsilon^k + \ldots + a_{k+l}\epsilon^{k+l}$ with $a_k, \ldots, a_{k+l} \ge 0$.

If $l \ge 2$ and $a_{k+l} \le a_k$,

$$\alpha = a_k\epsilon^k + \ldots + a_{k+l-1}\epsilon^{k+l-1} + a_{k+l}\epsilon^k(-1 + b_1\epsilon + \ldots + b_{l-1}\epsilon^{l-1})$$

$$= (a_k - a_{k+l})\epsilon^k + (a_{k+1} + a_{k+l}b_1)\epsilon^{k+1} + \ldots + (a_{k+l-1} + a_{k+l}b_{l-1})\epsilon^{k+l-1}.$$

If $l \ge 2$ and $a_{k+l} \ge a_k$,

$$\alpha = a_k\epsilon^k + \ldots + a_{k+l-1}\epsilon^{k+l-1} + (a_{k+l} - a_k)\epsilon^{k+l} + a_k\epsilon^k(-1 + b_1\epsilon + \ldots + b_{l-1}\epsilon^{l-1})$$

$$= (a_k + a_{k+l}b_1)\epsilon^{k+1} + \ldots + (a_k + a_{k+l}b_{l-1})\epsilon^{k+l-1} + (a_{k+l} - a_k)\epsilon^{k+l}.$$

Repeating the same process, we can obtain the desired expression of α. □

Proof of Theorem 1. If $\alpha \in \mathcal{O}_K^+$, by Lemma 4, $\alpha = a\epsilon^k + b\epsilon^{k+1}$ for some a, $b \in \mathbb{N}$ and $k \in \mathbb{Z}$. If k is even, by Lagrange's four square theorem, $a\epsilon^k$ is represented by $x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $b\epsilon^{k+1}$ is represented by $\epsilon x_5^2 + \epsilon x_6^2 + \epsilon x_7^2 + \epsilon x_8^2$. So f represents α. Similarly f represents α for odd k. Thus f is universal integral over K. □

Lemma 5. *There are infinitely many square free integers of the form* $n^2 - 1$.

Proof. If n is even, $n^2 - 1$ is square free if and only if both $n+1$ and $n-1$ are square free. It is known that [4] the number of positive square free integers which do not exceed x is $\frac{6x}{\pi^2} + O(\sqrt{x})$. So the number of positive integer n such that $n \leq x$ and both $n+1$ and $n-1$ are square free is larger than

$$(\frac{6x}{\pi^2} + O(\sqrt{x})) + (\frac{6x}{\pi^2} + O(\sqrt{x})) - x = \frac{12 - \pi^2}{\pi^2} x + O(\sqrt{x}).$$

Since $\frac{12-\pi^2}{\pi^2} > 0$, there are infinitely many n such that $n \leq x$ and $n^2 - 1$ is square free. □

Theorem 2. *There are infinitely many real quadratic fields that admit octonary universal forms.*

Proof. This is an immediate consequence of Theorem 1 and Lemma 5. □

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