On the Interior Regularity of Weak Solutions to Nonlinear Elliptic Systems of Second Order

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In this paper, which is a modified version of the thesis [4], we prove regularity for a weak solution (with gradient in the BMO-space) of the following nonlinear elliptic system (i = 1, ..., N):

\[-D_\alpha a_i^\alpha(x, u, Du) + a_i(x, u, Du) = -D_\alpha f_i(x) + f_i(x),\]

where \(x\) belongs to a bounded open set \(\Omega\) of \(\mathbb{R}^n\), \(u \geq 3\), \(u: \Omega \to \mathbb{R}^N\), \(N > 1\), \(u(x) = (u^1(x), ..., u^N(x))\) is a vector-valued function, \(Du = (D_1u, ..., D_Nu)\), \(D_\alpha = \partial/\partial x_\alpha\), we will use the summation convention over repeated indices.

In [6—9, 12] the so-called Liouville condition (L) is formulated in terms of the space \(L^\infty\). On the other hand, the proof of \(L^\infty\)-boundedness of the gradient of a weak solution for the system (0.1) has not yet been achieved in reasonably wide extent and the possibility of this proof is questionable.

The following definition is a generalized form of the Liouville property from [7, 8] and reads as follows.

Definition 0.1: The system (0.1) satisfies the Liouville property (L) if for every \(x^0 \in \Omega\) and every \(u \in \mathbb{R}^N\) the only solutions \(v \in \mathbb{R}^n\) to

\[-D_\alpha a_i^\alpha(x^0, u, Dv(x)) = 0, \quad (i = 1, ..., N)\]

with \(Dv \in \text{BMO}(\mathbb{R}^n)\) are polynomials of at most first degree.
The main result of this paper is the fact that if system (0.1) has property (L), then $Du$ is locally Hölder continuous in $\Omega$. To this effect it represents a generalization of [7, 8]. Because it is easier to verify that the gradient of the solution is an element of the BMO-space ($L^\infty \subseteq \text{BMO}$), the generalization reached in this paper has a fundamental meaning. The approach stated in this paper has been used in [15], which deals with quasilinear parabolic systems.

1. Notations and definitions

In the sequel $\Omega$ will be a bounded open set of $\mathbb{R}^n$ with Lipschitz boundary $\partial \Omega$. The meaning of $\Omega_0 \subseteq \subseteq \Omega$ is that the closure of $\Omega_0$ is contained in $\Omega$, i.e., $\overline{\Omega}_0 \subseteq \Omega$. For the sake of simplification we denote by $\cdot$ the norm and scalar product in $\mathbb{R}^n$ as well as in $\mathbb{R}^N$ and $\mathbb{R}^{nN}$. If $x \in \mathbb{R}^n$ and $r$ is a positive real number, we set $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$, $\Omega(x, r) = \Omega \cap B(x, r)$ and $Q(x, r)$ will be the cube in $\mathbb{R}^n$ with the center in the point $x$ and length of the side $r$.

By $\mathcal{P}_k$, $k \geq 0$ integer, we denote the set of all vector-valued polynomials $P = (P_1, \ldots, P_N)$ with real coefficients defined on $\mathbb{R}^n$ such that the degree of $P_i$ is less than $k$ for each $i = 1, \ldots, N$.

Beside the usually used Hölder and Sobolev spaces (for detailed information, see, e.g., [3, 6, 12]) we will use the following ones.

**Definition 1.1 (Campanato-Morrey spaces):** Let $\lambda \in [0, n]$, $p \in (1, \infty)$. The space $L^{p, \lambda}(\Omega)$ is the subspace of such functions $f \in L^p(\Omega)$ for which

$$
\|f\|_{L^{p, \lambda}(\Omega)} = \left\{ \sup_{x \in \Omega, r > 0} \frac{1}{r^{n - \lambda}} \int_{B(x, r)} |f(y)|^p \, dy \right\}^{1/p} < \infty.
$$

Let $k$ be a non-negative integer and $\lambda \in [0, n + (k - 1)p]$. The space $L_{\lambda}^{p, \lambda}(\Omega)$ is the subspace of such functions $f \in L^p(\Omega)$ for which

$$
\|f\|_{L_{\lambda}^{p, \lambda}(\Omega)} = \|f\|_{L^{p, \lambda}(\Omega)} + \|f\|_{L^{p, \lambda}(\Omega)} < \infty,
$$

where

$$
[f]_{L_{\lambda}^{p, \lambda}(\Omega)} = \left\{ \sup_{x \in \Omega, r > 0} \frac{1}{r^{n - \lambda}} \int_{B(x, r)} |f(y) - P(y)|^p \, dy \right\}^{1/p}.
$$

With the norms (1.1) and (1.2), $L^{p, \lambda}(\Omega)$ and $L_{\lambda}^{p, \lambda}(\Omega)$ are Banach spaces. We will work mainly with the spaces $L^{1,1}$, $L^{2,1}$, $L^{1,2}$ and $L^{2,2}$; instead of $L^{2,1}$ we will usually write $L_{1}^{2,1}$.

In our considerations we make use of the fact that for each function $u \in L^{2,1}(\Omega)$, each $x^0 \in \Omega$, $0 < r \leq \text{diam} \, \Omega$, there exists one and only one polynomial $P \in \mathcal{P}_k$, $P(x) = P(x, x^0, r, u)$ such that

$$
\inf_{P \in \mathcal{P}_k, \lambda \in \Omega(x, r)} \int_{B(x, r)} |u(x) - P(x, x^0, r, u)|^2 \, dx.
$$

For $k = 1$ we will write this polynomial $P$ in the form

$$
P(x, x^0, r, u) = b(x^0, r, u) + \sum_{s=1}^{n} b(x^0, r, u) (x_s - x^0) = b(x^0, r, u) + (b(x^0, r, u), (x - x^0)),
$$

and for $k = 0$ it equals the constant

$$
u_{x^0, r} = \int_{B(x^0, r)} u(y) \, dy = (\text{meas} \, B(x^0, r))^{-1} \int_{B(x^0, r)} u(y) \, dy.
$$
where \( \text{meas} B(x^0, r) \) means the \( n \)-dimensional Lebesgue measure. Denote further \( U(x^0, r) = \int |u(y) - u_{x^0}|^2 \, dy \), and define \( \text{BMO}(\mathbb{R}^n) \) as the set of all measurable functions \( u \) on \( \mathbb{R}^n \) for which the set \( \mathcal{U} = \{ U(x, r) : x \in \mathbb{R}^n, r > 0 \} \) is bounded, setting \( \|u\|_{\text{BMO}(\mathbb{R}^n)} = \sup \mathcal{U} \).

At last, let \( H^{1,1}(\Omega), \lambda \in [0, n] \) be the Banach space of all functions \( u \in H^1(\Omega), D_s u \in L^{2,1}(\Omega) \) with norm 

\[
\|u\|_{H^{1,\lambda}(\Omega)} = \|u\|_{L^2(\Omega)} + \sum_{s=1}^{n} \|D_s u\|_{L^2(\Omega)}.
\]

Proposition 1.1: We have the following important properties of the spaces defined above:

(a) \( L^{2,1}(\Omega) \supseteq L^{2,1}(\Omega), \lambda \in [0, n) \),
(b) \( L^{2,2}(\Omega) = L^{2,2}(\Omega), \lambda \in [0, n + 2) \),
(c) \( L^{2,n}(\Omega) \subset L^{2,1}(\Omega) \subset L^{2,1}(\Omega), 0 \leq \lambda < \lambda_1 < n \),
(d) \( L^{2,n}(\Omega) = L^\infty(\Omega) \supseteq L^{2,n}(\Omega) \),
(e) \( L^{p,n}(\Omega) = L^{p,n}(\Omega) = \text{BMO}(\Omega) \) for all \( p, s \in [1, \infty) \), \( \Omega \) being a cube,
(f) \( H^{1,\gamma}(\Omega) \subset C^{0,\gamma}(\Omega) \) for each \( \Omega_0 \subset \subset \Omega \), \( \gamma \in (0, 1) \) and

\[
\|\cdot\|_{C^{0,\gamma}(\Omega)} \leq c(n, \gamma, \text{diam } \Omega, \text{dist } (\Omega_0, \partial \Omega)) \|\|_{H^{1,\gamma}(\Omega)}.
\]

For the proofs and more detailed information about the Campanato-Morrey spaces see, e.g., [1-3, 6, 12]. In the sequel we will denote all important constants by the symbol \( c \) and other ones by \( \overline{c} \).

A function \( u \in H^1(\Omega) \) is called weak solution of (0.1) in \( \Omega \) if

\[
\int_{\Omega} a_s(x, u, Du) D_s \varphi(x) \, dx + \int_{\Omega} a_t(x, u, Du) \varphi(x) \, dx = \int_{\Omega} f_s(x) D_s \varphi(x) \, dx + \int_{\Omega} f_t(x) \varphi(x) \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega),
\]

(1.4)

where \( a_s^t, a_t, f_s^t, f_t \) are functions fulfilling for each \( (x, u, p) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \) with \( |u| \leq L \) the following conditions:

\[
a_s \in C^1(\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}),
\]

(1.5)

\[
|a_s^t(x, u, p)|, |a_t(x, u, p)| \leq \mathcal{C}_1(L) (1 + |p|),
\]

(1.6)

\[
|\partial a_s(x, u, p)/\partial p_i|, |\partial a_t(x, u, p)/\partial p_i| \leq \mathcal{C}_1(L),
\]

(1.7)

\[
|\partial a_s^t(x, u, p)/\partial u_i|, |\partial a_t^t(x, u, p)/\partial x_i| \leq \mathcal{C}_1(L) (1 + |p|),
\]

(1.8)

\[
\partial a_s^t(x, u, p)/\partial p_i \text{ is uniformly continuous on } \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN},
\]

(1.9)

\[
\partial a_t^t(x, u, p)/\partial p_i \to d_{ii}^t(x, u) \quad \text{as } |p| \to \infty, \quad \text{for all } (x, u) \in \Omega \times \mathbb{R}^N
\]

(1.10)

\[
f_t \in H^{1,\gamma}(\Omega), \quad f_t \in H^{1,\gamma}(\Omega), \quad \overline{q} > n,
\]

\[
\mathcal{C}_1(L) (1 + |p|),
\]

(1.11)

\[
\mathcal{C}_2,
\]

(1.12)

\[
\mathcal{C}_2 \mathcal{C}_2 \eta^2 \geq \nu(L) |\eta|^2
\]

(1.13)

It is known that \( u \in H^{1,\gamma}(\Omega) \) if the function \( u \) fulfills the conditions stated above (see, e.g., [3]).
2. The results

Principal result of this paper is the following theorem.

**Theorem 2.1:** Let \( u \in H^{1,(n)}(\Omega) \) be a weak solution of the system (0.1) and suppose that the conditions (1.5)\( - (1.13) \) hold. If the system (0.1) has the Liouville property (L), then \( u \in C^{1,\alpha}_{\text{loc}}(\Omega) \).

There arise two natural questions:

1. Do there exist systems of the form (0.1) with weak solutions in the space \( H^{1,(n)}(\Omega) \)?
2. Under which assumptions has the system of the form (0.1) the Liouville property (L)?

A partial answer on the first question is given in [5]. The problem of the \( H^{1,(n)} \)-regularity of weak solutions is studied in detail in [3]. The second question is positively answered in the case of \( n = 2 \) and \( N > 1 \) by the following

**Proposition 2.2:** Let the system (0.1) satisfy conditions (1.5)\( - (1.8), (1.11) - (1.13) \) and let \( n = 2 \). Then it has property (L).

In the case \( n \geq 3, N > 1 \) some conditions under which linear elliptic systems with \( L^\infty \)-coefficients, quasilinear or nonlinear systems, respectively, have property (L) are shown in [11], [13] and [10], respectively. From [14] it follows that there are nonlinear elliptic systems without property (L).

3. Lemmas

The following two lemmas concern the estimate of the coefficients of the polynomials from (1.5).

**Lemma 3.1** [1: pp. 140–144]: Let \( P \in \mathcal{P}_s \), \( s \in [1, \infty) \) and \( E \) be a measurable subset of the ball \( B(x_0, r) \subset \mathbb{R}^n \) satisfying the condition \( \text{meas } E \leq A r^n \), \( A \) a positive constant. Then there is a constant \( c = c(n, k, s, A) \) such that for each multiindex \( \alpha \) we have

\[
\| D_x P(x) \|_{L^\infty(E)} \leq (c/r^{\alpha + |\alpha|}) \int_E |P(x)|^s \, dx.
\]

**Lemma 3.2** [1: pp. 146]: Let \( u \in \mathcal{L}^{2,n+2}(\Omega) \). Then there exists a constant \( c = c(n) \) such that for every \( x \in \Omega \) and for all \( r, r_0, 0 < r \leq r_0 \leq \text{diam } \Omega \), we have

\[
|b^0(x, r_0) - b^0(x, r)| \leq c r \| u \|_{L^2,2,n+2(\Omega)},
\]

\[
|b^\alpha(x, r_0) - b^\alpha(x, r)| \leq c (1 + \ln (r_0/r)) \| u \|_{L^2,2,n+2(\Omega)}
\]

for all \( \alpha = 1, \ldots, n \), where \( b^0, b^\alpha \) are defined in (1.3).

Another important result needed for the proof of Theorem 2.1 is the following

**Proposition 3.3** [2: pp. 373]: Let \( \Omega \) be convex. Then there is a constant \( c = c(n, \text{diam } \Omega, \text{meas } \Omega) \) such that for each \( \lambda \in [0, n + 2] \) we have

\[
H^{1,\lambda}(\Omega) \subset \mathcal{L}^{2,2+n+2}(\Omega),
\]

\[
\| u \|_{L^{2,2+n+2}(\Omega)} \leq c \| u \|_{H^{1,\lambda}(\Omega)} \quad \text{for all } u \in H^{1,\lambda}(\Omega).
\]
Now we present a fundamental result concerning the partial regularity of weak solutions to the quasilinear elliptic systems of the type

\[ D_s[A^i_j(x, u) D_s u^i] + A^i_j(x, u) D_s p^j = -D_s g^s_i + g_i. \]  

(3.1)

Assume that the coefficients \( A^i_j \) are uniformly continuous, \( A^i_j \) are continuous in \( \Omega \times \mathbb{R}^N \), \( g_i \in L^q(\Omega) \), \( q > n \) and that \( c, \mu > 0 \) constants

\[ \sum |A^i_j| + \sum |A^i_j| + \sum |g_i|_{L^q} + \sum |g_i|_{L^q} \leq c, \]

\[ A^i_j(x, u) \xi_i \xi_j \geq \mu |\xi|^2 \quad \text{for all } (x, u) \in \Omega \times \mathbb{R}^N, \quad \xi \in \mathbb{R}^n. \]

Consider the solutions to the system (3.1) belonging to the space \( H^1_{loc}(\Omega) \).

Proposition 3.4 [12: pp. 147–149]: Let \( u \) be a weak solution of the system (3.1). Suppose that \( U(x, r) \to 0 \) as \( r \to 0^+ \) uniformly in each compact set \( K \subset \Omega \). Then \( u \in C^0(Q) \) with \( \alpha = 1 - n/q \) and the a-priori estimate \( ||u||_{C^0(K)} \leq c(\mu, \epsilon, K, \text{dist}(K, \partial \Omega)) \) holds.

4. Proof of the results

Let \( \Omega_0 \subset \Omega \), \( x^0 \in \Omega_0 \) be fixed, \( R_0 = \min \{1, \text{dist}(\Omega_0, \partial \Omega)\} \). For \( R \in (0, R_0) \) and \( u \in H^{1, \infty}(\Omega) \) (\( u \) is a weak solution of the system (0.1)) we define

\[ y = y(x) = (x - x^0)/R, \]

\[ u_0(y) = \left[ (x^0 + Ry) - b^0(x^0, R) - R(b(x^0, R), y) \right]/R, \]

where \( b^0(x^0, R) = b^0(x^0, R, u) \in \mathbb{R}^N \) and \( b(x^0, R) = b(x^0, R, u) \in \mathbb{R}^n \) are the coefficients of the polynomial \( P(x, x^0, R, u) \) from (1.3) since \( u \in L^{2n+2}(B(x^0, R)) \) for each \( B(x^0, R) \subset \Omega \) due to Proposition 3.3. From (4.1) it can be seen that for each \( a > 0 \) there exists \( R(a) \in (0, R_0) \) such that for all \( R \in (0, R(a)) \) we have \( B(0, 2a \sqrt{n}) \subset \subset \Omega \) (\( \Omega_0 \) is the image of \( \Omega \) through the transformation (4.1)). From (4.2) it follows that there exists a constant \( c > 0 \) such that for each \( r > 0 \), \( y_0 \in \mathbb{R}^n \) and all \( R \in (0, R(y_0)) \) \( R(y^0) = R_0 \) in the case \( y^0 = 0 \) we have

\[ \int_{B(y^0, r)} |Du_R(y) - (Du_R)y^0, y^0|^2 \, dy \leq c |Du|_{L^{2n}(\Omega)}, \]

and the equation (1.4) has the following form:

\[ \int_{B(y^0, r)} a^4(x^0 + Ry, b^0(x^0, R) - R(b(x^0, R), y)) \psi \, dy \]

\[ = \int_{B(y^0, r)} R^4(x^0 + Ry) D_s \psi \, dy + \int_{B(y^0, r)} Rf^4(x^0 + Ry) \psi \, dy \quad \text{for all } \psi \in C^\infty_{0}(\Omega), \]

As previously said, \( u \in H^1_{loc}(\Omega) \) and with respect to (4.2) also \( u_R \in H^1_{loc}(\Omega) \). Then it follows that \( v_R = D_s u_R \) satisfies the equation in variations

\[ \int_{O_n} (\partial a^4/\partial p^4) D_s p^4 \psi^4 + \partial a^4/\partial u^4 (b_k^4 + v_R^4) + \partial a^4/\partial x^4 D_s \psi^4 \psi \, dy \]

\[ + \int_{O_n} R^2 a_4 \partial p^4 D_s p^4 \psi^4 + \partial a_4/\partial u^4 (b_k^4 + v_R^4) + \partial a_4/\partial x^4 \psi \, dy \]

\[ = \int_{O_n} (\partial f^4/\partial x^4) D_s \psi^4 + \partial f^4/\partial x^4 \psi \, dy \quad \text{for all } \psi \in C^\infty_{0}(\Omega). \]
In what follows we are going to prove that for each \( a > 0 \) the set \( \mathcal{M}_0 = \{ u_R : 0 < R < R(a) \} \) is bounded in \( H^2(B(0, a)) \) by a constant depending only on \( a \). For this reason it is enough to prove the boundedness of sets \( \mathcal{M}_1 = \{ D^2 u_R : 0 < R < R(a) \} \) and \( \mathcal{M}_2 = \{ D^2 u_R : 0 < R < R(a) \} \) in \( L^2(B(0, a)) \). The set \( \mathcal{M}_1 = \{ D u_R : 0 < R < R(a) \} \) is then bounded according to the Gagliardo-Nirenberg Theorem (see, e.g., [3; pp. 25]).

First, let us prove the boundedness of \( \mathcal{M}_1 \). For \( a > 0 \) denote \( B(a) = B(0, a) \). Further choose \( \eta \in C_0^\infty(B(2a)) \) such that \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) on \( B(a) \) and \( |D\eta| \leq c/a \).

Substituting for \( \psi \) in the equation (4.5) the function \( \psi(y) = \eta^2(v_R(y) - (v_R)_{0,2a}) \), we have for each \( \epsilon > 0 \) from the assumptions (1.7), (1.8), (1.11)—(1.13), Young's inequality, Proposition 1.1 and properties of the function \( \eta \) that

\[
v(L) \int_{B(0,2a)} \eta^2 |Dv_R|^2 \, dy 
\leq c(L) \int_{B(0,2a)} \eta^2 |Dv_R|^2 \, dy + c(\epsilon, L) a^{-2} \int_{B(0,2a)} |v_R - (v_R)_{0,2a}|^2 \, dy 
+ c(\epsilon, L) \left\{ R^2(1 + |b(x^0, R)|^2) \int_{B(0,2a)} |D^2 u_R|^2 \, dy + R^2 \int_{B(0,2a)} |D^2 u_R|^4 \, dy \right. 
+ R^2(1 + |b(x^0, R)|^2 + |b(x^0, R)|^4) a^n + R^2 \int_{B(0,2a)} |D^4 I|^2 \, dy + R^4 \int_{B(0,2a)} |D^4 I|^4 \, dy \} 
= c(L) \{ A + B + C + D + E + F \}.
\]

Estimate now the individual terms in brackets. Since \( Du \in L^{2,n}(\Omega) \), we have

\[
A = a^{-2} \int_{R(x^*,2aR)} |\partial u/\partial x_y - (\partial u/\partial x_y)_{x^*,2aR}|^2 \, dx \leq c(Du)_{x^*,2,n(\Omega)} a^{n-2}.
\]

Further from Lemma 3.1, Lemma 3.2 and the fact that \( Du \in L^{2,1}(\Omega) \) for each \( \lambda \in [0, n) \) (according to Proposition 1.1(c)) we obtain

\[
B = (1 + |b(x^0, R)|^2) R^{-n+2} \int_{R(x^*,2aR)} |Du - b(x^0, R)|^2 \, dx 
\leq c[R^{1+2-n}(1 + |b(x^0, R)|^2) a^1 + R^2(|b(x^0, R)|^2 + |b(x^0, R)|^4) a^n 
\leq c(\lambda, R_0) (1 + ln R) R^{1+2-n}(a^1 + a^n) ||u||_{H^{1,1}(\Omega)} 
\leq c(\lambda, R_0 ||u||_{H^{1,1}(\Omega)}) (a^1 + a^n),
\]

where

\[
\int_{B(0,a)} |D^2 u_R|^2 \, dy 
\leq c(a) \{ A + B + C + D + E + F \}.
\]
where \( \lambda \in (n-2, n) \) is arbitrary. In estimating the term \( C \) we use the fact that 
\[ Du \in L^{\infty} Q \] 
for each cube \( Q \subseteq \Omega, \ s \in [1, \infty), \ \mu \in [0, n) \) (see Proposition 1.1(c)) and we proceed analogously as in the estimation of term \( B \) and obtain \( C \leq c(\lambda, R_0, \|u\|_{H^{1+s}(\Omega)}) \) \( \lambda \) is arbitrary. From Lemma 3.2 it follows that \( D \leq c(R_0) a^n \) and from the assumptions (1.11), (1.12) we have \( E \leq c(R_0, \varepsilon_2) a^{n+1-2/q} \), \( F \leq c(R_0, \varepsilon_2) a^{n+1} \) in case \( q > 4 \) and \( F \leq c(R_0, \varepsilon_2) \) in case \( q \leq 4 \). From these estimates it then follows

\[
\int_{B(0,a)} |Du|_2^2 \, dy \leq c(\varepsilon_2, R_0, \text{diam } \Omega, \|u\|_{H^{1+s}(\Omega), a}) \leq c(a)
\]

for each \( R \in (0, R(a)) \). Hence \( \int_{B(0,a)} |Du|_2^2 \, dy \leq c(a) \) for any \( R \in (0, R(a)) \) and the boundedness of the set \( M_\varepsilon \) is proved.

Now we are going to prove the boundedness of \( M_0 \). From Lemma 3.2, Proposition 3.3 and (4.1), (4.2) we have

\[
\int_{B(0,a)} |u_R(y)|_2^2 \, dy = R^{-n-2} \int_{B(0,a)} |u(x) - b^0(x, R) - (b(x, R), (x - x_0))|_2^2 \, dx
\]

\[
\leq 2a^{n+2}(aR)^{-n-2} \int_{B(a,x, aR)} |u(x) - b^0(x, aR) - (b(x, aR), (x - x_0))|_2^2 \, dx
\]

\[
+ 2R^{-n-2} \int_{B(0,a)} |b^0(x, aR) - b^0(x, R)|_2 \, dx
\]

\[
+ \left( |b(x, aR) - b(x, R), (x - x_0)|_2 \right)^2 \, dx
\]

\[
\leq c(u, \varepsilon_2, a^{n+2}, (B(0, x), \text{diam } \Omega), 1 + \text{ln}^2 a) \max \{a^n, a^{n+2}\} \leq [Du]_{X_{n+1}(\Omega)} \ c(a).
\]

Hence \( \int_{B(0,a)} |u_R(y)|_2^2 \, dy \leq c(a) \) for any \( R \in (0, R(a)) \) and the boundedness of \( M_0 \) in

\[ H^2(B(0, a)) \]

is proved.

Compactness of the imbedding of \( H^2(B(0, a)) \) into \( H^1(B(0, a)) \) allows us to choose a sequence \( R_k \to 0 \) such that \( u_{R_k} \to z \) in \( H^1(B(0, a)) \). Using the diagonal process we get a subsequence (we use the same notation for it) such that

\[
\lim_{k \to \infty} u_{R_k} = z \text{ in } H^1_0(\Omega), \quad \lim_{k \to \infty} Du_{R_k} = Dz \text{ a.e. in } \mathbb{R}^n.
\]

According to (4.3) we obtain that there exists a constant \( c > 0 \) such that for each \( y \in \mathbb{R}^n, \ r > 0 \) there holds

\[
\int_{B(y, r)} |Dz(y) - (Dz)_{y, r}|_2^2 \, dy \leq c[Du]_{X_{n+1}(\Omega)} \ r^n.
\]

Further we deduce from (4.4) the equation for the limit function \( z \). For passing to

the limit in equation (4.4) the behaviour of \( \sup \{b(x, R_k), k = 1, 2, \ldots\} \) is important. Remember for the following considerations that \( Rb(x, R) \to 0, b^0(x, R) \to B^0 \) as \( R \to 0+ \) exist due to Lemma 3.2 and from the definition of \( u_R \) follows boundedness of the set \( \{u_R: R > 0\} \) by a constant independent of \( R \).

(a) Let \( \sup \{b(x, R_k): k = 1, 2, \ldots\} \) be a finite number. In this case there exists a

subsequence (we use the same notation for it) \( \{b(x, R_k)\} \) such that \( \{b(x, R_k) \} \to B \in \mathbb{R}^{n\times n} \) as \( k \to \infty \). According to (1.6), (1.12), (4.7) and the Vitali Convergence Theorem we can pass to the limit with \( k \to \infty \) in the equation (4.4) (for the fixed function \( \psi \)). We see that the second integral on the left-hand side and the integrals on the right-hand side in (4.4) tend to zero. Thus we obtain that \( B + Dz(y) \) is a weak solution of the system

\[
\int_{\mathbb{R}^n} a_s(x, B_0, B + Dz) \, D_s \psi \, dy = 0 \quad \text{for all } \psi \in H^1_0(\mathbb{R}^n).
\]
Now from the Liouville property of the system (1.4) it follows that $z$ is a polynomial of at most first degree.

(b) Let $\sup \{ |b(x^0, R_k)| : k = 1, 2, \ldots \}$ be infinite. In this case we can suppose $|b(x^0, R_k)| \to \infty$ as $k \to \infty$. Denoting in the sequel $b_k = b(x^0, R_k)$, $b_k^0 = b^0(x^0, R_k)$, $u_k(x) = u_R(x) = u_k(u_R(x) + (b(x^0, R_k), y))$ we can rewrite equation (4.4) as follows:

$$
\int_{\mathbb{R}^n} \left[ a_i^{\alpha}(x^0 + R_k y, b_k^{\alpha} + w_k(y), b_k + Du_k(y)) - a_i^{\alpha}(x^0 + R_k y, b_k^{\alpha} + w_k(y), b_k) \right. \\
+ a_i^{\alpha}(x^0 + R_k y, b_k^{\alpha} + w_k(y), b_k) - a_i^{\alpha}(x^0 + R_k y, b_k^{\alpha}, b_k) \\
+ a_i^{\alpha}(x^0 + R_k y, b_k^{\alpha}, b_k) - a_i^{\alpha}(x^0, b_k^{\alpha}, b_k) \right] D_{\alpha} \psi^i(y) dy \\
+ R_k \int a_i(x^0 + R_k y, b_k^{\alpha} + w_k(y), b_k + Du_k(y)) \psi^i(y) dy \\
= \int \int_{\mathbb{R}^n} f_i(x^0 + R_k y) D_{\alpha} \psi^i(y) dy + R_k \int f_i(x^0 + R_k y) \psi^i(y) dy \\
\text{for all } \psi \in C_c^\infty(\mathbb{R}^n).
$$

Using the theorem on the mean value in the integrals from the previous system we can rewrite this system in the following form:

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial a_i^{\alpha}/\partial x^0(x^0 + R_k y, b_k^{\alpha} + w_k(y), b_k + t Du_k(y)) D_{t} u_k^{\alpha}(y) D_{t} \psi^i(y) dt dy \\
+ R_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial a_i^{\alpha}/\partial u^i(x^0 + R_k y, b_k^{\alpha} + t w_k(y), b_k) w_k^{\alpha}(y) D_{t} \psi^i(y) dt dy \\
+ R_k \int \int_{\mathbb{R}^n} \partial a_i^{\alpha}/\partial x^0(x^0 + R_k y, b_k^{\alpha}, b_k) y D_{t} \psi^i(y) dt dy \\
+ R_k \int a_i(x^0 + R_k y, b_k^{\alpha} + w_k(y), b_k + Du_k(y)) \psi^i(y) dy \\
= \int \int_{\mathbb{R}^n} f_i(x^0 + R_k y) D_{\alpha} \psi^i(y) dy + R_k \int f_i(x^0 + R_k y) \psi^i(y) dy \\
\text{for all } \psi \in C_c^\infty(\mathbb{R}^n).
$$

Taking into account (1.7), (1.9), (1.10), (1.12), (4.7) we can pass in the previous equation to the limit with $k \to \infty$ (for the fixed function $\psi$) and we have that the second, third and fourth integral in the left-hand side and the integrals on the right-hand side tend to zero. Due to (1.10) and the assumption $|b(x^0, R_k)| \to \infty$ as $k \to \infty$, we obtain that the function $z$ satisfies the equation

$$
\int_{\mathbb{R}^n} D_z^2 \psi^i(x^0, B^0) D_{\beta}^2 D_{\alpha} \psi^i dy = 0 \\
\text{for all } \psi \in C_c^\infty(\mathbb{R}^n).
$$

It is a linear elliptic system with the same constant of ellipticity and constant coefficients and by means of (4.8) we have that $Dz \in \text{BMO}(\mathbb{R}^n)$. In this case $z$ is a polynomial at most first degree again.

Returning to the $x$-coordinates, we prove that for each $x^0 \in \Omega_0$ there exists a sequence $R_k \to 0$ such that

$$
\lim_{R_k \to 0} \frac{1}{|B(x^0, R_k)|} \int_{B(x^0, R_k)} |Du(x) - (Du)_{x^0, R_k}|^2 dx = 0.
$$

(4.9)
We have

\[
\int_{B(z_0, R)} |Du(x) - (Du)_{x_*, R_k}|^2 \, dx = \int_{B(0, t)} |Du_{R_k}(y) - (Du_{R_k})_{0, 1}|^2 \, dy \\
\leq \int_{B(0, t)} |Du_{R_k} - I|^2 \, dy \quad \text{for all} \ t \in \mathbb{R}^{n,k}.
\]

Now we put \( t = Dz \) (\( Dz \) is a constant) and, passing to the limit, we see that (4.9) holds.

Now let us consider the equation in variations for the system (1.4) in \( \Omega_0 \). If we denote by \( v \), the derivative \( D_r u \), we get as before that

\[
\int_{B(z_0, R)} (\partial_i a_i^j / \partial p_j^k D_{\beta} \phi_j^k + \partial_i a_i^j \phi_j^k) \, D_{\sigma} \phi_i^j \, dx \\
+ \int_{B(z_0, R)} (\partial_i a_i^j \phi_j^k \phi_i^k + \partial_i a_i^j \phi_i^k) \, \phi_j^k \, dx \\
= \int_{B(z_0, R)} (\partial_i a_i^j \phi_j^k \phi_i^k \phi_j^k) \, D_{\sigma} \phi_i^j \, dx \
\text{for all} \ \phi \in C_0^\infty(\Omega_0), \ \gamma = 1, \ldots, n.
\]

Set

\[
A_{ij}^k(x, \sigma) = \partial_i a_i^j / \partial p_j^k \sigma(x, u(x), \sigma), \quad A_{ij}^k(x, \sigma) = \partial_i a_i^j / \partial p_j^k \sigma(x, u(x), \sigma),
\]

\[
g_i^j(x) = -\partial_i a_i^j / \partial k(x, u(x), Duv) v_j^k(x) - \partial_i a_i^j / \partial k(x, u(x), Duv) v_j^k(x) - \partial_i a_i^j / \partial k(x, u(x), Duv) v_j^k(x).
\]

From the assumption of the theorem it follows that \( A_{ij}^k \) are uniformly continuous and bounded in \( \Omega_0 \times \mathbb{R}^{n,k}, A_{ij}^k \) are continuous and bounded in \( \Omega_0 \times \mathbb{R}^{n,k}, g_i^j \in L^2(\Omega_0) \) and \( g_i^j \in L^2(\Omega_0) \). Then the system (4.10) can be rewritten as

\[
\int_{\Omega_0} \delta \eta \left[ A_{ij}^k(x, \sigma) D_{\beta} \phi_j^k + A_{ij}^k(x, \sigma) D_{\beta} \phi_j^k \right] \, dx \\
\int_{\Omega_0} \left[ g_i^j(x) D_{\sigma} \phi_j^k + g_i^j(x) \phi_j^k \right] \, dx \
\text{for all} \ \phi \in C_0^\infty(\Omega_0).
\]

Thus \( v \) is a solution of a quasilinear system of the type (3.1) for which partial regularity (Proposition 3.4) holds ((4.9) guarantees that the assumption of Proposition 3.4 is satisfied)

\[\text{Proof of Proposition 2.2: Let } \psi \in H^1_{loc}(\mathbb{R}^2) \text{ with } Dv \in \text{BMO}(\mathbb{R}^2) \text{ be a weak solution in } \mathbb{R}^2 \text{ of}
\]

\[
\int_{\mathbb{R}^2} \partial_i a_i^j(x, u, Duv) D_{\beta} \phi_j^k \, dx = 0 \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^2).
\]

The equation in variations is

\[
\int_{\mathbb{R}^2} \partial_i a_i^j / \partial p_j^k (x, u, Duv) D_{\beta} \phi_j^k \, dx = 0 \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^2),
\]

where \( v_i = D_i v \). Now we prove that \( Dv \in L^2(\mathbb{R}^2) \). Let \( y_0 \in \mathbb{R}^2, T > 0 \) be an arbitrary constant. Setting \( \gamma_0 = \eta_0^j(x) - (v_j^k \eta_0(x)) \), \( \eta \in C_0^\infty(B(y_0, 2T)), 0 \leq \eta \leq 1, \eta = 1 \) in \( B(y_0, T) \), \( |D\eta| \leq c/T \) in equation (4.11), we get

\[
\int_{B(y_0, T)} |Dv_\gamma| \, dx \leq c \quad \text{for } \gamma = 1, \ldots, n,
\]

where \( c \) is independent of \( y_0 \) and \( T \). It is known that a sequence \( \{ \phi_k \} \subset C_0^\infty(\mathbb{R}^2) \) exists such that \( D_{\phi_k} \rightarrow Dv \) in \( L^2(\mathbb{R}^2) \) and therefore from (4.11) we have

\[
\int_{\mathbb{R}^2} \partial_i a_i^j / \partial p_j^k (x, u, Duv) D_{\beta} \phi_j^k D_{\beta} \phi_j^k \, dx = 0
\]

and together with the condition of ellipticity (1.13) gives the result.
REFERENCES


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