A Class of Nonlinear Generalized Riemann-Hilbert-Poincaré Problems for Holomorphic Functions

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By means of the theory of pseudo-monotone operators the existence of a solution of a class of nonlinear generalized Riemann-Hilbert-Poincaré problems for a holomorphic function in the unit disk is proved.

Introduction

In recent papers of the author [11, 14] existence theorems of the theory of maximal monotone operators and of Hammerstein equations in $L_p$ spaces were applied to nonlinear Riemann-Hilbert, generalized Steklov, and generalized Poincaré problems for holomorphic functions in the unit disk. In the present paper the theory of pseudo-monotone operators in the Sobolev space $W^1_2$ is utilized for proving corresponding existence theorems by a class of nonlinear generalized Riemann-Hilbert-Poincaré problems (nonlinear Vekua's problems) involving derivatives up to second order of the boundary values. Besides, by means of related regularizing approximations also some types of noncoercive problems of this kind are dealt with. In particular, some existence theorems of Landesman-Lazer's type are derived completing the theorems of such type obtained for the Riemann-Hilbert, the generalized Steklov, and the generalized Poincaré problem in [12, 14].

For classical work on nonlinear generalized Riemann-Hilbert-Poincaré problems we refer to POGORZELSKI [8] and the papers quoted in the introduction of the monograph [4] by GUSEINOV and MUKHTAROV.

1. Statement of problem

Let $G: |z| < 1$ be the unit disk of the complex $z$ plane with boundary $\Gamma: t = e^{is}$ ($-\pi \leq s \leq \pi$). We deal with the following Problem A:

To find a holomorphic function $w(z) = u(z) + iv(z)$ in $G$, which satisfies the boundary condition

$$Lu, v = L_0[u, v] + L_1[v] + L_2[u] = f \text{ on } \Gamma,$$   \hspace{1cm} (1)
where
\[ L_0[u, v] = -\varepsilon u_{ss} + \kappa v_{ss}, \quad L_1[v] = \alpha v + \beta v_s, \]
and the additional condition
\[ v(0) = 0 \quad \text{in} \quad z^2 = 0. \] (2)

Here \( \varepsilon \geq 0, \kappa, \beta \geq 0, \) and \( \alpha \) are given real constants, \( \varphi \) and \( \psi \) are given continuous functions, and \( f \in L_1(\Gamma) \) is the given right-hand side.

Because of (2) the boundary values \( \nu = \nu(e^{i\alpha}) \) of \( u(z) \) and \( u = u(e^{i\alpha}) \) of \( u(z) \) on \( \Gamma \) are connected by the well-known relation \( \nu = -Hu \) with the Hilbert operator

\[ (Hu)(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{is}) \cot \frac{s}{2} ds. \]

We remark that an inhomogeneous additional condition \( v(0) = \gamma \) can be reduced to the homogeneous condition (2) introducing the unknown function \( w(z) = \gamma i \) instead of \( v(z) \).

In particular, we are interested in the special case
\[ -u_{ss} + \lambda u_s + \mu u = f \] (3)
of (1) with constants \( \lambda > 0, \mu \geq 0, \) which may be regarded as a steady analogue of the well-known Benjamin-Ono equation of the theory of long internal gravity waves in a stratified fluid with infinite depth in the spatially periodic case.

A holomorphic function \( u(z) = u(z) + iv(z) \) in \( G \) with boundary values \( u(s) = u(e^{is}) \) and \( v(s) = -(Hu)(s) \) is said to be a generalized solution of Problem A if \( u \in W^{1,2}_{2}(\Gamma) \)
satisfies the integral relation

\[ a_0(u, \eta) + a_1(u, \eta) + a_2(u, \eta) = b(\eta) \quad \text{for} \quad \eta \in W^{1,2}_{2}(\Gamma), \] (4)

where, for \( u, \eta \in W^{1,2}_{2}(\Gamma), \)
\[ b(\eta) = \int_{\Gamma} f \eta ds, \quad a_0(u, \eta) = \varepsilon \int_{\Gamma} u' \eta' ds + \kappa \int_{\Gamma} Hu' \eta' ds, \]
\[ a_1(u, \eta) = -\alpha \int_{\Gamma} Hu \cdot \eta ds - \beta \int_{\Gamma} Hu' \cdot \eta ds, \]
\[ a_2(u, \eta) = \int_{\Gamma} \varphi(u) u' \eta ds + \int_{\Gamma} \psi(u) \eta ds. \] (5)

Here the prime denotes derivatives with respect to \( s \).

Lemma: If \( \varepsilon + \kappa^2 > 0, \) a generalized solution \( u(z) \) of Problem A has boundary values \( u, v \in W^{1,2}_{2}(\Gamma) \) and the boundary condition (1) is fulfilled a.e. on \( \Gamma \).

Proof: Let \( u \in W^{1,2}_{2}(\Gamma) \) be a solution of (4). We put
\[ U = f - \varphi(u) u' - \psi(u) + \alpha Hu + \beta Hu' \in L_0(\Gamma). \]

From (4) with \( \eta = 1 \) we have \( \int_{\Gamma} U ds = 0. \) Therefore, the boundary value problem

\[ -\varepsilon u_{ss} + \kappa v_{ss} = U \quad \text{on} \quad \Gamma \] (7)

1) \( u \in W^{1,2}_{2}(\Gamma) \) includes that \( u \) is a (continuous) \( 2\pi \)-periodic function in \( s. \) Analogously, \( u \) and \( u_s \) are continuous \( 2\pi \)-periodic functions in \( s \) if \( u \in W^{1,2}_{2}(\Gamma). \)
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Nonlinear Riemann-Hilbert-Poincaré Problems

531

has a unique solution \( w_0(z) = u_0(z) + iv_0(z) \) with boundary values \( u_0, v_0 \in W^2_r(\Gamma) \) satisfying the additional condition \( w_0(0) = 0 \). This solution may be easily constructed in closed form by trigonometric Fourier expansion, for instance. The function \( u_0 \) fulfills the identity (see (4))

\[
    a_0(u_0, \eta) = b(\eta) - a_1(u, \eta) - a_2(u, \eta) \quad \text{for} \quad \eta \in W^2_r(\Gamma).
\]

Putting \( u_1 = u - u_0 \), from (4) and (8) we have the integral relation

\[
    a_0(u_1, \eta) = \int_{\Gamma} \left( \epsilon u_1' + xH u_1' \right) \eta' \, ds = 0 \quad \text{for} \quad \eta \in W^2_r(\Gamma).
\]

Hence it follows that the function \( \epsilon u_1' + xH u_1' \) has the generalized derivative zero on \( \Gamma \) and therefore is a constant. But then \( u_1' \) must be a constant and, due to its 2\pi-periodicity, also \( u_1 \) itself. That is, \( u = u_0 + u_1 \in W^2_r(\Gamma) \). Moreover, from (4) with \( u \in W^2_r(\Gamma) \) by partial integration we obtain the integral relation

\[
    \int_{\Gamma} \left( L(u, -Hu) - f \right) \eta \, ds = 0 \quad \text{for} \quad \eta \in W^2_r(\Gamma),
\]

which implies the validity of the boundary condition (1) a.e. on \( \Gamma \)

Remark: If \( f, \phi, \psi \in C^\alpha(\Gamma), 0 < \alpha < 1 \), are Hölder continuous functions with exponent \( \alpha \), then \( u, v \in C^{2,\alpha}(\Gamma) \) have Hölder continuous derivatives of second order with the same exponent \( \alpha \).

2. Basic existence theorem

We now prove the main theorem of this paper.

Theorem 1: Under the additional assumptions \( \epsilon > 0 \) and

\[
    u \psi(u) \geq \delta u^2 - D \quad (\delta > 0, \quad D \geq 0)
\]

Problem A possesses a generalized solution for any \( f \in L^2(\Gamma) \).

Proof: Problem A is equivalent to the operator equation

\[
    Au = b \quad \text{in} \quad X = W^2_r(\Gamma),
\]

where \( A = A_0 + A_1 + A_2 \) and the operators \( A_k; X = W^2_r(\Gamma) \to X^* = W^1_r(\Gamma) \) \((k = 0, 1, 2)\) are defined by \( a_k(u, \eta) = \langle A_k u, \eta \rangle_x \) for \( u, \eta \in X \) and \( b \in X^* = W^1_r(\Gamma) \) is defined by (5).

The linear operators \( A_0 \) and \( A_1 \) are continuous since the Hilbert operator \( H \) is a continuous linear operator in \( W^2_r(\Gamma) \). Besides, because of

\[
    a_0(u, u) = \epsilon \int_{\Gamma} u^2 \, ds \geq 0, \quad a_1(u, u) = \beta \int_{\Gamma} \frac{\partial u}{\partial r} \, u \, ds \geq 0,
\]

where \( \partial u/\partial r \) means the derivative of \( u \) in the direction of the polar radius \( r \), both operators \( A_0 \) and \( A_1 \) and therefore their sum \( A_1 = A_0 + A_1 \) are monotone.

The operator \( A_2 \) is completely continuous in the sense that it maps weakly convergent sequences into strongly convergent ones. Assuming \( u_n \rightharpoonup u \) in \( X \), we have \( \|u_n\|_X \leq K \) and, due to the compact embedding of \( X = W^2_r(\Gamma) \) in \( C(\Gamma) \), also \( u_n \rightharpoonup u \) in \( C(\Gamma) \). We have to show that

\[
    \|A_2 u_n - A_2 u\|_{X^*} = \sup_{\|\eta\|_X \leq 1} |\langle A_2 u_n - A_2 u, \eta \rangle| \to 0 \quad \text{as} \quad n \to \infty.
\]
By partial integration from (6) it follows that
\[ a_2(u, \eta) = \int \psi(u) \eta \, ds - \int \Phi(u) \eta' \, ds, \quad \Phi \text{ a primitive of } \varphi. \]

Since \( \psi(u_n) \to \psi(u) \) and \( \Phi(u_n) \to \Phi(u) \) in \( C(\Gamma) \), we have
\[
\sup_{\|\eta\|_{\infty} \leq 1} |A_2 u_n - A_2 u, \eta| \\
\leq \sup_{\|\eta\|_{\infty} \leq 1} \int |\psi(u_n) - \psi(u)| |\eta(s)| \, ds + \sup_{\|\eta\|_{\infty} \leq 1} \int |\Phi(u_n) - \Phi(u)| |\eta'(s)| \, ds \\
\leq \left( \int \left| \psi(u_n) - \psi(u) \right|^2 \, ds \right)^{1/2} + \left( \int \left| \Phi(u_n) - \Phi(u) \right|^2 \, ds \right)^{1/2} \to 0.
\]

This proves (12).

As the sum \( A = A_1 + A_2 \) of the continuous monotone operator \( A_1 \) and the completely continuous one \( A_2 \) the operator \( A \) is pseudomonotone. Since \( A_1 \) as a continuous linear operator and \( A_2 \) as a completely continuous one are bounded operators, so \( A \) is bounded, too. Finally, owing to the assumption (9) there is
\[ a_2(u, u) = \int w_2(u) \, ds \geq \delta \int w^2 \, ds - 2\pi D \]
and by (11)
\[ a_0(u, u) + a_1(u, u) \leq \epsilon \int w^2 \, ds. \]

Therefore, we have \( \langle Au, u \rangle_X \geq \min (\epsilon, \delta) \| u \|_X^2 - 2\pi D \), and because of the assumption \( \epsilon > 0 \) (and \( \delta > 0 \)) the operator \( A \) is coercitive.

The main theorem of the theory of pseudo-monotone operators by Brézis (cf. [16: Theorem 27.2]) now yields the existence of a solution \( u \) of the operator equation (10).

Remark: Since the operator \( A \) also satisfies the condition \( (S_\nu \otimes \nu) \) and hence \( S_\nu \), the solution \( u \) of (10) is strong limit of a subsequence of solutions of the Galerkin equations of (10) with respect to an arbitrary basis in \( X = W_2^1(\Gamma) \) (cf. [16: Theorem 27.1]). Further, we remark that the question of uniqueness of the solution is an open problem. Of course, the solution is unique in the particular case: \( \varphi = 0, \varphi \) a strictly increasing function.

### 3. Non-coercive problems

We now deal with the case \( \epsilon = 0 \), where the main term is \( L_0[v] = xv_{ss} \) in (1). The problem with the corresponding boundary condition
\[ L_0[v] + L_1[v] + L_2[u] = f \quad \text{on } \Gamma \]
and the additional condition (2) is named Problem B.

Theorem 2: Under the additional assumptions \( x \geq 0 \) \( (x \leq 0) \),
\[ \varphi(u) \geq v > 0 \quad \left( \varphi(u) \leq -v < 0 \right), \]
and (9) Problem B possesses a generalized solution for any \( f \in L_2(\Gamma) \).

Proof: We restrict ourselves to the case \( x \geq 0 \). In view of Theorem 1 the perturbed problem with the boundary condition
\[ -\epsilon u_{ss} + x w_{ss} + \alpha v + \beta w_s + \varphi(u) u_s + \psi(u) = f \quad \text{on } \Gamma \]
and the additional condition (2) possesses a generalized solution \( u_\epsilon(z) \) with \( u_\epsilon \in W_2^1(\Gamma) \) for any \( \epsilon > 0 \). By definition \( u_\epsilon \) satisfies the integral relation (4) and by the Lemma, \( u_\epsilon, v_\epsilon \in W_2^2(\Gamma) \) fulfil the boundary condition (17) a.e. on \( \Gamma \).
We show that the norms of $u_\varepsilon$ in $X = W^1_2(\Gamma)$ are bounded uniformly in $\varepsilon$. In view of (13), (14) from (4) for $\eta = u_\varepsilon$ we obtain the inequality
\[ \varepsilon \int u_\varepsilon'^2 \, ds + \delta \int u_\varepsilon^2 \, ds \leq \int f u_\varepsilon \, ds + 2\pi D. \]
Therefore we have $\delta \| u_\varepsilon \|^2 \leq \|f\|_2 \| u_\varepsilon \|_2 + 2\pi D^2$ which implies the norm boundedness in $L^2_2(\Gamma)$
\[ \| u_\varepsilon \|_2 \leq K_0 \quad \text{uniformly in } \varepsilon > 0. \]
(18)
Further, multiplying (17) for $u = u_\varepsilon$, $v = u_\varepsilon$ by $u_\varepsilon'$ and integrating over $\Gamma$ yields the relation
\[ \alpha \int u_\varepsilon' v'' \, ds + \alpha \int u_\varepsilon' v' \, ds + \int \varphi(u_\varepsilon) u_\varepsilon'' \, ds = \int f u_\varepsilon' \, ds. \]
Now there holds
\[ \int u_\varepsilon' v'' \, ds = \int u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} \, ds \geq 0 \quad \text{for } u \in W^2_2(\Gamma), \]
so that $\int u_\varepsilon' v'' \, ds \geq 0$ and, by (16), $\int \varphi(u_\varepsilon) u_\varepsilon'' \, ds \geq \nu \int u_\varepsilon'' \, ds$. Besides, $\|v\|_2 \leq \|v\|_2 \leq K_0$ by (18). Hence we infer the inequality $\nu \| u_\varepsilon \|_2^2 \leq \|f\|_2 + |\alpha| K_0 \| u_\varepsilon \|_2$ which implies the norm boundedness in $L^2_2(\Gamma)$
\[ \| u_\varepsilon \|_2 \leq K_1 \quad \text{uniformly in } \varepsilon > 0. \]
(19)
Finally, from (18) and (19) we have the estimate
\[ \| u_\varepsilon \|_2 \leq K' \quad \text{uniformly in } \varepsilon > 0. \]
(20)
Let $\varepsilon_n \to 0$ and put $u_n = u_{\varepsilon_n}$. Owing to (20) there exists a subsequence $\{u_{\varepsilon_n}\}$ of $\{u_\varepsilon\}$ converging weakly in $X = W^1_2(\Gamma)$ and therefore uniformly to a function $u \in W^1_2(\Gamma)$. Then also the functions $H u_{\varepsilon_n}$ converge weakly in $W^1_2(\Gamma)$ to $H u \in W^1_2(\Gamma)$. Performing the limit $\varepsilon_n \to 0$ in (4), we obtain the identity
\[ \int u_\varepsilon' \cdot \eta' \, ds + a_1(u, \eta) + a_2(u, \eta) = b(\eta) \quad (\eta \in W^1_2(\Gamma)) \]
for $u$, i.e., $u \in W^1_2(\Gamma)$ is generalized solution of Problem B.

Remark: If $\alpha \equiv 0$, by the Lemma $u \in W^2_2(\Gamma)$.

Further we drop the assumption (16) on $\varphi$ and prove

**Theorem 3:** Under the additional assumptions $\alpha \equiv 0$,
\[ |\varphi(u)| \leq E_1 |u|^\rho + D_1 \quad (0 < \rho < 2; E_1 \geq 0, D_1 \geq 0), \]
(21)
the assumption (9), and
\[ |\varphi(u)| \leq E_2 |u|^\sigma + D_2 \quad (0 < \sigma < 5; E_2 \geq 0, D_2 \geq 0) \]
(22)
Problem B possesses a generalized solution for any $f \in L^2_2(\Gamma)$.

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2) $\| \cdot \|_p$ denotes the norm in $L^p_2(\Gamma)$, $p > 1$.

35. Analysis Bd. 6, Heft 6 (1987)
Proof: We again consider the perturbed problem with the boundary condition (17) and (2). Due to the assumption (9) the estimate (18) holds again. Multiplying (17) for $u = u$, $v = v$, by $v$, and integrating over $I$ further yields the relation

$$-\alpha \int_I v_2^2 ds + \alpha \int_I v^2 ds + \int_I \phi(u) v u_v' ds + \int_I \psi(u) v ds = \int_\Gamma v ds.$$  

Now we have $\|v\|_2 \leq \|u\|_2 \leq K_0$ by (18) again,

$$\int_I \phi(u) v ds \leq E_2 \int_I |u|^p |v| ds + D_0 \int_I |v| ds \leq E_2 K_0 \|u\|_{\infty, I}^p + D_2 \sqrt{2\pi} K_0$$

by (22), and

$$\int_I \phi(u) v u_v' ds \leq E_1 \int_I |u|^p |u_v| |u_v'| ds + D_1 \int_I |v| |u_v'| ds \leq E_1 \|u\|_p^p \|v\|_2 \|u_v\|_2 + D_1 K_0 \|u_v\|_2$$

by (21). Furthermore, by Hölder's inequality

$$\|v\|_p^p \leq \|v\|_{2(p-1)}^p \leq A_p \|v\|_{2(p-1)}^p \leq A_{2(p-1)} \|v\|_{2(p-1)}^p$$

with $A_p$ the norm of the Hilbert operator in $L_\infty(\Gamma)$, $p > 1$. Therefore, there holds the estimation

$$|\alpha| \|u\|_2^2 = |\alpha| \|u_v\|_2^2 \leq K_0 \|\phi\|_2 + |\alpha| K_0^2 + D_2 \sqrt{2\pi} K_0 + E_2 K_0 \|u\|_{\infty, I}^p$$

by (22), and

$$\int_I \phi(u) v u_v' ds \leq E_1 \int_I |u|^p |u_v| |u_v'| ds + D_1 \int_I |v| |u_v'| ds \leq E_1 \|u\|_p^p \|v\|_2 \|u_v\|_2 + D_1 K_0 \|u_v\|_2$$

by (21). Furthermore, by Hölder's inequality

$$\|v\|_p^p \leq \|v\|_{2(p-1)}^p \leq A_p \|v\|_{2(p-1)}^p \leq A_{2(p-1)} \|v\|_{2(p-1)}^p$$

with $A_p$ the norm of the Hilbert operator in $L_\infty(\Gamma)$, $p > 1$. Therefore, there holds the estimation

$$|\alpha| \|u\|_2^2 = |\alpha| \|u_v\|_2^2 \leq K_0 \|\phi\|_2 + |\alpha| K_0^2 + D_2 \sqrt{2\pi} K_0 + E_2 K_0 \|u\|_{\infty, I}^p$$

by (22), and

$$\int_I \phi(u) v u_v' ds \leq E_1 \int_I |u|^p |u_v| |u_v'| ds + D_1 \int_I |v| |u_v'| ds \leq E_1 \|u\|_p^p \|v\|_2 \|u_v\|_2 + D_1 K_0 \|u_v\|_2$$

Finally, for any $U \in W_2(I)$ satisfying the relation $\int_I U ds = 0$ the well-known interpolation inequality

$$\|U\|_p \leq C \|U\|_p^\gamma \|U\|_2^{1-\gamma} \quad (C > 0)$$

where $p \geq 2$ and $\gamma = 1/2 - 1/p$, is valid (cf. [7: Chap. II, Th. 2.2]). Taking $U = u - \frac{1}{2\pi} \int u ds$, from (24) for an arbitrary function $u \in W_2(I)$ we obtain the inequality

$$\|u\|_p \leq c \|u_v\|_p^p \|u\|_2^{1-\gamma} + d \|u\|_2$$

with uniform positive constants $c$, $d$, depending only on $p$. Without loss of generality we suppose $\sigma \geq 1$ in (22) and choose $p = 2\sigma$ and $p = 2(\sigma + 1)$, respectively, in (25). Then we have

$$\|u\|_{2\sigma} \leq c_1 K_0^{1-\gamma_1} \|u_v\|_2^{\gamma_1} + d_1 K_0$$

with $\gamma_1 = 1/2 - 1/\sigma$ so that $\sigma \gamma_1 = (\sigma - 1)/2 < 2$ since $\sigma < 5$, and

$$\|u\|_{2(\sigma+1)} \leq c_2 K_0^{1-\gamma_2} \|u_v\|_2^{\gamma_2} + d_2 K_0$$

with $\gamma_2 = 1/2 - 1/[2(\sigma + 1)]$ so that $(\sigma + 1) \gamma_2 = \sigma/2 < 1$ since $\sigma < 2$.

Hence, on account of $\alpha = 0$, from (23), (26), (27) the uniform boundedness of the norms of $u_v$ in $L_\infty(I)$ follows, i.e., we again have the estimates (19) and (20). The rest of the proof is the same as in the proof of Theorem 2.

Example 1: The problem (3) with constants $\lambda > 0$ and $\mu > 0$, i.e., $\alpha = -1$, $\alpha = \beta = 0$, $\phi(u) = \lambda u$, $\psi(u) = \mu u$ fulfills the assumptions of Theorem 3.
4. Problems of Landesman-Lazer's type

Finally, we consider some problems without assuming the condition (9) for \( \varphi \). Firstly we make some brief remarks on the case \( \varphi = 0 \) which will be also dealt with below as a limit case of a Landesman-Lazer's type problem. In this case the condition

\[
\int_{\Gamma} f \, ds = 0
\]

is obviously necessary for the existence of a solution. If additionally \( \alpha = 0 \), Problem A can be reduced to the problem with the integrated boundary condition

\[
\lambda \frac{\partial u}{\partial r} - \varepsilon \frac{\partial u}{\partial s} + \beta v + \Phi(u) = \frac{\partial}{\partial s} \left( \frac{\partial}{\partial s} u \right) + F + C
\]

and (2), where \( C \) is a free constant,

\[
F(s) = \int_{\sigma} f(\sigma) \, d\sigma, \quad \Phi(u) = \int_{0}^{u} \varphi(\varphi) \, d\varphi.
\]

The problem described by the conditions (29) and (2) is a nonlinear generalized Poincaré problem and has been treated in the literature. We refer to the papers [14] for the general case, [1, 7, 9, 10, 13, 15] for the case \( \beta = 0 \), [3, 5, 6] for the case \( \varepsilon = \beta = 0 \), [11; 12] for the case \( \varepsilon = 0 \). See also [1, 10, 11, 13] for further references. Here we only consider two examples of this problem for illustration.

Example 2: The problem (3) with constants \( \lambda > 0 \) and \( \mu = 0 \) leads to the nonlinear Steklov problem

\[
\frac{\partial u}{\partial r} = \frac{\lambda}{2} u^2 - F - C \quad \text{on } \Gamma,
\]

which by [1: Example (2.6)] has a classical solution \( u \in C^2(G) \cap C^1(\overline{G}) \) for any Lipschitz continuous function \( F \) and constant \( C > -\min \{F(s): s \in \Gamma\} \). That means, the problem (3) with \( \lambda > 0 \) and \( \mu = 0 \) possesses a continuum of such solutions with bounded second derivative \( u'' \) of the boundary values \( u \) for any \( f \in L_2(\Gamma) \) which fulfills (28).

Example 3: By [12, 14] (cf. also the Remark to Theorem 4 below) the problem (29) with (2), where \( \alpha > 0 \) or \( \alpha = 0 \) with \( (\varepsilon \geq 0, \beta \geq 0 \text{ and } \varepsilon + \beta > 0) \), respectively, has a (suitably defined generalized) solution \( u \in L_2(\Gamma) \) if

\[
w \Phi(u) \geq -c \, |u| - d \quad (c \geq 0, d \geq 0)
\]

and

\[
\Phi_+ < \frac{1}{2\pi} \int_{\Gamma} \left[ F(s) + C \right] \, ds < \Phi_+,
\]

i.e. for any constant \( C \) with

\[
\Phi_+ < C + \frac{1}{2\pi} \int_{\Gamma} F(s) \, ds < \Phi_+, \quad \Phi_+ = \lim_{u \to +\infty} \inf \Phi(u), \quad \Phi_- = \lim_{u \to -\infty} \sup \Phi(u).
\]

If, additionally,

\[
|\Phi(u)| \leq a \, |u| + b \quad (a \geq 0, b \geq 0),
\]

then also \( u' \in L_2(\Gamma) \) and \( u \in W_2(\Gamma) \) is a generalized solution in the sense of point 1 above.

Therefore, under the assumptions (31) and (32) and \( \Phi_- < \Phi_+ \) the Problem A with \( \alpha > 0 \) or \( \alpha = 0, \varepsilon + \beta > 0 \) possesses a continuum of such generalized solutions \( u \in L_2(\Gamma) (u \in W_2(\Gamma)) \) for any \( f \in L_2(\Gamma) \) which satisfies (28). The assumptions (31) and \( \Phi_- < \Phi_+ \) are especially ful-
filled if \( \Phi \) is a non-constant monotone increasing function. I.e., under the above-mentioned restrictions on the parameters \( \varkappa, \epsilon, \beta \), Problem A has solutions if the condition \( \varphi(u) \geq 0 \) is satisfied (for \( \varphi(u) = 0 \), obviously, a solution \( u \) exists which is determined apart from an arbitrary additive constant).

In case \( \varphi \neq 0 \) there holds the following theorem of Landesman-Lazer's type for Problems A and B.

**Theorem 4:** Under the additional assumptions \( \varkappa \geq 0, \alpha \leq 0 (\varkappa \leq 0, \alpha \geq 0) \), (16), and

\[
\varphi(u) \geq -\delta_0 |u| - D_0 \quad (\delta_0 \geq 0, D_0 \geq 0)
\]

Problem A possesses a generalized solution for each \( f \in L_2(\Gamma) \) satisfying the inequality

\[
\psi_- < \frac{1}{2\pi} \int_\Gamma f(s) ds < \psi_+ \quad (34)
\]

where

\[
\psi_+ = \lim \inf_{u \to +\infty} \psi(u), \quad \psi_- = \lim \sup_{u \to -\infty} \psi(u).
\]

We remark that the condition (33) implies that \( -\infty \leq \psi_- \leq \delta_0, -\delta_0 \leq \psi_+ \leq +\infty \). Of course, for (34) to hold it is to assume that \( \psi_- < \psi_+ \).

**Proof:** We consider the perturbed problem with the boundary condition

\[
-\nu u_{6s} + \kappa v_{6z} + \alpha v + \beta v_0 + \varphi(u) u_0 + \delta u + \varphi(u) = f \quad \text{on} \ \Gamma.
\]

and the additional condition (2). By Theorems 1, 2 and the Lemma this problem has a generalized solution \( u_6(z) \) with \( u_6 \in W^2_2(\Gamma) \) for any \( \delta > 0 \). We again have to prove that the norms of \( u_6 \) in \( X = W^{1}_2(\Gamma) \) are uniformly bounded.

Multiplying (35) for \( u = u_6, v = v_0 \) by \( u_6' \) and integrating over \( \Gamma \) yields the relation

\[
\kappa \int_\Gamma u_6 v_0'' ds + \alpha \int_\Gamma u_6 v_0' ds + \int_\Gamma \varphi(u_6) u_0'^2 ds = \int_\Gamma f u_6' ds.
\]

Now there hold the inequalities

\[
\int_\Gamma u_6' v_0'' ds \geq 0 \quad \text{and} \quad \int_\Gamma u_6' v_0 ds \leq 0.
\]

On account of the assumption (16) we therefore have

\[
\nu \int_\Gamma u_6'^2 ds \leq \frac{1}{\kappa} \int_\Gamma f u_6' ds
\]

in cases \( \kappa \geq 0, \alpha \leq 0 \) and \( \kappa \leq 0, \alpha \geq 0 \), respectively. I.e., in both cases \( \nu \|u_6'\|_2^2 \leq \|f\|_2 \|u_0'\|_2 \). This implies the uniform boundedness of the norms of \( u_6' \) in \( L_2(\Gamma) \). It remains to show that also the norms of \( u_6 \) themselves in \( L_2(\Gamma) \) are uniformly bounded.

We decompose \( u_6 = C_6 + U_6 \), where \( C_6 \) are constants and \( \int_\Gamma U_6 ds = 0 \). Since \( U_6' = u_6' \) and the norms of \( u_6' \) in \( L_2(\Gamma) \) are uniformly bounded, the functions \( U_6 \) themselves are uniformly bounded:

\[
|U_6(s)| \leq L. \quad (36)
\]

We have to prove that also the constants \( C_6 \) are uniformly bounded. If this were not the case, there exists a sequence \( \{C_{6,\delta}\} \) going to \( +\infty \) or \( -\infty \) as \( \delta_0 \to 0 \). From (35)
we have the relation

\[ 2n\delta_0 C_{\delta_0} + \int f(C_{\delta_0} + U_{\delta_0}) \, ds = \int f \, ds. \]  

(37)

If now \( C_{\delta_0} \to \infty \) as \( \delta_0 \to 0 \), we apply Fatou's lemma to (37) taking into account that by (33) \( \psi(u) \leq -\mu_0 \) with a (positive) constant \( \mu_0 \) for sufficiently large \( u \), say \( u \geq \gamma_0 \), and by (36) \( C_{\delta_0} + U_{\delta_0} \geq \gamma_0 \) for sufficiently large \( u \). Hence we obtain the inequality

\[ \int f \, ds \geq \liminf_{n \to \infty} \int f(C_{\delta_0} + U_{\delta_0}) \, ds \geq \int \liminf_{n \to \infty} (C_{\delta_0} + U_{\delta_0}) \, ds \geq 2\pi \psi, \]

which is a contradiction to the right-hand side of (34). In the same way, the assumption \( C_{\delta_0} \to -\infty \) as \( \delta_0 \to 0 \) leads to a contradiction to the left-hand side of (34).

**Corollary:** If \( \psi \leq \psi(u) \leq \psi_+ \) for all \( u \in \mathbb{R} \), in particular, for a monotone non-decreasing function \( \psi \), the condition (34) with \( \leq \) instead of \( < \) is obviously necessary for the solvability of Problem A because of the relation

\[ \int \psi(u) \, ds = \int f \, ds \]

following from (4) with \( \eta = 1 \). In the limit case \( \psi = 0 \) the above proof also goes through with the assumption (34) replaced by (28) since for \( \psi = 0 \) from (37) and (28) it follows that all constants \( C_{\delta_0} \) vanish. The condition (28) is therefore necessary and sufficient in this case.

**Remark:** In the particular case \( \epsilon = \pi = \beta = 0 \), and \( \varphi = 1 \) \( (\varphi = -1) \) the existence assertion of Theorem 4 also holds true under the more general conditions that \( \alpha \in \{1, 2, \ldots\} \) \( (\alpha \in \{-1, -2, \ldots\}) \) and the Carathéodory function \( \psi = \psi(u, s) \) satisfies the assumption (33) with non-negative functions \( \delta_0 \in L_2(\Gamma); D_0 \in L_1(\Gamma) \) and the assumption

\[ \sup_{|u| \leq R} |\psi(u, s)| \in L_1(\Gamma) \quad \text{for any } R > 0. \]

This follows from Remark 111.3 to Theorem 111.6 of [2] like in the corresponding proof in [12], but there only the Theorem 111.6 of [2] itself has been used. The solution \( u \) lies in \( L_2(\Gamma) \) with \( \pm u' + \alpha v \in L_1(\Gamma) \). Finally, the same statement is true also in the case \( \epsilon = \pi = 0, \beta + 0, \alpha \) arbitrary and \( \varphi = \pm 1 \), where \( u \in L_2(\Gamma) \) with \( \beta v' \pm u' + \alpha v \in L_1(\Gamma) \) (cf. [14]).

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