On the Stability Property for a General Form of Variational Inequalities

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Stability criteria for a general form of variational inequalities in reflexive Banach spaces are established under assumptions on the monotonicity, concavity, continuity and boundedness of the parameter-dependent problem. Some special cases are considered.

1. Introduction

For a parameter-dependent problem \( (P_t) \) it is natural to raise the problem: Assume that Problem \( (P_{t_0}) \) admits a solution, when does a neighbourhood \( U \) of \( t_0 \) then exist such that for each \( t \in U \) Problem \( (P_t) \) also admits a solution? What information about the solution set of Problem \( (P_t) \), \( t \in U \), can be obtained?

In the case where \( (P_t) \) is an optimization problem, the above problem (stability problem) was investigated by many authors, e.g. Kirsch [7], GoLLAN [5], BANK et al. [1]. For parametric optimization problems there was furthermore a lot of researches devoted to the extremal value function, e.g. GAUVIN and Tolle [4], LEMPIO and MAURER [9], EKELAND and TEMAM [3], LEVITIN [10]. In the case of generalized equations, the stability problem and related problems were treated by ROBINSON [13], HOANG TUY [6], KUMMER [8]. A survey of parameter-dependent problems can be found in BANK et al. [1].

In this paper we are concerned with the qualitative stability problem in the case of a general form of variational inequalities. Specifically, the Problem \( (P_t) \) is here the following:

\[
\begin{cases}
\text{Find } x \in C \text{ such that} \\
f(x, y, t) \leq 0 \text{ for all } y \in C; t \in T
\end{cases}
\]

\( (P_t) \)

where \( C \) is a closed convex subset of a reflexive Banach space \( X \); \( T \) is a metric space and \( f : C \times C \times T \to \mathbb{R} \) is a function with certain properties of monotonicity, concavity and continuity (\( \mathbb{R} \) is the set of all real numbers).

Throughout this paper, we denote by \( X \) a reflexive Banach space, by \( X^* \) the dual space of \( X \), by \( C \subset X \) a closed convex set, by \( T \) a metric space, by \( t_0 \in T \) an accumulation point and by \( f \) a real-valued function on \( C \times C \times T \). The compact-
ness, closure, openness of a set in $X$ and the continuity of a real-valued function on $C$ are understood in the sense of the weak topology. The continuity of a real-valued function on $C \times T$ is understood in the sense of the weak topology on $X$ and the metric topology on $T$.

2. Definitions and main results

In this section, some definitions used for the investigation below and stability criterions established for the parameter-dependent problem

$$\begin{cases} x \in C \\ f(x, y, t) \leq 0 \text{ for all } y \in C \end{cases} \quad (P_t)$$

will be given. The function $g: C \times C \to \mathbb{R}$ is said to be monotone if $g(x, x) \leq 0$ and $g(x, y) + g(y, x) \geq 0$ for all $x, y \in C$. $g$ is said to be hemicontinuous if for arbitrary given $x, y \in C$ the function $g(x + \lambda(y - x), y)$ of the real variable $\lambda \in [0, 1]$ is lower semicontinuous (Mosco [12]). The set-valued mapping $\Gamma: T \to 2^C$ is said to be upper semicontinuous at $t_0 \in T$ if for each open set $\Omega \supset \Gamma(t_0)$ there is a neighbourhood $V$ of $t_0$ such that $\Omega \supset \Gamma(t)$ for all $t \in V$ (Berge [2]). The solution set mapping $S: T \to 2^C$ is defined by

$$S(t) = \{ x \in C : f(x, y, t) \leq 0 \text{ for all } y \in C \}.$$  

Problem $(P_t)$ is said to be stable at $t_0$ if there is a neighbourhood $U$ of $t_0$ such that $S(t) \neq \emptyset$ for each $t \in U$ and the mapping $S: U \to 2^C$ is upper semicontinuous at $t_0$.

We introduce now the following assumptions (the topology considered on $X$ is the weak topology, see Introduction).

Assumption 2.1: For each $t \in T$, $f(\cdot, \cdot, t)$ is a monotone and hemicontinuous function; for each $x \in C$, $f(x, \cdot, \cdot)$ is an upper semicontinuous function; for each $(x, t) \in C \times T$, $f(x, \cdot, t)$ is a concave function.

Assumption 2.2: There is a point $y_0 \in C$ such that the image set $N(t_0)$ of the mapping $N: T \to 2^C$ defined by

$$N(t) = \{ x \in C : f(y_0, x, t) \geq 0 \}$$

is bounded.

In order to formulate and prove the main stability theorem, we need some preliminary considerations.

Theorem 2.1: Under Assumptions 2.1 and 2.2 the following conditions are equivalent:

(i) There is a neighbourhood $U$ of $t_0$ such that $N(U)$ is bounded in $X$.

(ii) $N$ is upper semicontinuous at $t_0$.

(iii) There is an open set $\Omega \supset N(t_0)$ and a neighbourhood $V$ of $t_0$ such that $\Omega \cap N(V)$ is bounded in $X$.

The following lemmas are used for the proof of this theorem.

Lemma 2.1: Let Assumption 2.1 and condition (iii) be satisfied. Then for any sequence $\{t_k\} \subset T$, $t_k \to t_0$, the sequence $\{x_k\} \subset X$, $x_k \in N(t_k) \setminus N(t_0)$, has an accumulation point contained in $N(t_0)$.
Lemma 2.2: Let \( \Gamma : T \to 2^X \) be an upper semicontinuous mapping at \( \ell^0 \) with the closed and bounded image set \( \Gamma(\ell^0) \). Then there is a neighbourhood \( V \) of \( \ell^0 \) such that \( \Gamma(V) \) is bounded in \( X \).

Lemma 2.3 (BANK et al. [1: Lemma 2.2.2]): Let \( \Gamma : T \to 2^X \) be a mapping with the closed image set \( \Gamma(\ell^0) \). Then \( \Gamma \) is upper semicontinuous at \( \ell^0 \) if and only if for any sequence \( \{t_k\} \subset T \), \( t_k \to \ell^0 \), the sequence \( \{x_k\} \subset X \), \( x_k \in \Gamma(t_k) \setminus \Gamma(\ell^0) \), has an accumulation point contained in \( \Gamma(\ell^0) \).

Proof of Lemma 2.1: First we show that \( \{x_k\} \) is bounded. Assume the contrary. Then there is a subsequence \( \{x_{k'}\} \) with \( \|x_{k'}\| \to \infty \). From \( \{x_{k'}\} \) we now construct a bounded sequence \( \{x_{k''}\} \),

\[
x_{k''} = \frac{d}{\|x_{k'}\|} x_{k'} + \left(1 - \frac{d}{\|x_{k'}\|}\right) y_0
\]

(1.1)

where \( d \) is an arbitrary fixed number with \( \|y_0\| < d \leq \|x_{k'}\| \) (\( y_0 \) is the point given in the assumption). Such a construction of \( \{x_{k''}\} \) is always feasible since \( \|x_{k'}\| \to \infty \). It is easy to check that

\[
d - \|y_0\| \leq \|x_{k''}\| \leq d + \|y_0\|.
\]

(1.2)

Since \( t_{k'} \to \ell^0 \), we have \( t_{k'} \in V \) for all \( k' \geq k_0 \) enough large (\( V \) is the neighbourhood given in (iii)). By \( \{t_n\} \) we denote the sequence of \( \{t_{k'}\} \) with \( k' \geq k_0 \). So we get \( \{t_n\} \subset V \) and \( t_n \to \ell^0 \). Since by the assumption \( \Omega \cap N(V) \) is bounded, from (1.2) we can assume that \( \{x_{n(k')}\} \subset \Omega \) for \( d_0 \) enough large. By (1.2) \( \{x_{n(k')}\} \) has a convergent subsequence \( \{x_{n(k''')}\} \). Let \( x_{n(k''')} \to \bar{x} \). Since \( \Omega \) is open and \( x_{n(k''')} \in \Omega \) for all \( n', \bar{x} \notin \Omega \). Hence, \( \Omega \cap N(V) \subset \Omega \) follows \( \bar{x} \notin N(V) \).

On the other hand \( f(y_0, y_0, \ell^0) = 0 \) (by the monotonicity of \( f \)). So we can write \( f(y_0, y_0, \ell^0) > -\epsilon \) for each \( \epsilon > 0 \). Since the function \( f(y_0, y_0, \cdot) \) is continuous at \( \ell^0 \), there is an index \( k'(\epsilon) \) of the index set of the sequence \( \{t_{k'}\} \) such that \( f(y_0, y_0, t_{k'}) > -\epsilon \) for all \( k' \geq k'(\epsilon) \). By \( x_{k'} \in N(t_{k'}) \) we have \( f(y_0, x_{k'}, t_{k'}) \geq 0 \). From (1.1), the last two inequalities and the concavity of \( f \) it follows \( f(y_0, x_{n(\cdot)}, t_{k'}) > -\epsilon \). The upper semicontinuity of \( f \) then implies \( f(y_0, \bar{x}, \ell^0) \geq \epsilon \) and since \( \epsilon > 0 \) is arbitrary we get \( f(y_0, \bar{x}, \ell^0) \geq 0 \). By the assumption that means \( \bar{x} \notin N(\ell^0) \). This contradicts \( \bar{x} \in N(\ell^0) \).

Hence, the sequence \( \{x_{k'}\} \) is bounded.

Because of the boundedness \( \{x_{k'}\} \) has a convergent subsequence \( \{x_{k''}\} \). Let \( x_{k''} \to \bar{x} \). Since \( x_{k''} \in N(t_{k''}) \) we have \( f(y_0, x_{k''}, t_{k''}) \geq 0 \). By the upper semicontinuity of \( f \) then follows \( f(y_0, \bar{x}, \ell^0) \geq 0 \), i.e. \( \bar{x} \in N(\ell^0) \).

Proof of Lemma 2.2: Assume the contrary: that for all neighbourhoods \( V \) of \( \ell^0 \), \( \Gamma(V) \) is unbounded. Since \( \Gamma(\ell^0) \) is bounded, we can then construct sequences \( \{t_k\} \) and \( \{x_k\} \) as follows:

Let \( V_1 = B(\ell^0, r) \subset T \) be the ball with center \( \ell^0 \) and radius \( r \). Since \( \Gamma(V_1) \) is unbounded, we can take \( x_1 \in \Gamma(V_1) \setminus \Gamma(\ell^0) \). There is then a point \( t_1 \in V_1 \) with \( x_1 \in \Gamma(t_1) \). So, we have

\[
t_1 \in V_1, \quad x_1 \in \Gamma(t_1) \setminus \Gamma(\ell^0).
\]

Let \( V_2 = \{t \in T : 2d(t, \ell^0) \leq d(t_1, \ell^0)\} \) (here \( d(\cdot, \cdot) \) denotes the distance function in the metric space \( T \)). Since \( \Gamma(V_2) \) is unbounded, we can take \( x_2 \in \Gamma(V_2) \setminus (\Gamma(\ell^0) \cup \{x \in C : \|x\| \leq 2 \|x_1\|\}) \). There is then a point \( t_2 \in V_2 \) with \( x_2 \in \Gamma(t_2) \). So, we have

\[
t_2 \in \{t \in T : 2d(t, \ell^0) \leq d(t_1, \ell^0)\},
\]

\[
x_2 \in \Gamma(t_2) \setminus (\Gamma(\ell^0) \cup \{x \in C : \|x\| \leq 2 \|x_1\|\}).
\]
Continuing this process, we then obtain for \( k = 1, 2, \ldots \)
\[ t_{k+1} \in \{ t : (k + 1) \ d(t, \ell^0) \leq d(t_k, \ell^0) \} , \]
\[ x_{k+1} \in \Gamma(t_{k+1}) \setminus \{ \Gamma(\ell^0) \cup \{ x \in C : ||x|| \leq (k + 1) \ ||x|| \} \} . \]
By the above constructed sequences \( \{ t_k \} \subset T \) and \( \{ x_k \} \subset X \), \( x_k \in \Gamma(t_k) \setminus \Gamma(\ell^0) \), it is easy to check that \( t_k \to \ell^0 \) and \( \{ x_k \} \) has no accumulation point. Hence, by Lemma 2.3 \( \Gamma \) is not upper semicontinuous at \( \ell^0 \). But this contradicts the assumption.

**Proof of Theorem 2.1:** (iii) \( \Rightarrow \) (ii): Let \( \{ t_k \} \subset T \) and \( \{ x_k \} \subset X \) be sequences with \( t_k \to \ell^0 \) and \( x_k \in N(t_k) \setminus N(\ell^0) \). By Lemma 2.1 \( \{ x_k \} \) has an accumulation point contained in \( N(\ell^0) \). By the upper semicontinuity of \( f \) it is easy to see that \( N(\ell^0) \) is closed. Hence, by Lemma 2.3 \( N \) is upper semicontinuous at \( \ell^0 \). (ii) \( \Rightarrow \) (i): Since \( N(\ell^0) \) is bounded (Assumption 2.2) and closed, the assertion follows from Lemma 2.2.

(i) \( \Rightarrow \) (iii) is obvious.

**Remark 2.1:** As seen in the proof of Theorem 2.1, (iii) \( \Rightarrow \) (ii) and the proof of Lemma 2.1, the upper semicontinuity of the set-valued mapping \( N \) at \( \ell^0 \) follows from Assumption 2.1 and condition (iii). From this fact it is easy to derive the following criterion for the upper semicontinuity of a set-valued mapping:

Let \( \varphi \) be a real-valued and upper semicontinuous function on \( C \times T \) such that \( \varphi(\cdot, t) \) is concave for each \( t \in T \). Let the set-valued mapping \( M \) be defined by \( M(t) = \{ x \in C : \varphi(x, t) \geq 0 \} \). Suppose that there is an open set \( \Omega \subset X \) containing \( M(\ell^0) \) and a neighbourhood \( V \) of \( \ell^0 \) such that \( \Omega \cap M(V) \) is a bounded set in \( X \). Then \( M \) is upper semicontinuous at \( \ell^0 \).

Using Theorem 2.1 we now establish stability criterions for Problem (P) above. We shall prove the following main stability theorem.

**Theorem 2.2:** Let Assumptions 2.1 and 2.2 and one of the conditions (i)—(iii) of Theorem 2.1 be satisfied. Then Problem (P) is stable at \( \ell^0 \).

The following results of Mosco [12] are used for the proof.

**Lemma 2.4** [12: Theorem 3.1]: Let \( g : C \times C \to \mathbb{R} \) be a monotone and hemicontinuous function such that \( g(x, \cdot) \) is concave and upper semicontinuous for each \( x \in C \). Suppose that there exist a compact set \( B \subset C \) and a point \( y_0 \in B \) such that \( f(x, y_0) > 0 \) for each \( x \in C \setminus B \) (the coerciveness condition). Then the solution set of the problem

\[ \{ x \in C : g(x, y) \leq 0 \quad \text{for all} \quad y \in C \} \]

is non-empty, convex and compact.

**Lemma 2.5** [12: Lemma 3.1]: Let \( g : C \times C \to \mathbb{R} \) be a monotone and hemicontinuous function such that \( g(x, \cdot) \) is concave and upper semicontinuous for each \( x \in C \). Let

\[ G(y) = \{ x \in C : g(x, y) \leq 0 \} \quad \text{and} \quad H(y) = \{ x \in C : g(y, x) \geq 0 \} . \]

Then \( \cap_{y \in C} G(y) = \cap_{y \in C} H(y) \).

**Proof of Theorem 2.2:** By Theorem 2.1 it is enough to show that Assumptions 2.1 and 2.2 and condition (i) imply the stability of Problem (P) at \( \ell^0 \). Since \( N(U) \subset C \) is bounded (condition (i)), we can assume that \( C(U) \) is contained in a compact set \( B \subset C \). By the monotonicity of \( f \) we have \( \{ x \in C : f(x, y_0, t) \leq 0 \} \subset N(t) \). For each \( t \in U \) it is then easy to see that \( f(x, y_0, t) > 0 \) for all \( x \in C \setminus B \), i.e. the coerciveness
condition in Lemma 2.4 is satisfied for Problem (P₁). By applying this Lemma it implies that the solution set $S(\theta)$ of (P₁) is non-empty, convex and compact.

We now show the upper semicontinuity of the mapping $S : U \to 2^B$ at $\theta$. Let $\{t_k\} \subset U$ and $\{x_k\} \subset B$ be sequences with $t_k \to \theta$ and $x_k \in S(t_k) \setminus S(\theta)$. Since $B$ is compact, $\{x_k\}$ has a convergent subsequence $\{x_{k'}\}$. Let $x_{k'} \to x$. Since $x_{k'} \in S(t_{k'})$ we have $f(x_{k'}, y, t_{k'}) \leq 0$ for all $y \in C$ and hence, by the monotonicity of $f$, $f(y, x_{k'}, t_{k'}) \geq 0$. From the upper semicontinuity of $f$ follows $f(y, x, \theta) \leq 0$ for all $y \in C$, i.e. $x \in S(\theta)$. $S(\theta)$ is here closed. Therefore, by Lemma 2.3, $S$ is upper semicontinuous.

Remark 2.2: It is easy to see that Assumption 2.2 is contained in condition (iii). Hence, by Theorem 2.2, in the case where this condition is satisfied, Problem (P₁) is stable at $\theta$ if Assumption 2.1 is satisfied.

As we see in the proofs of Theorems 2.1 and 2.2, the set-valued mapping $N : T \to C$ plays an essential role. We are here interested in the question under which conditions the “level set” $N(U)$ is bounded for a neighbourhood $U$ of $\theta$. Let us now consider a case where the set $N(U)$ is bounded.

Let $K \subset X$ be a cone with vertex $a$. $K$ is said to be pointed if $a \notin \overline{co}(K \setminus B(a, 1))$, where $B(a, 1)$ denotes the ball with center $a$ and radius 1. In the following lemma we give a property of the pointed cone, used for the stability consideration below.

Lemma 2.6: If $K$ is a pointed cone with vertex $a$, then there is a functional $l' \in X^*$ such that $l'(x) > l'(a)$ for all $x \in K \setminus \{a\}$ and the intersection of each hyperplane $\{x \in X : l'(x) = \beta, \beta \geq l'(a)\}$ with $K$ is a bounded set.

Proof: Since $K$ is a pointed cone we have $a \notin \overline{co}(K \setminus B(a, 1))$. Hence, there is a functional $l' \in X^*$ separating $a$ and $\overline{co}(K \setminus B(a, 1))$ strictly such that with a suitable $\alpha$ we have

$$l'(a) < x < l'(x) \quad \text{for all } x \in \overline{co}(K \setminus B(a, 1)). \tag{1.3}$$

Now we show that $l'(x) > l'(a)$, for all $x \in K \setminus \{a\}$. It is easy to see that $K \cap \{x \in X : l'(x) < l'(a)\} = \emptyset$. Assume the contrary: there is a point $x$ of this intersection. Then, by a property of the cone we have $a + \lambda(\overline{x} - a) \in K \cap \{x \in X : l'(x) < l'(a)\} < l'(a)$, $\lambda > 0$, i.e. there is an $x \in K \setminus B(a, 1)$ with $l'(x) < l'(a)$. This contradicts (1.3). By an analogous argument we get $(K \setminus \{a\}) \cap \{l' : l'(a)\} = \emptyset$. So, that means:

$l'(x) > l'(a)$ for all $x \in K \setminus \{a\}$.

Since, by (1.3), the hyperplane $(l', a)$ separates $a$ and $K \setminus B(a, 1)$ strictly, the intersection of $(l', a)$ with $K$ cannot be contained in $K \setminus B(a, 1)$; it is contained in $K \cap B(a, 1)$. Hence, $(l', a) \cap K$ is bounded. Now it is not difficult to show that $(l', a) \cap K$ for $a \geq l'(a)$ is bounded, too. We assume here that $\beta > l'(a)$ (in the case $\beta = l'(a)$ it is easy to see that $(l', \beta) \cap K = \{a\}$ and hence bounded). Let $c = (\beta - l'(a))/(a - l'(a))$, we have $c > 0$. Since $K - a$ is a cone with vertex 0, it follows then that $K - a = c(K - a)$. Since $(l', a) - a = \{x \in X : l'(x) = a - l'(a)\}$ it is easy to check that $(l', a) - a = c(l', a) - a$. We then have

$$(l', \beta) \cap K - a = (l', \beta) - a \cap (K - a) = c(l', a) \cap K - a.$$  

Since $(l', a) \cap K$ is bounded, $(l', \beta) \cap K$ is bounded.

Using Lemma 2.6 we can now prove the following theorem for the stability of Problem (P₁).
Theorem 2.3: Let Assumptions 2.1 and 2.2 be satisfied. Suppose that \( N(U) \) is contained in a pointed cone \( K \). Then Problem (P) is stable at \( t^0 \).

Proof: We show that the condition (iii) (given in Theorem 2.1) is here satisfied and hence the assertion follows from Theorem 2.2. Let \( l' \in X^* \) be the functional which exists by Lemma 2.6 for the pointed cone \( K \). Since \( N(t^0) \) is bounded (Assumption 2.2) we have \( \gamma = \sup \{ l'(x): x \in N(t^0) \} < +\infty \). Consider the hyperplane \( (l', \alpha) \) with \( \alpha > \gamma \). It is then easy to see that \( l'(x) < \alpha \) for all \( x \in N(t^0) \) where \( \alpha \) is the vertex of \( K \). By Lemma 2.6 \( (l', \alpha) \cap K \) is bounded and hence \( Q = \{ x \in K: l'(x) < \alpha \} \) is also bounded. Since \( N(U) \subseteq K \), it is then easy to see that \( N(U) \cap \{ x \in X: l'(x) < \alpha \} \subseteq Q \) is also bounded. Thus, condition (iii) is satisfied by taking the open set \( \Omega = \{ x \in X: l'(x) < \alpha \} \) and the neighbourhood \( V = U \).

Remark 2.3: According to Remark 2.1 it is here worth noticing that by using the pointed cone defined above we can derive the following criterion for the upper semicontinuity of a set-valued mapping:

Let \( \varphi \) be a real-valued and upper semicontinuous function on \( C \times T \) such that \( \varphi(\cdot, t) \) is concave for each \( t \in T \). Let the set-valued mapping \( M \) be defined by \( M(t) = \{ x \in C: A(x, t) = \varphi(x, t) \} \). Suppose that \( M(t^0) \) is bounded and there is a neighbourhood \( V \) of \( t^0 \) such that \( M(V) \) is contained in a pointed cone. Then \( M \) is upper semicontinuous.

By an argument analogous to that used in the proof of Theorem 2.3 it is easy to see that there is an open set \( \Omega \) such that \( \Omega \cap M(V) \) is bounded. Hence, by Remark 2.1, follows the assertion.

The above-considered mapping \( M \) is a mapping with special structure. About criterions for the upper semicontinuity (and also lower semicontinuity) of general set-valued mappings we refer the reader, for example, to Berge [2] and Bank et al. [1].

3. Special cases

In this section, stability criterions for some special cases will be given by using Theorems 2.2 and 2.3. Consider the following family of variational inequalities

\[
\begin{cases}
  x \in C \\
  \langle A(x, t) - v', x - y \rangle + \varphi(x, t) - \varphi(y, t) \leq 0 \quad \text{for all } y \in C,
\end{cases}
\]

where \( A \) is an operator from \( C \times T \) into \( X^* \), \( \varphi \) is a real-valued function on \( C \times T \) and \( v' \in X^* \), is a given functional.

Proposition 3.1: Let \( A(\cdot, t) \) be monotone and hemicontinuous for each \( t \in T \), \( A(x, \cdot) \) continuous for each \( x \in C \), \( \varphi \) lower semi-continuous, \( \varphi(\cdot, t) \) convex for each \( t \in T \). Suppose that there is a point \( y_0 \in C \) such that the image set \( N(t^0) \) of the set-valued mapping \( N \) defined by

\[
N(t) = \{ x \in C: \langle A(y_0, t) - v', y_0 - x \rangle + \varphi(y_0, t) - \varphi(x, t) \geq 0 \}
\]

is bounded. Moreover, suppose that there is a neighbourhood \( U \) of \( t^0 \) such that \( N(U) \) is contained in a pointed cone.

Then Problem (3.1) is stable at \( t^0 \).

Proof: The assertion follows immediately from Theorem 2.3.

Corollary 3.1: Let \( A: C \to X^* \) be a monotone and hemicontinuous operator, \( \varphi: C \times T \to \mathbb{R} \) a lower semicontinuous function such that \( \varphi(\cdot, t) \) is convex for each \( t \), and \( \alpha: T \to \mathbb{R} \) a continuous function. Suppose that \( \varphi(x, t^0) \to +\infty \) as \( ||x|| \to \infty \) and
there is a point $y_0 \in C$ satisfying $Ay_0 = 0$, such that the set \( \{ x \in C : \varphi(x, t) \leq \varphi(y_0, t) \} \) is contained in a pointed cone for all $t$ in a neighbourhood of $t^0$.

Then the problem

$$
\begin{cases}
    x \in C \\
    \alpha(t) \langle Ax, x - y \rangle + \varphi(x, t) - \varphi(y, t) \leq 0 \quad \text{for all } y \in C
\end{cases}
$$

is stable at $t^0$.

**Proof:** Apply Proposition 3.1 with $N(t) = \{ x \in C : \varphi(x, t) \leq \varphi(y_0, t) \}$ and $v' = 0$. Since $\varphi(x, t^0) \to +\infty$ as $\|x\| \to \infty$, $N(t^0)$ is bounded. The other assumptions are satisfied too.

**Corollary 3.2:** Let $C$ be a pointed cone $K$ with vertex 0 and let $v'$ be a functional with $v'(x) \leq 0$ for all $x \in K$. Let the operator $A$ and function $\varphi$ be given as in Proposition 3.1. Suppose that $\varphi(x, t^0) \to +\infty$ as $\|x\| \to \infty$ and there is a point $y_0 \in C$ satisfying $A(y_0, t^0) = 0$.

Then Problem (3.1) is stable at $t^0$.

**Proof:** Apply Proposition 3.1 with $N(t) = \{ x \in C : \langle v', x - y_0 \rangle + \varphi(y_0, t) - \varphi(x, t) \geq 0 \}$. By the property of $v'$ and $\varphi$ it is easy to see that $N(t^0)$ is bounded. The other assumptions are satisfied too.

Let us now consider the following family of optimization problems

$$
\min \{ \varphi(x, t) : x \in C \}, \quad (3.2)
$$

where as above $\varphi$ is a real-valued function on $C \times T$. We write this problem in the form

$$
\begin{cases}
    x \in C \\
    \varphi(x, t) - \varphi(y, t) \leq 0 \quad \text{for all } y \in C.
\end{cases} \quad (3.2')
$$

Problem (3.2) is said to be stable at $t^0$ if Problem (3.2') is stable at $t^0$. From Proposition 3.1 it is easy to derive

**Corollary 3.3:** Let $\varphi$ be a lower semicontinuous function such that $\varphi(\cdot, t)$ is convex for each $t$. Suppose that $\varphi(x, t^0) \to +\infty$ as $\|x\| \to \infty$ and there is a point $y_0 \in C$ such that the set

$$
N(t) = \{ x \in C : \varphi(x, t) \leq \varphi(y_0, t) \}
$$

is contained in a pointed cone for all $t$ in a neighbourhood of $t^0$.

Then Problem (3.2) is stable at $t^0$.

The stability criterion in Corollary 3.3 is given only for the solution set mapping of convex optimization problems. A general stability theory for general optimization problems is given in Bank et al. [1], Gollan [2] and Kirsch [7].

**Proposition 3.2:** Let $X$ be a finite-dimensional space and let Assumptions 2.1 and 2.2 be satisfied. Then Problem $(P_t)$ is stable at $t^0$.

**Proof:** Since in a finite-dimensional space an open set is weakly open, condition (iii) given in Theorem 2.1 is satisfied by taking $\Omega = \{ x \in X : \|x\| < \tau \} \supseteq N(t^0)$ for $\tau > 0$ enough large and $V \subset T$ to be an arbitrary neighbourhood of $t^0$. $\Omega \cap N(V)$ is then bounded. Thus, by Theorem 2.2 Problem $(P_t)$ is stable at $t^0$.

**Example:** Let $(Q_t)$ be the following family of nonlinear complementarity problem

$$
\begin{cases}
    x \in \mathbb{R}^n, \quad M(x, t) \in \mathbb{R}^n \\
    x'M(x, t) = 0
\end{cases} \quad (Q_t)
$$
where $\mathbb{R}_+^n = \{ x \in \mathbb{R}^n : x_1, x_2, \ldots, x_n \geq 0 \}$, $M: \mathbb{R}_+^n \times T \to \mathbb{R}^n$ is an operator such that for each $t \in T$, $M(\cdot, t)$ is monotone and hemicontinuous and for each $x \in \mathbb{R}_+^n$, $M(x, \cdot)$ is continuous ($x'$ is the transposed vector of $x$). Assume that there is $y_0 \in \mathbb{R}_+^n$ such that the set $\{ x \in \mathbb{R}_+^n : x'M(y_0, t_0) \leq \alpha \}$, $\alpha := y_0'M(y_0, t_0)$, is bounded. Then there exists a neighbourhood $V$ of $t_0$ such that for each $t \in V$ the complementarity problem (Q1) has a solution. If we denote by $\Gamma(t)$ the solution set of (Q1), $t \in V$, then the set-valued mapping $\Gamma: t \to \Gamma(t)$ is upper semicontinuous at $t_0$.

It is here easy to see that $\mathbb{R}_+^n$ is a pointed cone. Hence, by applying Proposition 3.2 with $C = \mathbb{R}_+^n$, $f(x, y) = (x' - y')M(x, t)$ it implies that Problem (P1) in this case is stable at $t_0$. The assertion follows then by the fact that a point $x \in \mathbb{R}_+^n$ is a solution of Problem (P1) (in this case) if and only if it is a solution of the complementarity problem (Q1), $t \in V$ (see Lüthi [11]).

Some applications (e.g., to the obstacle problem, the free boundary problem) will be studied in another paper.

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