Bifurcation and Stability of Cellular States in Magnetic Fluids

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Prof. Dr. H. Beckert to his 65th birthday

In der vorliegenden Arbeit werden die Untersuchungen des Autors zu Verzweigungs- und Stabilitätsverhältnissen periodischer Gleichgewichtszustände magnetischer Flüssigkeiten in einem vertikalen Magnetfeld auf den Fall endlich tiefer Flüssigkeitsschichten ausgedehnt.

В продолжении исследований автора в работе изучаются устойчивость и бифуркации периодических равновесных состояний магнитной жидкости в вертикальном магнитном поле. Дается распространение на случай жидкости конечной глубины.

This paper continues earlier work by the author concerning bifurcation and stability of periodic equilibrium states of a magnetic fluid subjected to a vertical magnetic field. Here the treatment is extended to cover the case of a fluid of finite depth.

Consider a magnetic fluid in a vertical magnetic field under the influence of gravity and surface tension. This paper continues earlier work [2, 3] by the author on this subject. Here the treatment is extended to cover the case of a fluid of finite depth. Let $-h \leq z \leq Z(x, y)$ be a layer of magnetic fluid. Any steady-state equilibrium position of its upper free surface $z = Z(x, y)$ is characterized by the variational principle $\delta E = 0$, $E$ being the potential energy of the system. Clearly the plane horizontal interface, which may be taken to be the $(x, y)$-plane, always represents an equilibrium state. As the exterior field $H$ increases past a certain critical value $H_{cr}$ this basic solution loses its stability and the system moves into a new nontrivial state.

As in [3] we look for periodic $\Gamma$ with hexagonal lattice structure $\Lambda$. Our approach via Lyapunov-Schmidt procedure is based on analytic expansion of $E$ relative to the Sobolev spaces $H_s$ of $\Lambda$-periodic functions with mean zero as defined by (1.19). It turns out that (provided $s \geq 5/2$) the first variation $DF$ of the magnetic energy $F$ acts as an analytic map from $H_s$ into $H_{s-1}$, this improving a corresponding result of [3]. As an immediate consequence this implies analyticity of $DE$ as a mapping from $H_s$ into $H_{s-2}$.

An outline of the paper is as follows. In § 1 our objective is to compute the Taylor series of $F$ (resp. $E$), essentially up to fourth order terms in $\Gamma$. Particularly: $\dim N = 6$ at criticality where $N$ denotes the kernel. In § 2, using the symmetries of $E$, we solve the branching equations for three types of solutions $I$, $II^\pm$. Tested against disturbances in the lattice class $\Lambda$ the transcritical branch $II^+$ turns out to be stable only. Having (2.15) in mind; this indicates hysteresis at $H_{cr}$ (cf. [5, 7]). The final § 3 is devoted to the proof of Theorem 2.1.

It should be remarked that a further supercritical branch can be determined by means of scaling techniques. Bifurcating solutions to the nonlinear problem (infinite depth) were first constructed formally by Gailitis [5]. For a detailed discussion of bifurcation phenomena in the presence of a symmetry group see the expository paper [7].
§ I

Consider the upper free surface $F: z = Z(x, y)$ separating a layer $-h \leq z \leq Z(x, y)$ of an incompressible magnetic fluid of depth $h > 0$ from a vacuum. Subjected to the action of surface tension $\beta$, gravity $(0, 0, -g)$ and an exterior vertical magnetic field $\mathcal{H}$ the plane horizontal interface — say $z = 0$ — always represents an equilibrium state. As $\mathcal{H}$ increases past a certain critical value $H_{cr}$ this basic solution loses its stability and the system moves into a new nontrivial state.

As in [3] we look for periodic states. Choose dimensionless coordinates $(x_1, x_2, x_3) = (x, y, z)/l$ where $l > 0$ measures the wavelength to be specified later on. If in the $(x_1, x_2, x_3)$ reference-system labelled by

$$
\Gamma: x_3 = \zeta(x_1, x_2) = Z(x, y)/l
$$

we restrict the interfaces $\Gamma$ to be $\Lambda$-periodic with respect to the hexagonal lattice

$\Lambda = \{k_1\omega_1 + k_2\omega_2 : k_1, k_2 \in \mathbb{Z}\}$ generated by $\omega_1 = 2\pi(1, 0)$, $\omega_2 = 2\pi(1/2, \sqrt{3}/2)$. Let $\mathcal{P}(0, \omega_1, \omega_2, \omega_1 + \omega_2)$ be the fundamental parallelogram of the lattice. On $\mathcal{P}$ we assume the fluid/vacuum to occupy the regions

$$
\Omega^- : (x_1, x_2) \in \mathcal{P}, \quad -q < x_3 < \zeta(x_1, x_2); \quad q := h/l > 0
$$

resp.

$$
\Omega^+ : (x_1, x_2) \in \mathcal{P}, \quad x_3 > \zeta(x_1, x_2)
$$

(lower/upper vacuum part); let $\Omega = \Omega^- \cup \Omega^+$. If necessary, in the following we shall distinguish the corresponding fields accordingly by indices $\"-\"$ ($\"+\"$).

By definition an equilibrium state $\zeta$ has to satisfy the variational equation $(DE(\zeta), h) = 0$ for all admissible variations $h$ where $E$ denotes the energy functional of our system. Considering incompressibility we impose $\zeta$ and $h$ to have mean zero:

$$
\int_{\mathcal{P}} \zeta \, dx_1 \, dx_2 = 0.
$$

If the magnetic field $\mathcal{H} = H \nabla \psi$:

$$
\psi^\Pi(x, y, z) = \frac{z}{\mu} + l \frac{1 - \mu}{\mu} u^\Pi(x_1, x_2, x_3) \quad \text{on } \Omega^\Pi,
$$

$$
\psi^\pm(x, y, z) = z + l \frac{1 - \mu}{\mu} u^\pm(x_1, x_2, x_3) \quad \text{on } \Omega^\pm
$$

is permitted to vary in a neighbourhood of $H[\nabla(z^\Pi/\mu), \nabla z^\pm]$ we get

$$
E = \int_{\mathcal{P}} \sqrt{1 + |\nabla \zeta|^2} \, dx_1 \, dx_2 + \frac{q^2_2}{2} \int_{\mathcal{P}} \zeta^2 \, dx_1 \, dx_2 - q_1 q_2 \frac{1 + \mu}{\mu} F
$$

for the energy (per unit area) measured in units of $\beta$ (surface tension), see [3]. Here $F = F(\zeta)$ is defined to be the minimal value to the quadratic variational problem

$$
\int_{\mathcal{P}} |\nabla u|^2 \, dV + \mu \int_{\mathcal{P}} |\nabla u|^2 \, dV \to \min
$$

$$
\mathcal{P}
$$

(1.5)
\( (dV = dx_1 \, dx_2 \, dx_3) \) which is to solve subject to boundary and periodicity conditions

\[
    u = (u^n, u^\pm) \text{ } \Lambda \text{-periodic,} \\
    u^+ - u^n = x_3 + \text{const. on } \Gamma, \quad u^n - u^- = \text{const. on } x_3 = -q. \tag{1.6}
\]

The dimensionless parameters \( q_1, q_2 \) are defined by

\[
    8\pi \mu (1 + \mu) q_1 = (eg\beta)^{-1/2} \left( \mu - 1 \right)^2 H^2, \quad \beta^{1/2} q_2 = l(eg)^{1/2}
\]

where \( q > 0 \) is the density and \( \mu > 0 \) the magnetic permeability of the fluid (\( \mu = 1 \) in \( \Omega^\pm \)). Note \( q = h/l = h \sqrt{eg/q_2} \sqrt{\beta} \).

To begin, we compute the derivatives of \( E \) — at the present stage on a somewhat formal way. Consider, in addition to \( \Gamma \), a family of neighbouring surfaces \( \Gamma_t \): \( x_3 = \zeta(x_1, x_2) + th(x_1, x_2), \Gamma_0 = \Gamma \). Let \( \Omega_t^\pm, \Omega_t^n \) be the corresponding family of domains (1.1). Solving (1.5) relative to \( \Omega_t^\pm, \Omega_t^n \) gives rise to fields \( u(t; x_1, x_2, x_3) \). Let a dot denote differentiation with respect to \( t \) at \( t = 0 \): \( \dot{u} = \partial u/\partial t \) (0; ; ; ;). Differentiation of \( F \) yields

\[
\begin{align*}
    (DF, h) &= \frac{d}{dt} F(t + th) |_{t=0} \\
    &= 2 \int_{\Omega^n} \nabla u \nabla \dot{u} \, dV + 2\mu \int_{\Omega^n} \nabla u \nabla \dot{u} \, dV \\
    &\quad + \int_{\Gamma} (\mu |\nabla u^n|^2 - |\nabla u^+|^2) h \, dx_1 \, dx_2,
\end{align*}
\tag{1.7}
\]

the last term due to varying the boundary. Note \( \Delta u = 0 \) in \( \Omega^n \) (resp. \( \Omega^\pm \)) due to (1.5). From (1.6) we get by differentiation

\[
    \dot{u}^+ - \dot{u}^n = (1 - u^+_{z_3} - uh h)_{z_3} + \text{const. on } \Gamma. \tag{1.8}
\]

Therefore (1.7) leads to

\[
    (DF, h) = \int_{\Omega^n} (\mu |\nabla u^n|^2 - |\nabla u^+|^2) h \, dx_1 \, dx_2
    + 2 \int_{\Gamma} \frac{\partial u^+}{\partial n} (u^+_{z_3} - u^n_{z_3}) - 1) h \, d\Gamma, \tag{1.9}
\]

when integrated by parts (note that \( u, \dot{u} \in O\left(\exp (-2|x_3|/\sqrt{3})\right)) \). In (1.9) the normal \( n \) has to be taken directed to \( \Omega^+ \).

Remark 1.1: Remembering (1.3) we get after retransformation

\[
    (DE, h) = (\beta l)^{-1} \int_{\Omega} \left( -\beta \text{ div} \frac{\nabla Z}{\sqrt{1 + |\nabla Z|^2}} + egZ \right.
    + \frac{1 - \mu}{8\pi} \left( |\hat{\xi}_t^n|^2 + \mu |\hat{\xi}_n^n|^2 \right) \left( \frac{x}{l}, \frac{y}{l} \right) \, dx \, dy,
\]

\( \hat{\xi}_t \) (resp. \( \hat{\xi}_n \)) being the tangential (resp. normal) component of \( \xi \). Because of (1.2) this implies

\[
    -\beta \text{ div} \frac{\nabla Z}{\sqrt{1 + |\nabla Z|^2}} + egZ + \frac{1 - \mu}{8\pi} \left( |\hat{\xi}_t^n|^2 + \mu |\hat{\xi}_n^n|^2 \right) = \text{const.}
\]

along an equilibrium interface \( \Gamma \).
Further differentiation gives

\[ D^2 F(\zeta) \{ h^2 \} = \frac{d^2}{dt^2} F(\zeta + th)|_{t=0} \]
\[ = 2 \int_{\Omega} (|\nabla \tilde{u}|^2 + \nabla u \nabla \tilde{u}) \, dV + 2\mu \int_{\Omega} (|\nabla \tilde{u}|^2 + \nabla u \nabla \tilde{u}) \, dV \]
\[ + 4 \int_{\Gamma} (\mu \nabla^T u^n \nabla u^n - \nabla u^+ \nabla \tilde{u}^+) \, h \, dx_1 \, dx_2 \]
\[ + 2 \int_{\Gamma} (\mu \nabla^T u^n \nabla u^n - \nabla u^+ \nabla \tilde{u}^+) \, h^2 \, dx_1 \, dx_2. \tag{1.10} \]

Its value at \( \zeta = 0 \):

\[ D^2 F(0) \{ h^2 \} = 2 \int_{\Omega} |\nabla \tilde{u}|^2 \, dV + 2\mu \int_{\Omega} |\nabla \tilde{u}|^2 \, dV \]
\[ \tag{1.11} \]

is of particular interest.

For simplicity we adopt the following notation:

\[ a(u, v) = \int_{\Omega} \nabla u \nabla v \, dV + \mu \int_{\Omega} \nabla u \nabla v \, dV, \]
\[ \hat{a}(u, v) = \int_{\Gamma} (\mu \nabla^T u^n \nabla v^n - \nabla u^+ \nabla v^+) \, h \, dx_1 \, dx_2, \tag{1.12} \]
\[ \hat{a}(u, v) = \int_{\Gamma} \frac{\partial}{\partial x_3} (\mu \nabla^T u^n \nabla v^n - \nabla u^+ \nabla v^+) \, h^2 \, dx_1 \, dx_2. \]

If we keep \( \zeta \) (hence \( \Omega^0, \Omega^\perp \)) and \( h \) both fixed then we have to think of (1.12) as of bilinear forms in \( u; v \). Now (1.10), (1.11) reads

\[ D^2 F(\zeta) \{ h^2 \} = 2a(\tilde{u}, \tilde{u}) + 2a(u, \tilde{u}) + 4\hat{a}(u, \tilde{u}) + \hat{a}(u, u), \tag{1.13} \]
\[ D^2 F(0) \{ h^2 \} = 2a(\tilde{u}, \tilde{u}) \]

As above we get by repeated differentiation

\[ D^3 F(\zeta) \{ h^3 \} = 6a(\tilde{u}, \tilde{u}) + 6\hat{a}(u, \tilde{u}), \tag{1.14} \]
\[ D^4 F(0) \{ h^4 \} = 8a(\tilde{u}, u^3) + 6a(\tilde{u}, \tilde{u}) + 24\hat{a}(u, \tilde{u}) + 12\hat{a}(u, u). \]

We still have to determine the derivatives of \( u \). We start with differentiating the variational equation \( a(u, \varphi) = 0 \) to (1.5) choosing the test function \( \varphi \) to be sufficiently regular. This yields

\[ a(u, \varphi) + \hat{a}(u, \varphi) = 0 \text{ for all } \varphi \Lambda \text{-periodic.} \]

In addition, \( \tilde{u} \) has to satisfy (1.6) resp. (1.8). At \( \zeta = 0 \) this particularly reduces to

\[ a(\tilde{u}, \varphi) = 0 \text{ for all } \varphi \Lambda \text{-periodic;} \]
\[ \tilde{u}^+ - \tilde{u}^- = h + \text{const. along } x_3 = 0, \tag{1.15} \]
\[ \tilde{u}^+ - \tilde{u}^- = \text{const. along } x_3 = -q. \]

Similarly by repeated differentiation

\[ a(\tilde{u}, \varphi) + 2\hat{a}(u, \varphi) = 0, \text{ } \tilde{u} \Lambda \text{-periodic, for all } \varphi \Lambda \text{-periodic;} \]
\[ \tilde{u}^+ - \tilde{u}^- = -2(\tilde{u}_{x_3} - \tilde{u}_{x_3}^0) \, h + \text{const. along } x_3 = 0, \tag{1.16} \]
\[ \tilde{u}^+ - \tilde{u}^- = \text{const. along } x_3 = -q. \]
at $\zeta = 0$. Integrated by parts this leads to
\[ \Delta \hat{u}^n = 0 \quad \text{on } \Omega^n, \quad \Delta \hat{u}^\pm = 0 \quad \text{on } \Omega^\pm; \]
\[ \hat{u}^\pm_{z_1} - \mu \hat{u}^n_{z_1} = 0 \quad \text{along } x_3 = 0 \quad (x_3 = -q) \quad \text{(1.17)} \]
resp.
\[ \Delta \hat{u}^n = 0 \quad \text{on } \Omega^n, \quad \Delta \hat{u}^\pm = 0 \quad \text{on } \Omega^\pm; \]
\[ \hat{u}^+_z = - \mu \hat{u}^n_z = 2 \left( \frac{\partial}{\partial x_1} \hat{h}(\hat{u}^+_{z_1} - \mu \hat{u}^n_{z_1}) + \frac{\partial}{\partial x_2} \hat{h}(\hat{u}^+_{z_2} - \mu \hat{u}^n_{z_2}) \right) \quad \text{along } x_3 = 0, \]
\[ \hat{u}^-_{z_3} = - \mu \hat{u}^n_{z_3} = 0 \quad \text{along } x_3 = -q. \quad \text{(1.18)} \]

From now let $\zeta = 0$ be fixed. To solve (1.17) resp. (1.18) expand $\hat{h}$ in a Fourier series
\[ \hat{h} = \sum_{\omega \in \Lambda'^*} \hat{h}_\omega \exp(i \omega \cdot x), \quad \hat{h}_\omega = \overline{\hat{h}_\omega}. \quad \text{(1.19)} \]

Here $\Lambda' = \{ k_1 \omega_1' + k_2 \omega_2' : k_1, k_2 \in \mathbb{Z} \}$ is the dual lattice to $\Lambda$ which is generated by $\omega_1' = 2/\sqrt{3}(\sqrt{3}/2, -1/2)$, $\omega_2' = 2/\sqrt{3} (0, 1)$ and $\exp(\omega)$ denotes the scalar product of $\omega \in \Lambda'$ and $x = (x_1, x_2)$. In the following Lemma we consider $\hat{u}, \hat{u}$ to be dependent on $\mu$ also.

**Lemma 1.1:** (i) Let $\zeta = 0$, then
\[ \hat{u}^- = - \frac{2 \mu}{(\mu + 1)^2} \sum_{\omega \in \Lambda'} \frac{\hat{h}_\omega \exp(i \omega \cdot x)}{1 - \left( \frac{\mu - 1}{\mu + 1} \right)^2 e^{-2q|\omega|}}, \]
\[ \hat{u}^n = - \frac{1}{\mu + 1} \sum_{\omega \in \Lambda'} \frac{\hat{h}_\omega \exp(i \omega \cdot x)}{1 - \left( \frac{\mu - 1}{\mu + 1} \right)^2 e^{-2q|\omega|}} \left( \exp(i \omega \cdot x) + \frac{\mu - 1}{\mu + 1} e^{-(\mu|\omega|(2q + x_3))} \right), \]
\[ \hat{u}^+ = \frac{\mu}{\mu + 1} \sum_{\omega \in \Lambda'^*} \frac{1 - \mu - 1}{\mu + 1} e^{-2q|\omega|} \hat{h}_\omega \exp(i \omega \cdot x - |\omega| x_3). \]

(ii) If in addition $\mu = 1$, then
\[ \hat{u}^+(x_1, x_2, 0) = \hat{u}^n(x_1, x_2, 0) = \frac{1}{2} A(h^2), \]
\[ \hat{u}^+_{z_1}(x_1, x_2, 0) = - \hat{u}^n_{z_1}(x_1, x_2, 0) = \frac{1}{2} \Delta(h^2) \]

where $A$ denotes the map
\[ h \rightarrow Ah = \sum_{\omega \in \Lambda} |\omega| \hat{h}_\omega \exp(i \omega \cdot x). \]

**Proof:** (i) is easily verified when inserted in (1.15), (1.17). Let $\mu = 1$, then in view of (i)
\[ \hat{u}^+(x_1, x_2, 0) = - \hat{u}^n(x_1, x_2, 0) = \frac{h}{2}, \]
\[ \hat{u}^+_{z_1}(x_1, x_2, 0) = \hat{u}^n_{z_1}(x_1, x_2, 0) = - \frac{1}{2} Ah. \]
Consequently (1.16), (1.18) reduces to

\[ \ddot{u}^+(x_1, x_2, 0) - \ddot{u}^I(x_1, x_2, 0) = \text{const.} \]
\[ \ddot{u}_{x_3}^+(x_1, x_2, 0) - \ddot{u}_{x_3}^I(x_1, x_2, 0) = \Delta (h^2). \]  

(1.20)

Now consider the harmonic function \( v \) on \( \Omega^+ \) with boundary values \( h^2 \)-along \( x_3 = 0 \), and whose Dirichlet integral extended over \( \Omega^+ \) is finite. Obviously \( \ddot{u}^+ = -\frac{1}{2} v_{x_3} \), \( \ddot{u}^I, \ddot{u}^-(x_1, x_2, x_3) = -\frac{1}{2} v_{x_3}(x_1, x_2, -x_3) \) represents the desired solution of (1.20). This immediately implies (ii).

Inserting (i) in (1.13) we get after integration by parts

\[ D^2 F(0; h^2) = -2 \int_\mathcal{P} \ddot{u}^+_1(x_1, x_2, 0) \, h \, dx_1 \, dx_2 \]
\[ + \frac{2\mu}{\mu + 1} |\mathcal{P}| \sum_{\omega \in \mathcal{A}'} 1 - \frac{\mu - 1}{\mu + 1} \text{e}^{-2\omega |\mathcal{P}|} |\mathcal{P}| |h_\omega|^2, \]  

(1.21)

\(|\mathcal{P}| = 2 \sqrt{3} \pi^2 \). To stress the dependence on the additional parameters in the following, we use the notation \( F(\xi; \mu, q), E(\xi; \mu, q_1, q_2) \). As above we get from (1.14) by Lemma 1.1

\[ D^3 F(0; \mu, q) \{ h^3 \} = 6 \int_\mathcal{P} \ddot{u}_{x_3}^+(\dot{u}^+ - \ddot{u}^+)|_{x_3=0} \, dx_1 \, dx_2 + 6\partial(\dot{u}, \ddot{u}) \]
\[ = \frac{3}{2} (\mu - 1) \left( \left( A h, h Ah - \frac{1}{2} A h^2 \right) - (h^2, \Delta e^{-2q A h}) \right) \]
\[ + O(\mu - 1^2), \]  

(1.22)

where \((\cdot, \cdot)\) denotes the \( L_2 \)-scalar product on \( \mathcal{P} \) and

\[ e^{-2q A h} = \sum_{\omega \in \mathcal{A}'} h_\omega \text{e}^{-2\omega |\mathcal{P}|} \text{e}^{i\omega x}. \]

We point out that \( D^3 F(0; 1, q) = 0 \).

In order to obtain an analogous expression for \( D^4 F(0; 1, q) \) we differentiate (1.8) twice in \( t \). Setting \( \xi = 0, \mu = 1 \), this in view of Lemma 1.1 leads to

\[ u^{(3)} - u^{(3)} = -3(\ddot{u}_{x_3}^+ - \ddot{u}^{I}_{x_3}) \, h - 3(\ddot{u}^+ - \ddot{u}^I) \, h^2 + \text{const.} \]
\[ = 3(-h \Delta h^2 + h^2 \Delta h) + \text{const.} = -\Delta h^2 + \text{const.} \]

along \( x_3 = 0 \). Now, from (1.14) we get by Lemma 1.1 and the previous formula

\[ D^4 F(0; 1, q) \{ h^4 \} = 8 \int_\mathcal{P} \ddot{u}_{x_3}^+(u^{(3)} - u^{(3)})|_{x_3=0} \, dx_1 \, dx_2 \]
\[ + 6 \int_\mathcal{P} (\ddot{u}^I \ddot{u}^{I}_{x_3} - \ddot{u}^+ \ddot{u}^{+}_{x_3})|_{x_3=0} \, dx_1 \, dx_2 + 24\partial(\dot{u}, \ddot{u}) + 12\partial(\ddot{u}, \dddot{u}) \]
\[ = 4(h, A^3 h^2) - 3(h^2, A^3 h^2). \]  

(1.23)
Stability of the unperturbed state $\zeta = 0$ is determined by the second variation $D^2E(0; \mu, q_1, q_2)$ which we proceed to study. Let

$$Q(\vartheta, \mu, q_1) = \vartheta^2 + 1 - 2q_1\vartheta \frac{1 - \frac{\mu - 1}{\mu + 1} e^{-2\vartheta\sqrt{\alpha}/\sqrt{\beta}}}{1 - \left(\frac{\mu - 1}{\mu + 1}\right)^2 e^{-2\vartheta\sqrt{\alpha}/\sqrt{\beta}}}$$

then in view of (1.4) and (1.21)

$$D^2E(0; \mu, q_1, q_2) = \int \left(\|\nabla h\|^2 + q_2^2 h^2\right) dx_1 dx_2$$

$$- q_1q_2 \frac{1 + \mu}{\mu} D^2F(0; \mu, q_2)$$

$$= q_2^2 \left|\mathcal{D}\right| \sum_{\omega \in \Lambda^*} Q\left(\frac{\omega}{q_2}, \mu, q_1\right) |h_\omega|^2. \quad (1.24)$$

**Lemma 1.2:** For $\mu$ in a neighbourhood of $\mu = 1$ there exist analytic $\vartheta^{cr}, q_1^{cr}, \theta > 0$ such that for all $\vartheta \geq 0$

$$Q(\vartheta; \mu, q_1) > 0 \text{ if } 0 \leq q_1 < q_1^{cr}$$

and $Q(\vartheta^{cr}; \mu, q_1^{cr}) = 0.\text{ Moreover } Q(\vartheta; \mu, q_1^{cr}) > 0 \text{ if } \vartheta \neq \vartheta^{cr}$.

**Proof:** The critical values $\vartheta^{cr}, q_1^{cr}$ are to be determined from $Q = \partial Q/\partial \vartheta = 0$. Eliminating $q_1$ leads to

$$4 \left[\frac{\mu - 1}{(\mu + 1)^2} \left(1 - \frac{\mu - 1}{\mu + 1} e^{-2\vartheta}\right) \left(1 - \left(\frac{\mu - 1}{\mu + 1}\right)^2 e^{-2\vartheta}\right)^2\right] = \frac{1}{x \vartheta} \frac{\vartheta^2 - 1}{\vartheta^2 + 1}$$

where $\alpha = h \sqrt{\alpha}/\sqrt{\beta}$. For $\mu$ near to 1 this is easily seen to be uniquely solvable for $\vartheta$. Power series expansion shows

$$\vartheta^{cr}(\mu) = 1 + \alpha e^{-2\vartheta}(\mu - 1) + O((\mu - 1)^2),$$

$$q_1^{cr}(\mu) = 1 + \frac{1}{2} e^{-2\vartheta}(\mu - 1) + O((\mu - 1)^2) .$$

(1.25)

Let $H_s (s \text{ real})$ be the Sobolev space of $\Lambda$-periodic functions (resp. distributions) (1.19) with finite norm

$$||h||_s^2 = |h_0|^2 + \sum_{\omega \in \Lambda^*} |\omega|^{2s} |h_\omega|^2$$

and $\tilde{H}_s$ that subspace of functions in $H_s$ satisfying (1.2). Obviously $D^2E(0; \mu, q_1, q_2)$ is continuous on $\tilde{H}_1 \times \tilde{H}_1$.

If we define the critical "wavelength" to be

$$q_2^{cr}(\mu) = \frac{2}{\sqrt{3}} \vartheta^{cr} = \frac{2}{\sqrt{3}} \left(1 - \alpha e^{-2\vartheta}(\mu - 1) + O((\mu - 1)^2)\right) .$$

(1.26)
then Lemma 1.2 implies positivity of $D^2E(0; \mu, q_1, q_2)$ on $\hat{H}_1 \times \hat{H}_1$ as long as $0 \leq q_1 < q_1^{cr}$, whereas

$$D^2E(0; \mu, q_1^{cr}, q_2^{cr}) \{ h^2 \} = (q_2^{cr})^2 \sum_{|\omega| > 2/\sqrt{3}} Q \left( \frac{|\omega|}{q_2^{cr}}, \mu, q_1^{cr} \right) |h_\omega|^2$$

possesses the six-dimensional kernel

$$N_6 : h = \sum_{|\omega| = 2/\sqrt{3}} h_\omega e^{i\omega x}, \quad h_{-\omega} = \overline{h_\omega}.$$ 

Accordingly $\zeta = 0$ loses its stability as $q_1$ crosses $q_1^{cr}$.

§ 2

In this section, assuming $s \geq 5/2$, we look at $E$ as a functional on the spaces $\hat{H}_s$.

**Theorem 2.1**: Assume $s \geq 5/2$, then (i) $F(\zeta; \mu, q)$ as defined by (1.5), (1.6) is analytic as a map of a neighbourhood $\mathcal{N}$ of any $(0, 1, q_0)$ in $\hat{H}_s \times \mathbb{R}^2$ into $\mathbb{R}$ and (ii) its derivative $DF$ (with respect to $\zeta$) maps $\mathcal{N}$ into $\hat{H}_{s-1}$ analytically.

This Theorem is proved in § 3. As an immediate consequence of (ii) and Lemma 1.2 we get,

**Corollary 2.1**: Let $s \geq 5/2$, (i) $E(\zeta; \mu, (1 + \epsilon) q_1^{cr}(\mu), q_2^{cr}(\mu))$ is analytic from a neighbourhood of $(\zeta; \epsilon, \mu) = (0; 0, 1)$ in $\hat{H}_s \times \mathbb{R}^2$ into $\mathbb{R}$, (ii) $D E(\zeta; \mu, (1 + \epsilon) q_1^{cr}(\mu), q_2^{cr}(\mu))$ considered as a map from $\hat{H}_s \times \mathbb{R}^2$ into $\hat{H}_{s-2}$ is analytic at $(0; 0, 1)$.

Corollary 2.1 implies by interpolation

**Corollary 2.2**: Let $s \geq 5/2$, then $D^2E(\zeta; \mu, (1 + \epsilon) q_1^{cr}(\mu), q_2^{cr}(\mu))$ — originally considered on $\hat{H}_s \times \hat{H}_s$ — is continuous on $\hat{H}_s \times \hat{H}_1$. Its continuous extension on $\hat{H}_s \times \hat{H}_1$ considered as a map from $\hat{H}_s \times \mathbb{R}^2$ into $L(\hat{H}_1, \hat{H}_1; \mathbb{R})$ is analytic at $(0; 0, 1)$.

**Proof**: Let

$$E(\zeta; \mu, (1 + \epsilon) q_1^{cr}(\mu), q_2^{cr}(\mu)) = \sum_{i,j,k \geq 0} \epsilon^i (\mu - 1)^j E_{ijk+2}(\zeta^{k+2})$$

be the power series expansion of $E$; $E_{ijk+2}$ denoting certain symmetric and continuous $(k + 2)$-linear forms in $\zeta^{k+2} = (\zeta, \ldots, \zeta) \in \hat{H}_{s+2}$. Analyticity of

$$DE(\zeta; \mu, (1 + \epsilon) q_1^{cr}, q_2^{cr}) = \sum_{i,j,k \geq 0} (k + 2) \epsilon^i (\mu - 1)^j E_{ijk+2}(\zeta^{k+1}, \cdot)$$

as a mapping from $\hat{H}_s \times \mathbb{R}^2$ into $\hat{H}_{s-2}$ — as referred to in Corollary 2.1 — by definition means convergence of

$$\sum_{i,j,k \geq 0} (k + 2) \|E_{ijk+2}\| \epsilon^i (\mu - 1)^j \zeta^{k+1}$$ 

in some neighbourhood of $(0; 0, 1)$ in $\mathbb{R}^3$ where $\|E_{ijk+2}\|$ is defined by

$$\|E_{ijk+2}\| = \sup_{\|h\|_1 < 1} |E_{ijk+2}(\zeta^{k+1}, h)|.$$
Since
\[ |E_{ijk+2}(\zeta^k, h_1, h_2)| \leq \frac{(k + 1)^{k+1}}{(k + 1)!} \|E_{ijk+2}\| \|\zeta^k\| \|h_1\| \|h_2\|_{l-3}, \]
(cf. [4]) we get by interpolation
\[ |E_{ijk+2}| = \sup_{\|\zeta\| \|h_1\| \|h_2\| \leq 1} |E_{ijk+2}(\zeta^k, h_1, h_2)| \leq C \frac{(k + 1)^{k+1}}{(k + 1)!} \|E_{ijk+2}\| \tag{2.2} \]
where the constant $C$ is independent of $i$, $j$, $k$ (see e.g. [6]). From (2.1), (2.2) we deduce the convergence of
\[ \sum_{i,j,k \geq 0} (k + 2) (k + 1) |E_{ijk+2}| \varepsilon^j (\mu - 1)^j \zeta^k \]
in a neighbourhood of $(0; 0, 1) \in \mathbb{R}^3$ and hence the analyticity of
\[ D^2E(\zeta^k; \mu, (1 + \varepsilon) q_1^{cr}, q_2^{cr}) = \sum_{i,j,k \geq 0} (k + 2) (k + 1) \varepsilon^j (\mu - 1)^j E_{ijk+2}(\zeta^k, \cdot, \cdot) \]
at $(0; 0, 1) \in \hat{H} \times \mathbb{R}^2$ considered as a mapping from $\hat{H} \times \mathbb{R}^3$ into $L(\hat{H}_1, \hat{H}_1; \mathbb{R})$.

In the following let $Lk = -\Delta k + \frac{4}{3} k - \frac{4}{\sqrt{3}} Ah$ denote the linear operator defined by the quadratic form (1.24) at $(\mu, q_1, q_2) = (1, q_1^{cr}(1), q_2^{cr}(1)) = (1, 1, 2/\sqrt{3})$. Obviously $L \in L(\hat{H}_s, \hat{H}_{s-2})$ for any real $s \geq 2$, its range in $\hat{H}_{s-2}$ being $\hat{H}_{s-2} \ominus N_6$. Further:
$L$ acts as an isomorphism onto $\hat{H}_{s-2} \ominus N_6$ when restricted to $\hat{H}_s \ominus N_6$. (*)

We are now in position to solve the equilibrium condition:
\[ \langle DE(\zeta; \mu, (1 + \varepsilon) q_1^{cr}(\mu), q_2^{cr}(\mu)), h \rangle = 0, \quad \zeta \in \hat{H}_s, \quad \forall h \in \hat{H}_s. \tag{2.3} \]
near $(\zeta; \varepsilon, \mu) = (0; 0, 1)$ for $\zeta$. According to Corollary 2.1
\[ E(\zeta; \mu, (1 + \varepsilon) q_1^{cr}(\mu), q_2^{cr}(\mu)) \]
\[ = \frac{1}{2} (Lk, \zeta) + \sum_{i+j>0} \varepsilon^j (\mu - 1)^j E_{ij2}(\zeta^2) + \sum_{i+j,k \geq 0} \varepsilon^j (\mu - 1)^j E_{ijk+3}(\zeta^{k+3}) \tag{2.4} \]
where $E_{003} = E_{103} = 0$ and
\[ E_{003}^2 \varepsilon^j (\mu - 1)^j E_{ij2}(\zeta^2) \]
\[ = \frac{(q_2^{cr})^2}{2} |\mathcal{P}| \sum_{\omega \in \Lambda} Q \left( \frac{\omega}{q_2^{cr}}, \mu, (1 + \varepsilon) q_1^{cr} \right) |\zeta_\omega|^2 - \frac{1}{2} (Lk, \zeta), \]
\[ E_{013} = \frac{1}{\sqrt{3}} \left( \left( \frac{1}{2} A \xi^2 - \zeta A \xi, A \xi \right) + (\zeta^2, \Delta e^{-\sqrt{3} A} \xi) \right), \tag{2.5} \]
\[ E_{004} = \frac{1}{6} \sqrt{3} \left( 3(\zeta^2, A^3 \xi^2) - 4(\zeta, A^3 \xi^3) \right) - \frac{1}{8} \int_{\mathcal{P}} |\nabla \xi|^4 \, dx_1 \, dx_2, \]
cf.
(1.22)—(1.26).
Remark 2.1: Note that (i): \( \sum_{i+j \geq 0} \epsilon^i (\mu - 1)^j E_{ij2}(\zeta, h) = \frac{(q_2 c r)^2}{2} Q(\theta c r, \mu, (1 + \epsilon) q_1 c r)(\zeta, h) \)

\[ = -\frac{2}{\sqrt{3}} (\zeta, h) q_1 c r q_2 c r \cdot e^{\frac{1 - \mu}{\mu + 1} e^{-2\theta c r}} \]

1 - \left( \frac{\mu - 1}{\mu + 1} \right)^2 e^{-2\theta c r} \]

if \( \zeta, h \in N_6 \) as a consequence of (2.5) and Lemma 1.2.

Let denote:

\[ E_{\text{red}} = \sum_{i+j \geq 0} \epsilon^i (\mu - 1)^j E_{ij2}(\zeta^2) + (\mu - 1) E_{013}(\zeta^3) + E_{004}(\zeta^4) \]

and \( E_{\text{res}} \) the higher order terms \((i + j + k \geq 2)\) in (2.4):

\[ E(\zeta; \mu, (1 + \epsilon) q_1 c r, q_2 c r) = \frac{1}{2} (L \zeta, \zeta) + E_{\text{red}} + E_{\text{res}}. \]

Setting \( \zeta = \zeta_1 + \zeta_2 \) \((\zeta_1 \in N_6, \zeta_2 \in \hat{H}_3) \) then (2.3) will be equivalent to

\[ (L \zeta_2, h) = -\langle (D E_{\text{red}} + D E_{\text{res}})(\zeta_1 + \zeta_2; \epsilon, \mu), h \rangle \quad \forall \ h \in \hat{H}_3 \subseteq N_6, \]

\[ 0 = \langle (D E_{\text{red}} + D E_{\text{res}})(\zeta_1 + \zeta_2; \epsilon, \mu), h \rangle \quad \forall \ h \in N_6. \]

Because of Corollary 2.1 the linear functional on the right-hand side actually belongs to \( \hat{H}_3 \). Thus, according to (*) equation (2.6) can be solved for \( \zeta_2 \) via the contraction mapping theorem:

\[ \zeta_2 = \sum_{i+j+k \geq 1} \epsilon^i (\mu - 1)^j Z_{ijk}(\zeta^k). \]

where \( Z_{ij1} = Z_{002} = 0 \) \((i, j \geq 0)\) by comparison of coefficients (cf. Remark 2.1).

Substituting (2.8) into (2.7) we get the bifurcation equations

\[ \langle D E_{\text{red}}(\zeta_1; \epsilon, \mu), h \rangle + \text{h.o.t.} = 0 \quad \forall \ h \in N_6. \]

the higher order terms (h.o.t.) being of order \( \epsilon^i (\mu - 1)^j \| \zeta_1 \|^{k+2} \) \((i + j + k \geq 2)\).

Introducing on \( N_6 \) the real valued Fourier-coefficients \( \zeta_{a, i} = a_1 - i b_1, \zeta_{a, i} = a_2 - i b_2, \zeta_{a, i} = a_3 - i b_3 \),

\[ a_2 = a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2, \]

\[ a_3 = a_1 a_2 a_3 - a_1 b_2 b_3 - a_2 b_3 b_1 - a_3 b_1 b_2, \]

\[ a_4^{(1)} = (a_2^2 + b_2^2)^2 + (a_1^2 + b_1^2)^2 + (a_2^2 + b_2^2)^2 + (a_3^2 + b_3^2)^2, \]

\[ a_4^{(2)} = (a_1^2 + b_1^2) (a_2^2 + b_2^2) (a_3^2 + b_3^2) \]

\[ + (a_1^2 + b_1^2) (a_3^2 + b_3^2), \]

it is easy to check that for \( \zeta_1 \in N_6 \)

\[ E_{\text{red}} = \frac{2}{3} \left( 1 + O(\mu - 1) \right) \epsilon \sigma_2 \]

\[ - \frac{8}{\sqrt{3}} (1 + 2e^{-2\theta})(\mu - 1) \sigma_3 + \frac{20}{9} a_4^{(1)} + \frac{8}{3} \left( \frac{4}{\sqrt{3}} - 5 \right) a_4^{(2)} \]

(2.10)
We now establish the existence of two types of solutions for which (2.9) reduces to a scalar equation. Considering the translational invariance of the energy functional we may assume $b_1 = b_2 = 0$ without loss in generality:

(I): $a_2 = a_3 = b_3 = 0$, then (2.9) reduces to

$$\frac{\partial}{\partial a_1} E_{\text{red}}(\zeta_1; e, \mu) + \text{h.o.t.} = 0, \quad \zeta_1 = 2a_1 \cos \omega_1 x,$$

(2.11)

(II): $a_1 = a_2 = a_3, b_3 = 0$, then (2.9) reduces to

$$\frac{\partial}{\partial a_1} E_{\text{red}}(\zeta_1; e, \mu) + \text{h.o.t.} = 0,$$

$$\zeta_1 = 2a_1(\cos \omega_1 x + \cos \omega_2 x + \cos (\omega_1' + \omega_2') x).$$

(2.12)

Remark 2.2: As a consequence of the invariance of $E$ under the group of rigid motions every solution of (2.11) (resp. (2.12)) satisfies the complete equilibrium conditions. For, let $T$ be the representation of the translational group defined on $\hat{H}_s$ as usual. Then

$$\langle DE(\zeta), \zeta \rangle = 0 \quad \text{for all } \zeta \in \hat{H}_s;$$

$$\langle DE(\zeta), h \rangle = \langle DE(T\zeta), T,h \rangle \quad \text{for all } \zeta, h \in \hat{H}_s,$$

hence $\zeta_2(T_1\zeta_1) = T_1\zeta_2(\zeta_1)$ in (2.8). In particular, if $\zeta_1$ is a solution of (2.11) then

$$\langle DE(\zeta_1 + \zeta_2), 2a_1 \sin \omega_1 x \rangle = -\left(\frac{\partial \zeta_1}{\partial x} + \frac{\partial \zeta_2}{\partial x}\right) = 0$$

in virtue of $\frac{\partial \zeta_2}{\partial x} \in \hat{H}_{s-1} \ominus N_4$. Further, if $\tau: x \rightarrow x + \frac{\omega_2}{2}$

$$\langle DE(\zeta_1 + \zeta_2), e^{i\omega_2 x} \rangle = \langle DE(\zeta_1 + \zeta_2), e^{i\omega_2 x} \rangle = 0,$$

whence $\langle DE(\zeta_1 + \zeta_2), e^{i\omega_2 x} \rangle = 0$ as desired. Similarly $\langle DE(\zeta_1 + \zeta_2), e^{-i(\omega_1' + \omega_2')}x \rangle = 0$

Similar considerations apply to solutions of (2.12).

Returning to (2.11), (2.12) and applying the Weierstrass Preparation Theorem we arrive at the "reduced" bifurcation equation

$$\frac{\partial}{\partial a_1} E_{\text{red}}(\zeta_1; e, \mu) = 0.$$  

(2.13)

Solving (2.13) we get in view of (2.10)

(I): $a_1^\pm \approx \pm \sqrt{\frac{3}{5}} e, (e \geq 0)$,

(2.14)

(II): $a_1^\pm \approx 0.078(1 + 2e^{-2a})(\mu - 1)$

$$\pm \{0.078^2(1 + 2e^{-2a})^2(\mu - 1)^2 + 0.181e\}^{1/2}$$

(2.15)

in cases (I), (II) respectively. In fact both solutions (2.14) coincide under translation $x \rightarrow x + \frac{\omega_1}{2}$. 
Remark 2.3: In (2.14), (2.15) those terms from (2.13) have been dropped which carry no information about the actual solution. Observe that (2.15) is sufficient as an approximation to (2.13) only in a restricted neighbourhood \( |\varepsilon| < \varepsilon_0 (\mu - 1)^2 \) of \((0, 1)\). Further: Suitable resealing of (2.11) resp. (2.12) via the Implicit Function Theorem leads to power series expansions

\[
\begin{align*}
(I): & \quad a_{1}^\pm = \pm \sqrt{3} \varepsilon \left\{ \sqrt{\frac{3}{5}} + \sum_{i+j>0} a_{ij} \varepsilon (\mu - 1)^j \right\}, \\
(II): & \quad a_{1}^+ = (\mu - 1) \left( 0.156 (1 + 2e^{-2\varepsilon}) + \sum_{i+j>0} a_{ij} \left( \frac{\varepsilon}{(\mu - 1)^2} \right)^i (\mu - 1)^j \right) \\
& \quad + \sum_{i+j>0} a_{ij} \left( \frac{\varepsilon}{(\mu - 1)^2} \right)^i (\mu - 1)^j.
\end{align*}
\]

We conclude this section with a stability result. By definition stability of a solution \( \zeta \in \dot{H}_s \) means

\[ D^2E(\zeta) (h^2) > 0 \quad \text{for all } h \in \dot{H}_s, h \neq 0. \quad (2.16) \]

Considering the translational invariance of \( E \) which implies

\[ D^2E(\zeta) (\zeta_x, h) = D^2E(\zeta) (\zeta_y, h) = 0 \quad \text{for all } h \in \dot{H}_s \]

we have to impose some additional constraint, e.g. \( h \perp \frac{\partial \zeta_1}{\partial x}, \frac{\partial \zeta_1}{\partial y} \) in (2.16).

**Theorem 2.2:** Solutions of type \( \text{II}^- \) lead to unstable equilibria, whereas the branch \( \text{II}^+ \) is stable in the sense above.

**Proof:** We first study the branch \( \text{II}^+ \). Inspecting the expansion of the second variation \( D^2E \) along our solution \( \zeta = \zeta_1 + \zeta_2 (\zeta_1 \in N_6, \zeta_2 \in \dot{H}_s \cap N_6) \) yields

\[
D^2E(\zeta, \mu; (1 + \varepsilon) q_1^{cr}(\mu), q_2^{cr}(\mu)) (h^2) = (Lh, h) + 6(\mu - 1) E_{013}(\zeta_1, h^2) + 12E_{004}(\zeta_1^2, h^2) + \text{h.o.t.,}
\]

the higher order terms being of order \( \left( \frac{\varepsilon}{(\mu - 1)^2} \right)^{i+j} ; i + j > 0 \). Let \( h = h_1 + h_2 (h_1 \in N_6, h_2 \in \dot{H}_s \cap N_6) \), then

\[
(Lh, h) \geq c ||h_2||^2, \quad c > 0,
\]

\[
| (\mu - 1) E_{013}(\zeta_1, h_1, h_2) | | E_{004}(\zeta_1^2, h_1, h_2) |
\]

\[
\leq C(\mu - 1)^2 ||h_1|| ||h_2|| \leq \frac{C}{2} (|\mu - 1|^3 ||h_1||^2 + |\mu - 1||h_2||^2).
\]

(cf. Corollary 2.2). Thus, for \((\varepsilon, \mu)\) in a sufficiently small neighbourhood \( |\varepsilon| \leq \varepsilon_0 \times (\mu - 1)^2 \) of \((0, 1)\), positivity of (2.17) is implied by that of

\[
6(\mu - 1) E_{013}(\zeta_1, h_1^2) + 12E_{004}(\zeta_1^2, h_1^2)
\]

(2.18)

this being true even if we replace \( \zeta_1 \) by its first approximation \( 0.312(1 + 2e^{-2\varepsilon}) \times (\mu - 1)(\cos \omega_1 x + \cos \omega_2 x + \cos (\omega_1' + \omega_2') x). \) Since the eigenvalues

\[
(\lambda_1, \lambda_{23}, \lambda_4, \lambda_{56}) = |\mathcal{P}| (0.72, 1.37, 2.17, \theta) (1 + 2e^{-2\varepsilon})^2 (\mu - 1)^2
\]

\[ \lambda_1, \lambda_{23}, \lambda_4, \lambda_{56} = |\mathcal{P}| (0.72, 1.37, 2.17, \theta) (1 + 2e^{-2\varepsilon})^2 (\mu - 1)^2 \]
of the so modified form (2.18) are positive (with the exception of \( \lambda_5, 6 \)) the branch \( \Pi^+ \) is stable on both sides of criticality.

Similarly

\[
D^2E(\zeta; \mu, (1 + \varepsilon) q_1^{cr}, q_2^{cr}) \{ h^2 \} = (Lh, h) \\
+ 2\varepsilon E_{102}(h^2) + 6(\mu - 1) E_{013}(\zeta_1, h^2) + \text{h.o.t.}
\]

along the branch \( \Pi^- \), the higher order terms now being of order \( \left( \frac{\varepsilon}{(\mu - 1)^2} \right)^{1+i} \times (\mu - 1)^{2+i}, i + j > 0. \) Thus, on a neighbourhood of \((0, 1)\) as above, its sign is determined by that of

\[
2\varepsilon E_{102}(h_1^2) + 6(\mu - 1) E_{013}(\zeta_1, h_1^2).
\]

Replacing \( \zeta_1 \) in (2.19) by

\[
-(\frac{4}{\sqrt{3}} (1 + 2e^{-2\varepsilon})^{-1} (\mu - 1)^{-1} \cos \omega_1 x + \cos \omega_2 x + \cos (\omega_1 + \omega_2) x)
\]

we get,

\[
(\lambda_1, \lambda_{2,3}, \lambda_4, \lambda_5, 6) = \frac{16}{3} |\mathcal{P}| \begin{pmatrix} 1, -2, -3, 0 \end{pmatrix} \varepsilon
\]

for the corresponding eigenvalues. Hence the solution \( \Pi^- \) turns out to be unstable always.

Concerning the branch \( \Pi^- \) one finds

\[
D^2E(\zeta; \mu, (1 + \varepsilon) q_1^{cr}, q_2^{cr}) \{ h^2 \} = (Lh, h) \\
+ 6(\mu - 1) E_{013}(\zeta_1, h^2) + 12 E_{004}(\zeta_1^2, h^2) + \text{h.o.t.}
\]

with h.o.t. of order \( \sqrt{\varepsilon^{1+i}(\mu - 1)^i}, i + j > 1. \) Thus its stability is determined by the eigenvalues of \( 2\varepsilon E_{102}(h_1^2) + 6(\mu - 1) E_{013}(\zeta_1, h_1^2) + 12 E_{004}(\zeta_1^2, h_1^2). \) After replacing \( \zeta_1 \) by (2.14) we get for their values

\[
(\lambda_1, \lambda_{2,3}, \lambda_4, \lambda_5, 6) = |\mathcal{P}| \begin{pmatrix} 1, 10.66, 0.83\varepsilon \pm 3.57 (1 + 2e^{-2\varepsilon}) \sqrt{\varepsilon} (\mu - 1), 0.83\varepsilon \mp 3.57(1 + 2e^{-2\varepsilon}) \sqrt{\varepsilon} (\mu - 1), 0 \end{pmatrix},
\]

showing the (supercritical) solution \( \Pi^- \) to be unstable.

Remark 2.4: Concerning the value of \( E \) at a solution \( \Pi^+ \) we get

\[
E(\zeta; \mu, (1 + \varepsilon) q_1^{cr}, q_2^{cr}) = (\mu - 1) E_{013}(\zeta_1^3) + E_{004}(\zeta_1^4) + \text{h.o.t.}
\]

\[
= -0.004 |\mathcal{P}| (1 + 2e^{-2\varepsilon})^4 (\mu - 1)^4 + \text{h.o.t.} < 0
\]

with higher order terms \( \left( \frac{\varepsilon}{(\mu - 1)^2} \right)^{1+i} (\mu - 1)^{4+i}, i + j > 0. \)

\section{3}

We now pass to the proof of Theorem 2.1. We adopt the following notation: For any open interval \( I \) on the \( z \)-axis let \( \|u\|_I \) denote the norm of \( u \) in \( L_2(I) \). Let

\[
W_{m, i} = \left\{ v \in L_2(I, H_m) : v^{(m)} = \frac{\partial^m v}{\partial z^m} \in L_2(I, H_0) \right\}
\]
be the Sobolev space of $\Lambda$-periodic functions
\[ v(x, y, z) = \sum_{\omega \in \Lambda} v_{\omega}(z) e^{i\omega z} \]
with distributional derivatives up to order $m$ in $L_2(\mathcal{P} \times I)$, the derivatives up to order $m - 1$ being $\Lambda$-periodic again. Choose
\[ \|v\|_{m, I}^2 = \|v_0\|^2 + \sum_{\omega \in \Lambda'} (|\omega|^{2m} \|v_\omega\|^2 + \|v^{(m)}_\omega\|^2) \]
to be the norm in $W_{m, I}$. Similarly, for $m \geq 1$, let
\[ V_{m, I} = \{ v \in \mathcal{D}'(I, H_m) : v', v^{(m)} \in L_2(I, H_0) \} \]
normed by
\[ \|v\|_{m, I}^2 = \|v_0\|^2 + \sum_{\omega \in \Lambda'} (|\omega|^{2m} \|v_\omega\|^2 + \|v^{(m)}_\omega\|^2). \quad (3.1) \]

In the following for $I^- = (-\infty, -\eta_0)$, $I^n = (-\eta_0, 0)$, $I^+ = (0, +\infty)$ we shall consider the various spaces $L_2(I^-) \times L_2(I^n) \times L_2(I^+), W_m = W_{m, I^-} \times W_{m, I^n} \times W_{m, I^+}, V_m = V_{m, I^-} \times V_{m, I^n} \times V_{m, I^+}$, the corresponding norms of which we denote by $\|\|_m$, $\cdot \|_m$ resp. Further let $\mathcal{F}^\pm = \mathcal{P} \times I^\pm, \mathcal{F}^n = \mathcal{P} \times I^n$. Accordingly, we write $v = (v', v^n, v^+) = \mathcal{V}_m$ for a function belonging to $W_m$ (resp. $V_m$).

Lemma 3.1: Let $\mu > 0$ and $\mathcal{F} \in \{W_m\}^3$, then the unique $v \in V_\mu$ which satisfies $v^+(x, y, 0) - v^n(x, y, 0) = \text{const.}$, $v^n(x, y, -\eta_0) - v^-(x, y, -\eta_0) = \text{const.}$ and
\[ \int_{\mathcal{F}^\pm} \nabla v \cdot \nabla \varphi \, dV + \mu \int_{\mathcal{F}^n} v' \nabla \varphi \, dV = \int_{\mathcal{F}^n} \nabla \varphi \, dV + \int_{\mathcal{F}^\pm} \nabla \varphi \, dV \]
for every $\varphi \in V_1$ (satisfying the homogeneous jump conditions) belongs to $V_{m+1}$. Moreover
\[ \|v\|_{m+1} \leq C \|\|_m \]
with a constant $C$ independent of $\mathcal{F}$.

Proof: For convenience we assume $\mu = 1$. Getting the estimate (3.3) for general $\mu > 0$ requires minor modifications only. Let $\mathcal{F} = (f_1, f_2, f_3)$ and
\[ f_j = \sum_{\omega \in \Lambda'} f_{j, \omega}(z) e^{i\omega z} \quad (j = 1, 2, 3) \]
its Fourier expansion (notice: $f_j = (f_j^-, f_j^n, f_j^+)$). We set $f_{j, \omega} = (f_{j, \omega}^-, f_{j, \omega}^0, f_{j, \omega}^+, f_{j, \omega}^{II, \omega})$. Obviously the Fourier coefficients $v_{\omega}(z)$ of our solution have to satisfy the variational equations
\[ \int_{I^{-}} \left( v_{\omega} \varphi_{\omega}' + |\omega|^2 v_{\omega} \varphi_{\omega} \right) dz + \int_{I^{+}} \left( v_{\omega} \varphi_{\omega}' + |\omega|^2 v_{\omega} \varphi_{\omega} \right) dz \\
= \int_{I^{-}} \left( f_{3, \omega} \varphi_{\omega}' - i \omega f_{2, \omega} \varphi_{\omega} \right) dz + \int_{I^{+}} \left( f_{3, \omega} \varphi_{\omega}' - i \omega f_{2, \omega} \varphi_{\omega} \right) dz \quad (3.4) \]
subject to the jump conditions $v_0^+(0) - v_0^n(0) = \text{const.}, v_0^n(-\eta_0) - v_0^-(\eta_0) = \text{const.}$, resp. $v_{\omega}^+(0) - v_{\omega}^0(0) = 0, v_{\omega}^0(-\eta_0) - v_{\omega}^-(\eta_0) = 0$ if $\omega \neq 0$. Choosing the test function $\varphi_{\omega}$ in (3.4) to be $v_{\omega}$ and applying Schwarz's inequality we get
\[ \|v_{\omega}'\|^2 + |\omega|^2 \|v_{\omega}\|^2 \leq \|\varphi_{\omega}\| \|v_{\omega}\|, \]
whence
\[ \|v_{\omega}'\|^2 + |\omega|^2 \|v_{\omega}\|^2 \leq 2 \||\omega||v_{\omega}\|^2 \]
This proves (3.3) for $m = 0$. Likewise by differentiating the Euler-Lagrange equations to (3.4) we get
\[ -v_{\omega}^{(m+1)} + |\omega|^2 v_{\omega}^{(m-1)} = -i \omega f_{\omega}^{(m-1)} - f_{3, \omega}^{(m)} \quad (m \geq 1). \]
Thus
\[ \|v_{\omega}^{(m+1)}\|^2 \leq 3(|\omega|^4 \|v_{\omega}^{(m-1)}\|^2 + |\omega|^2 \|f_{\omega}^{(m-1)}\|^2 + \|f_{3, \omega}\|^2). \]
Applying the well known inequality
\[ \|u_\omega^{(m)}\|^2 \leq \text{const.} \left( \varepsilon^k \|u_\omega^{m+k}\|^2 + \frac{1}{\varepsilon^{m-l}} \|u_\omega^{(l)}\|^2 \right), \quad \varepsilon > 0 \]
and recalling (3.1), (3.5) gives the desired estimate for \( m \geq 1 \).

**Proof of Theorem 2.1:** The strategy is to transform (1.5) into a variational problem posed on the fixed domain \((\mathcal{F}^+, \mathcal{F}^-)\). Note that by interpolation it is sufficient to assume \( s = m + 1/2, 2 \leq m \in \mathbb{N} \).

Now, for \((\zeta, q) \in \dot{H}_{m+1/2} \times \mathbb{R}\) in a neighbourhood of \((0, -q_0)\) let
\[ x_1 = x, x_2 = y, x_3 = z + w(x, y, z) \]  
(3.6)
define a diffeomorphism from \(\mathcal{F}^-\) (resp. \(\mathcal{F}^+\)) as defined above onto \(\Omega^-\) (resp. \(\Omega^+, \Omega^+\)) cf. (1.1). As the example
\[ w^- = q_0 - q, w^+ = \sum_{\omega \neq 0} \zeta_{\omega} e^{\omega x - |\omega| z}, \]
\[ w^{fl} = \frac{q_0 + q}{-q_0} z + \sum_{\omega \neq 0} \zeta_{\omega} e^{\omega x} \left( \frac{e^{-|\omega| z}}{1 - e^{2|\omega| z}} + \frac{e^{\omega |z|}}{1 - e^{-2|\omega| z}} \right) \]
suggests, it is always possible to require: 1st the transition function \(w = (w^-, w^{fl}, w^+)\) to belong to \(V_{m+1}\) and 2nd the map
\[ (\zeta, q) \in \dot{H}_{m+1/2} \times \mathbb{R} \rightarrow w \in V_{m+1} \]
to be analytic at \((0, -q_0)\). Let \(w = 0\) when \((\zeta, q) = (0, -q_0)\).

According to (3.6) the variational problem (1.5), (1.6) transforms into
\[ \int_{\mathcal{F}^+ \cup \mathcal{F}^-} \left( |\nabla v|^2 (1 + w) - 2v \nabla v \nabla w + v^2 \frac{|\nabla w|^2}{1 + w} \right) dV \]
\[ + \mu \int_{\mathcal{F}^+} \left( |\nabla v|^2 (1 + w) - 2v \nabla v \nabla w + v^2 \frac{|\nabla w|^2}{1 + w} \right) dV \rightarrow \min \]
(3.7)
\((dV = dx dy dz)\), which has to be solved for \(v(x, y, z) = u(x_1, x_2, x_3)\) subject to
\[ v^+(x, y, 0) - v^{fl}(x, y, 0) = \zeta(x, y) + \text{const.}, \quad v^{fl}(x, y, -q_0) - v^-(x, y, -q_0) = \text{const.} \]
To show, in a first step, the analytic dependence of its solution \(v\) on \((\zeta, \mu, q)\) near \((0, 1, q_0)\) set \(v = v_1 + v_2\) where
\[ v_1 = \frac{1}{2} \text{sgn } z \sum_{\omega \neq 0} \zeta_{\omega} e^{\omega x + |\omega| |z|} \]
is the solution to
\[ \int_{\mathcal{F}^+ \cup \mathcal{F}^-} |\nabla v_1|^2 dV + \int_{\mathcal{F}^+} |\nabla v_1|^2 dV \rightarrow \min \]
subject to \(v^{+1}(x, y, 0) - v^{1fl}(x, y, 0) = \zeta(x, y) + \text{const.}, \quad v^{1fl}(x, y, -q_0) - v^{-1}(x, y, -q_0) = \text{const.} \)
Notice \(|v_1|_{m+1} = \frac{1}{\sqrt{2}} \|\zeta\|_{m+1/2} \). Then, collecting higher order terms
\[ f = f(\zeta, \mu, q; v_2) \]and denoting \(e_2 = (0, 0, 1)\):
\[ f^+ = -w_z \nabla v + v_z \nabla w + (\nabla v \nabla w - \frac{v_z |\nabla w|^2}{1 + w_z}) e_z, \]
\[ f^{fl} = -(\mu - 1) \nabla v_1 - \mu w_z \nabla v + \mu v_z \nabla w + \mu \left( \nabla v \nabla w - \frac{v_z |\nabla w|^2}{1 + w_z} \right) e_z \]  
(3.8)
on the right-hand side, \( v_2 \) has to satisfy a variational equation subject to homogeneous jump conditions as referred to in Lemma 3.1. If \( T \in L(W_m^3, V_{m+1}) \) denotes the solution map for (3.2) this equation will be equivalent to

\[
P_2 = T[i(\xi, \mu, q; v_2)].
\]

Recall that the spaces \( W_s \) form Banach algebras provided that \( s > 3/2 \) (see [1]). Hence, under the assumption \( m \geq 2 \), mapping \( \tilde{f} \) which transforms \((\xi, \mu, q; v_2) \in \dot{H}_{m+1/2} \times \mathbb{R}^2 \times V_{m+1} \) according to (3.8) into \( i(\xi, \mu, q; v_2) \in W_m \) turns out to be analytic at \((0, 1, q_0, 0)\). Thus we can solve (3.9) via the contraction mapping theorem for \( v_2 \in \tilde{V}_{m+1} \) as an analytic function of \((\xi, \mu, q) \in \dot{H}_{m+1/2} \times \mathbb{R}^2 \) near \((0, 1, q_0)\). Obviously this implies \( v = v_1 + v_2 \) to be analytic too.

We proceed by expanding the minimal value (3.7). Its analytic dependence on \((v, w, \mu) \in V_1 \times \tilde{V}_{m+1} \times \mathbb{R}\) is easily seen by Sobolev’s embedding theorem. Replacing \( v, w \) by its power series expansions we get analyticity of (3.7) as a function of \((\xi, \mu, q) \in \dot{H}_{m+1/2} \times \mathbb{R}^2 \). This proves part (i) of the theorem.

The remaining part (ii) now follows in a few lines. Observe the earlier formula (1.9) — obtained in § 1 by formal differentiation — actually to be valid in virtue of the present hypothesis. By transforming (1.9) according to (3.5) we get

\[
\langle DF(\xi), h \rangle = \int_{\mathcal{P}} \Phi(\nabla \xi, \nabla v|_{z=0}, \nabla w|_{z=0}) h \, dx \, dy
\]

where the integrand is analytic in its arguments. By the trace mapping theorem \( \nabla v|_{z=0}, \nabla w|_{z=0} \in H_{m-1/2} \). Consequently \( \Phi \in H_{m-1/2} \), since the spaces \( H_s \) are Banach algebras provided that \( s > 1 \). This finishes the proof.

**REFERENCES**


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