Linear $q$-Difference Equations

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Abstract. We prove that a linear $q$-difference equation of order $n$ has a fundamental set of $n$-linearly independent solutions. A $q$-type Wronskian is derived for the $n$th order case extending the results of Swarttouw–Meijer (1994) in the regular case. Fundamental systems of solutions are constructed for the $n$-th order linear $q$-difference equation with constant coefficients. A basic analog of the method of variation of parameters is established.

Keywords. $q$-Difference equations, $q$-Wronskian, $q$-type Liouville’s formula

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1. Introduction and basic definitions

In the following, $q$ is a positive number, $0 < q < 1$, and $I$ is an open interval containing zero. Now we state the basic definitions used in this article, cf. [4, 9]. Then we introduce a brief account about the $q$-calculus established in [3]. Let $n \in \mathbb{N}$. The $q$-shifted factorial $(a;q)_n$ of $a \in \mathbb{C}$ is defined by

$$(a;q)_0 := 1 \quad \text{and, for } n > 0, \quad (a;q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}).$$

The multiple $q$-shifted factorial for complex numbers $a_1, \ldots, a_k$ is defined by

$$(a_1, a_2, \ldots, a_k; q)_n := \prod_{j=1}^{k} (a_j; q)_n.$$
The limit \( \lim_{n \to \infty} (a; q)_n \) exists and is denoted by \((a; q)_\infty\). The third type of the \(q\)-Bessel functions of Jackson of order \(\nu\) is defined to be, see [10],

\[
J_{\nu}^{(3)}(x; q) = x^\nu \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(\nu+1)} \frac{x^{2n}}{(q; q)_n(q^{\nu+1}; q)_n}.
\]

We denote this function by \(J_{\nu}(x; q)\) instead of \(J_{\nu}^{(3)}(x; q)\) for simplicity. In some literature this function is called the Hahn–Exton \(q\)-Bessel function, see [12, 14].

The functions \(\cos_q x\) and \(\sin_q x\) are defined for \(x \in \mathbb{C}, \ |x|(1-q) < 1\), by

\[
\cos_q x := \sum_{n=0}^{\infty} (-1)^n \frac{(x(1-q))^{2n}}{(q; q)_2n},
\]

\[
\sin_q x := \sum_{n=0}^{\infty} (-1)^n \frac{(x(1-q))^{2n+1}}{(q; q)_{2n+1}}.
\]

The functions \(\cos(x; q)\) and \(\sin(x; q)\) are defined in \(\mathbb{C}\) by

\[
\cos(x; q) := \sum_{n=0}^{\infty} (-1)^n q^n (x(1-q))^{2n} \frac{1}{(q; q)_{2n}},
\]

\[
\sin(x; q) := \sum_{n=0}^{\infty} (-1)^n q^{n+1} (x(1-q))^{2n+1} \frac{1}{(q; q)_{2n+1}},
\]

and they are \(q\)-analogs of the sine and cosine functions, [4, 9]. See also [1, 2], [5]–[7] for a study of the zeros and completeness of \(q\)-trigonometric and \(q\)-Bessel systems.

Let \(\mu \in \mathbb{R}\) be fixed. A set \(A \subseteq \mathbb{R}\) is called a \(\mu\)-geometric set if for \(x \in A, \mu x \in A\). Now, we define the \(q\)-difference operator of Heine. Let \(f\) be a function defined on a \(q\)-geometric set \(A \subseteq \mathbb{R}\). The \(q\)-difference operator is defined by the formula

\[
D_q f(x) := \frac{f(x) - f(qx)}{x - qx}, \quad x \in A \setminus \{0\}.
\]

If \(0 \in A\), we say that \(f\) has \(q\)-derivative at zero if the limit

\[
\lim_{n \to \infty} \frac{f(xq^n) - f(0)}{xq^n}, \quad x \in A
\]

exists and does not depend on \(x\). In this case, we shall denote this limit by \(D_q f(0)\). In some literature the \(q\)-derivative at zero is defined to be \(f'(0)\) if it exists, cf. [12, 14], but the above definition is more suitable for our approach.

The non-symmetric Leibniz’ rule

\[
D_q(fg)(x) = g(x)D_q f(x) + f(qx)D_q g(x)
\] (1.1)
holds. Relation (1.1) can be symmetrized using the relation \( f(qx) = f(x) - x(1-q)D_q f(x) \), giving the additional term \(-x(1-q)D_q f(x)D_q g(x)\). The q-integration of F. H. Jackson [11] is defined for a function \( f \) defined on a q-geometric set \( A \) to be

\[
\int_a^b f(t) \, dq t = \int_0^b f(t) \, dq t - \int_a^b f(t) \, dq t, \quad a, b \in A,
\]

where

\[
\int_0^x f(t) \, dq t = \sum_{n=0}^{\infty} x^q^n (1-q) f(x^q^n), \quad x \in A,
\]

provided that the series converge.

**Theorem 1.1** ([3]). The q-integral (1.2) exists only if \( \lim_{k \to \infty} x^q_k f(x^q_k) = 0 \).

Consider the non-homogeneous q-difference equation of order \( n \)

\[
a_0(x)D_q^n y(x) + a_1(x)D_q^{n-1} y(x) + \cdots + a_n(x)y(x) = b(x), \quad x \in I,
\]

for which \( a_i, 0 \leq i \leq n \), and \( b \) are continuous at zero functions defined on \( I \) and \( a_0(x) \neq 0 \) for all \( x \in I \). Equation (1.3) together with the initial conditions

\[
D_q^{i-1} y(0) = b_i, \quad b_i \in \mathbb{C}, \quad i = 1, \ldots, n,
\]

form a q-type Cauchy problem. By a solution of problem (1.3)–(1.4), we mean a continuous at zero function which satisfies (1.3) subject to the initial conditions (1.4). According to [3], there exists a unique solution of (1.3)–(1.4) in a subinterval \( J \) of \( I \), \( J = [-h, h], h > 0 \). In the next section, we shall study the \( n \)-th order homogeneous linear equation

\[
a_0(x)D_q^n y(x) + a_1(x)D_q^{n-1} y(x) + \cdots + a_n(x)y(x) = 0, \quad x \in I.
\]

A fundamental set of solutions for (1.5) when the coefficients are constants is derived in §2. In §3, a q-type Wronskian for the solutions of (1.5) is introduced and it is proved that it satisfies a first order q-difference equation and its solution is given. This extends the results of Swarttouw–Meijer [15] in the regular case. As applications, a formula for a solution of (1.3) in terms of a fundamental set of solutions of (1.5) will be given in §4 by using a q-analog of the method of variation of parameters.

### 2. Linear homogeneous q-difference equations

Let \( M \) denote the set of solutions of (1.5) valid in a subset \( J \subseteq I \) which contains zero. Then it is easy to see that \( M \) is a linear space over \( \mathbb{C} \). Also from the existence and uniqueness of the solutions, cf. [3], if \( \phi \in M \) and \( D_q^i \phi(0) = 0, 0 \leq i \leq n-1 \), then \( \phi(x) \equiv 0 \) on \( J \). Moreover, cf. [3], \( \{D_q^i \phi\}_{i=0}^{n-1}, 0 \leq i \leq n-1 \), are continuous at zero for any \( \phi \in M \). A set of \( n \) solutions of (1.5) is said to
be a fundamental set (f.s.) for (1.5) valid in $J$ or a f.s. of $M$ if it is linearly independent in $J$. Moreover, as in differential equations, if $b_{ij}, 1 \leq i, j \leq n,$ are numbers, and, for each $j$, $\phi_j$ is the unique solution of (1.5) which satisfies the initial conditions

$$D_q^{i-1} \phi_j(0) = b_{ij}, \quad 1 \leq i \leq n,$$

then $\{\phi_j\}_{j=1}^n$ is a f.s. of (1.5) if and only if $\det(b_{ij}) \neq 0$. Hence $M$ is a linear space of dimension $n$. In the following we are concerned with constructing a f.s. for (1.5) when it has constant coefficients, $a_r, 0 \leq r \leq n$. Set $L := a_0 D_q^n + a_1 D_q^{n-1} + \cdots + a_n$. Then, (1.5) can be written as

$$L y(x) = a_0 D_q^n y(x) + a_1 D_q^{n-1} y(x) + \cdots + a_n y(x) = 0.$$  \hfill (2.1)

The characteristic polynomial $P(\lambda)$ of (2.1) is defined by

$$P(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n, \quad \lambda \in \mathbb{C}.$$ 

Let $\lambda_i, 1 \leq i \leq k$, denote the distinct roots of $P(\lambda)$ and $m_i$ denotes the multiplicity of $\lambda_i$, so that $\sum_{i=1}^k m_i = n$. Corresponding to each $\lambda_i$ we define an $m_i$-dimensional subspace $M_i$ by

$$M_i = \{v \in M : (D_q - \lambda_i)^{m_i} v = 0\}.$$ 

The construction of a f.s. of (2.1) depends on the fact that, cf. [13],

$$M = M_1 \oplus \cdots \oplus M_k.$$  \hfill (2.2)

**Lemma 2.1.** Let $(X, \mathbb{K})$ be a vector space, and let $T$ be a linear operator on $X$. For any $\lambda \in \mathbb{K}$, if there exist $y_0, y_1, \ldots, y_{m-1}$ in $X$ such that

$$T y_0 = \lambda y_0, \quad y_0 \neq 0$$

$$T y_i = \lambda y_i + y_{i-1}, \quad 1 \leq i \leq m - 1,$$

then $y_1, \ldots, y_{m-1}$ are linearly independent.

**Proof.** By induction on $i$, $0 \leq i \leq m - 1$. \hfill $\square$

**Lemma 2.2.** If $\lambda_i \neq 0$, then the initial value problem

$$D_q \phi_{0,i} = \lambda_i \phi_{0,i}, \quad \phi_{0,i}(0) = 1, r$$

$$D_q \phi_{r,i} = \lambda_i \phi_{r,i} + \phi_{r-1,i}, \quad \phi_{r,i}(0) = 0, \quad r = 1, \ldots, m_i - 1,$$

has the solution

$$\phi_{r,i}(x) = \begin{cases} 
\frac{e_q(\lambda_i x)}{\lambda_i^{r+1}} \sum_{k=0}^\infty \frac{(\lambda_i x)^k}{(q;q)_k}, & r = 0 \\
\frac{1}{\lambda_i^r} \sum_{k=r}^{\infty} \frac{\Gamma(k-1) \cdots (k-r+1) (\lambda_i x)^k}{(q;q)_k}, & r = 1, 2, \ldots, m_i - 1, \end{cases} \hfill (2.3)$$

which is valid for $|x| < \frac{1}{\lambda_i (1-q)}$. If $\lambda_i = 0$, then

$$\phi_{r,i}(x) = \frac{x^r (1 - q)^r}{(q;q)_r}, \quad r = 0, 1, \ldots, m_i - 1.$$  \hfill (2.4)
Proof. The proof follows by direct computations. \( \square \)

One can see that \((D_q - \lambda_i)^{m_i} \phi_{r,i} = 0, r = 0, 1, \ldots, m_i - 1. \) Thus, \( \phi_{r,i} \in M_i \) for \( r = 0, 1, \ldots, m_i - 1. \) Therefore, these functions form a basis for \( M_i \) since they are linearly independent by Lemma 2.1. This fact and (2.2) above imply the following theorem.

**Theorem 2.3.** The set \( \{ \phi_{i,r} \}_{r=0}^{m_i-1} \) of (2.3) when \( \lambda_i \neq 0 \) or of (2.4) when \( \lambda_i = 0 \) is a linearly independent set of solutions of (2.1). Moreover, \( \bigcup_{i=1}^{k} \{ \phi_{i,r} \}_{r=0}^{m_i-1} \) is a fundamental set of solutions of (2.1).

**Example 2.4.** The \( q \)-difference equation

\[ D_q^3 y(x) - 4D_q^2 y(x) + 5D_q y(x) - 2y(x) = 0, \]

has the functions \( e_q(2x), e_q(x) \) and \( \sum_{k=1}^{\infty} \frac{k^2 (1-q)^k x^k}{(q^2)^k} \) as a f.s..  

**3. A \( q \)-type Wronskian**  

This section contains a \( q \)-analog of the Wronskian of linear differential equations, we prove that the \( q \)-analog satisfies a first order \( q \)-difference equation and we derive its solution. We also derive a \( q \)-type Liouville’s formula for the \( q \)-Wronskian.

**Definition 3.1.** Let \( y_i, 1 \leq i \leq n, \) be functions defined on a \( q \)-geometric set \( A. \) The \( q \)-Wronskian of the functions \( y_i \) which will be denoted by \( W_q(y_1, \ldots, y_n)(x) \) is defined to be

\[ W_q(y_1, \ldots, y_n)(x) := \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ D_qy_1(x) & \cdots & D_qy_n(x) \\ \vdots & \ddots & \vdots \\ D_q^{n-1}y_1(x) & \cdots & D_q^{n-1}y_n(x) \end{vmatrix}, \]

provided that the derivatives exist in \( I. \) For convenience we write \( W_q(x) \) instead of \( W_q(y_1, \ldots, y_n)(x). \)

**Lemma 3.2.** Let \( y_1, \ldots, y_n \) be functions defined on a \( q \)-geometric set \( A. \) Then for any \( x \in A, x \neq 0, \)

\[ D_q W_q(y_1, y_2, \ldots, y_n)(x) = \begin{vmatrix} y_1(qx) & y_2(qx) & \cdots & y_n(qx) \\ (D_qy_1)(qx) & (D_qy_2)(qx) & \cdots & (D_qy_n)(qx) \\ \vdots & \ddots & \ddots & \vdots \\ (D_q^{n-1}y_1)(qx) & (D_q^{n-1}y_2)(qx) & \cdots & (D_q^{n-1}y_n)(qx) \\ D_q^n y_1(x) & D_q^n y_2(x) & \cdots & D_q^n y_n(x) \end{vmatrix}. \]  

\( \text{(3.1)} \)
Assume that (3.1) holds at $j$ in terms of the first row we obtain

$$W_q(y_1, y_2, \ldots, y_k+1)(x) = \sum_{j=1}^{k+1} (-1)^{j+1} y_j(x) W_q^{(j)}(x),$$

where

$$W_q^{(j)} := \begin{cases} W_q(D_q y_2, \ldots, D_q y_{k+1}), & j = 1 \\ W_q(D_q y_1, \ldots, D_q y_{j-1}, D_q y_{j+1}, \ldots, D_q y_k), & 1 < j \leq k + 1 \\ W_q(D_q y_1, \ldots, D_q y_k), & j = k + 1. \end{cases}$$

Consequently, from (1.1),

$$D_qW_q(y_1, y_2, \ldots, y_k+1)(x) = \sum_{j=1}^{k+1} (-1)^{j+1} D_q y_j(x) W_q^{(j)}(x) + \sum_{j=1}^{k+1} (-1)^{j+1} y_j(qx) D_q W_q^{(j)}(x).$$

Now

$$\sum_{j=1}^{k+1} (-1)^{j+1} D_q y_j(x) W_q^{(j)}(x) = \begin{vmatrix} D_q y_1(x) & \cdots & D_q y_{k+1}(x) \\ D_q y_1(x) & \cdots & D_q y_{k+1}(x) \\ \vdots & \ddots & \vdots \\ D_q^{k-1} y_1(x) & \cdots & D_q^{k-1} y_{k+1}(x) \end{vmatrix} = 0,$$

and from the induction hypothesis,

$$\sum_{j=1}^{k+1} (-1)^{j+1} y_j(qx) D_q W_q^{(j)}(x) = \sum_{j=1}^{k+1} (-1)^{j+1} y_j(qx) \times \begin{vmatrix} (D_q y_1)(qx) & \cdots & (D_q y_{j-1})(qx) & (D_q y_j)(qx) & \cdots & (D_q y_{k+1})(qx) \\ (D_q^2 y_1)(qx) & \cdots & (D_q^2 y_{j-1})(qx) & (D_q^2 y_j)(qx) & \cdots & (D_q^2 y_{k+1})(qx) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ (D_q^{k-1} y_1)(qx) & \cdots & (D_q^{k-1} y_{j-1})(qx) & (D_q^{k-1} y_j)(qx) & \cdots & (D_q^{k-1} y_{k+1})(qx) \\ D_q^{k+1} y_1(x) & \cdots & D_q^{k+1} y_{j-1}(x) & D_q^{k+1} y_j(x) & \cdots & D_q^{k+1} y_{k+1}(x) \end{vmatrix} = (3.2)$$

where when $i = 1$ the determinant of (3.2) starts with $D_q y_2(qx)$ and when $j = k + 1$, the determinant ends with $D_q^{k+1} y_k(x)$. Thus

$$\sum_{j=1}^{k+1} (-1)^{j+1} y_j(qx) D_q W_q^{(j)}(x) = \begin{vmatrix} y_1(qx) & \cdots & y_{k+1}(qx) \\ (D_q y_1)(qx) & \cdots & (D_q y_{k+1})(qx) \\ (D_q^2 y_1)(qx) & \cdots & (D_q^2 y_{k+1})(qx) \\ \vdots & \ddots & \vdots \\ (D_q^{k-1} y_1)(qx) & \cdots & (D_q^{k-1} y_{k+1})(qx) \\ D_q^{k+1} y_1(x) & \cdots & D_q^{k+1} y_{k+1}(x) \end{vmatrix}.$$
proving (3.1) for \( n = k + 1 \) and hence all \( k \in \mathbb{N} \). \( \square \)

**Theorem 3.3.** If \( y_1, y_2, \ldots, y_n \) are solutions of (1.5) in \( J \subseteq I \), then their \( q \)-Wronskian satisfies the first order \( q \)-difference equation

\[
D_q W_q(x) = -R(x)W_q(x), \quad x \in J \setminus \{0\}
\]

where

\[
D = D_q = \begin{pmatrix}
D_q W_q(x) &= -R(x)W_q(x), \quad x \in J \setminus \{0\}
\end{pmatrix}
\]

**Proof.** From the definition of the operator \( D_q \), we have

\[
(D_q^m y)(qx) = D_q^m y(x) - x(1 - q)D_q^{m+1} y(x), \quad m \in \mathbb{N}.
\]

Substituting in (3.1) yields

\[
D_q W_q(y_1, \ldots, y_n)(x) = y_1(x) - x(1 - q)D_q y_1(x) \quad \cdots \quad y_n(x) - x(1 - q)D_q y_n(x)
\]

We shall prove by induction on \( n \) that

\[
D_q W_q(y_1, \ldots, y_n)(x) = \sum_{k=1}^{n} (-1)^{k-1}(x - q)^{k-1} D_q^{n-k} y_1(x) \quad \cdots \quad D_q^{n-k} y_n(x)
\]

If (3.4) holds at \( n = m \), then

\[
D_q W_q(y_1, y_2, \ldots, y_{m+1})(x) = \sum_{j=1}^{m+1} (-1)^{j+1}(y_j(x) - x(1 - q)D_q y_j(x)) A_{1j},
\]

where

\[
A_{1j} = D_q W_q(D_q y_1, \ldots, D_q y_{j-1}, D_q y_{j+1}, \ldots, D_q y_{m+1}), \quad j = 1, 2, \ldots, m.
\]
Hence from the previous hypothesis we obtain
\[ D_q W_q(y_1, y_2, \ldots, y_{m+1})(x) \]
\[ = \sum_{j=1}^{m+1} (-1)^{j+1} y_j(x) - x(1 - q) D_q y_j(x) \sum_{k=1}^{m} (-1)^{k-1} (x(1 - q))^{k-1} B_{jk} \]
\[ = \sum_{k=1}^{m} (-1)^{k-1} (x(1 - q))^{k-1} \sum_{j=1}^{m+1} (-1)^{j+1} y_j(x) B_{jk} \]
\[ + \sum_{k=1}^{m} (-1)^{k} (x(1 - q))^{k} \sum_{j=1}^{m+1} (-1)^{j+1} D_q y_j(x) B_{jk}, \]

with
\[ B_{jk} := \begin{vmatrix}
D_q y_1(x) & \cdots & D_q y_j(x) & D_q y_{j-1}(x) & D_q y_{j+1}(x) & \cdots & D_q y_{m+1}(x) \\
D_q^2 y_1(x) & \cdots & D_q^2 y_j(x) & D_q^2 y_{j-1}(x) & D_q^2 y_{j+1}(x) & \cdots & D_q^2 y_{m+1}(x) \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
D_q^{m-k} y_1(x) & \cdots & D_q^{m-k} y_j(x) & D_q^{m-k} y_{j-1}(x) & D_q^{m-k} y_{j+1}(x) & \cdots & D_q^{m-k} y_{m+1}(x) \\
D_q^{m-k+2} y_1(x) & \cdots & D_q^{m-k+2} y_j(x) & D_q^{m-k+2} y_{j-1}(x) & D_q^{m-k+2} y_{j+1}(x) & \cdots & D_q^{m-k+2} y_{m+1}(x) \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
D_q^{m+1} y_1(x) & \cdots & D_q^{m+1} y_j(x) & D_q^{m+1} y_{j-1}(x) & D_q^{m+1} y_{j+1}(x) & \cdots & D_q^{m+1} y_{m+1}(x)
\end{vmatrix}, \]

where \( k = 1, 2, \ldots, m \), when \( j = 1 \) the determinant \( B_{1k} \) start with \( D_q y_2(x) \) and when \( j = m + 1 \), the determinant \( B_{(m+1)k} \) ends with \( D_q^{m+1} y_{m}(x) \). From the properties of the determinants we conclude that

\[ \sum_{j=1}^{m+1} (-1)^{j+1} y_j(x) B_{jk} = \begin{vmatrix}
y_1(x) & \cdots & y_{m+1}(x) \\
D_q y_1(x) & \cdots & D_q y_{m+1}(x) \\
\vdots & \ddots & \vdots \\
D_q^{m-k} y_1(x) & \cdots & D_q^{m-k} y_{m+1}(x) \\
D_q^{m-k+2} y_1(x) & \cdots & D_q^{m-k+2} y_{m+1}(x) \\
\vdots & \ddots & \vdots \\
D_q^{m+1} y_1(x) & \cdots & D_q^{m+1} y_{m+1}(x)
\end{vmatrix}, \tag{3.5} \]
\[ \sum_{j=1}^{m+1} (-1)^{j+1} D_q y_j(x) B_{jk} = 0, \quad \text{for } k = 1, 2, \ldots, m - 1, \tag{3.6} \]

and

\[ \sum_{j=1}^{m+1} (-1)^{j+1} D_q y_j(x) B_{jm} = \begin{vmatrix}
D_q y_1(x) & \cdots & D_q y_{m+1}(x) \\
D_q^2 y_1(x) & \cdots & D_q^2 y_{m+1}(x) \\
\vdots & \ddots & \vdots \\
D_q^{m+1} y_1(x) & \cdots & D_q^{m+1} y_{m+1}(x)
\end{vmatrix}. \tag{3.7} \]
Combining equations (3.5)–(3.7) with (3), we obtain (3.4) when \( n = m + 1 \). One can easily see that (3.4) holds at \( n = 1 \). Consequently it holds for all \( n \in \mathbb{N} \).

From (1.5), we have

\[
D^q_n y_j(x) = - \sum_{i=1}^{i=n} \frac{a_i(x)}{a_0(x)} D^q_{n-i} y_j(x), \quad j = 1, 2, \ldots, n.
\]

Then (3.4) is nothing but

\[
D^q W(x) = - \sum_{k=0}^{k=n-1} (x - qx)^k \frac{a_{k+1}(x)}{a_0(x)} W(x) = -R(x)W(x).
\]

This completes the proof of the theorem.

The next theorem gives a \( q \)-type Liouville’s formula for the \( q \)-Wronskian.

**Theorem 3.4.** Let \( x(1-q)R(x) \neq -1 \) for all \( x \in J \). Then the \( q \)-Wronskian of any set of solutions \( \{\phi_i\}_{i=1}^n \) of equation (1.5) is given by

\[
W_q(x) = W_q(\phi_1, \ldots, \phi_n)(x) = \frac{1}{\prod_{k=0}^{k=\infty} (1 + x(1-q)q^k R(xq^k))} W_q(0), \quad x \in J.
\] (3.8)

**Proof.** Equation (3.3) is

\[
\frac{W_q(x) - W_q(qx)}{x - qx} = -R(x)W_q(x), \quad x \neq 0,
\]

i.e., \( W_q(x) - W_q(qx) = -x(1-q)R(x)W_q(x) \). Hence, under the assumption \( 1 + x(1-q)R(x) \neq 0 \), we obtain \( W_q(x) = \frac{W_q(qx)}{1+x(1-q)R(x)} \). Therefore,

\[
W_q(x) = \frac{W_q(xq^{m+1})}{\prod_{k=0}^{k=m} (1 + x(1-q)q^k R(xq^k))}, \quad \text{for all } m \in \mathbb{N} \text{ and } x \in I.
\]

Since all functions \( \frac{a_i}{a_0} \) are continuous at zero, then \( \sum_{k=0}^{\infty} q^k |R(xq^k)| \) is convergent. Consequently, \( \prod_{k=0}^{k=\infty} (1 + x(1-q)q^k R(xq^k)) \) converges for every \( x \in I \). Thus, using the continuity of \( W_q(x) \) at zero, (3.8) follows.

**Corollary 3.5.** Let \( \{\phi_i\}_{i=1}^n \) be a set of solutions of (1.5) in some subinterval \( J \) of \( I \) which contains zero. Then \( W_q(x) \) is either never zero or identically zero in \( I \). The first case occurs when \( \{\phi_i\}_{i=1}^n \) is a fundamental set of (1.5) and the second when it is not.
Proof. A set of solutions \( \{\phi_i\}_{i=1}^n \) forms a f.s. of (1.5) if and only if
\[
W_q(0) = \det \left( D_q^{i-1}\phi_j(0) \right)_{i,j=1}^n \neq 0,
\]
cf. [3]. This proves the corollary since from Theorem 3.4, \( W_q(x) \neq 0 \) for all \( x \in J \) if and only if \( W_q(0) \neq 0 \).

**Example 3.6.** In this example we calculate the \( q \)-Wronskian of
\[
-\frac{1}{q} D_q^{-1} D_q y(x) + y(x) = 0, \quad x \in \mathbb{R}.
\]
The solutions of (3.9) subject to the initial conditions
\[
y(0) = 0, \quad D_q y(0) = 1 \quad \text{and} \quad y(0) = 1, \quad D_q y(0) = 0,
\]
are \( \sin(x; q) \), \( \cos(x; q) \), \( x \in \mathbb{R} \), respectively. Since (3.9) can be written as
\[
D_q^2 y(x) + qx(1-q) D_q y(x) - q y(x) = 0.
\]
Then \( a_0(x) \equiv 1 \), \( a_1(x) = qx(1-q) \) and \( a_2(x) = -q \). Thus \( R(x) \equiv 0 \) on \( \mathbb{R} \) and \( W_q(x) \equiv W_q(0) \).

Then,
\[
W_q(0) = W_q(0) = \prod_{n=0}^{\infty} \left( 1 + q^{2n} \{ x(1-q) \}^2 \right)^{-1}, \quad |x|(1-q) < 1.
\]

**Example 3.7.** We calculate the \( q \)-Wronskian of the solutions of the \( q \)-difference equations
\[
-\frac{2}{q} y(x) + y(x) = 0, \quad x \in \mathbb{R}.
\]
The functions \( \sin_q x \), \( \cos_q x \), \( |x|(1-q) < 1 \), are solutions of (3.10) subject to the initial conditions
\[
y(0) = 0, \quad D_q y(0) = 1 \quad \text{and} \quad y(0) = 1, \quad D_q y(0) = 0,
\]
respectively. Here \( R(x) = x(1-q) \). So, \( x(1-q)R(x) \neq -1 \) for all \( x \) in \( \mathbb{R} \).

Hence,
\[
W_q(x) = \prod_{n=0}^{\infty} \left( 1 + q^{2n} \{ x(1-q) \}^2 \right)^{-1}, \quad |x|(1-q) < 1.
\]

But
\[
W_q(0) = W_q(0) = \prod_{n=0}^{\infty} \left( 1 + q^{2n} \{ x(1-q) \}^2 \right)^{-1}, \quad |x|(1-q) < 1.
\]

Therefore, \( W_q(x) \equiv \left( \prod_{n=0}^{\infty} \left( 1 + q^{2n} \{ x(1-q) \}^2 \right) \right)^{-1}, \quad |x|(1-q) < 1. \)
Remarks. 1. Theorem 3.3 might be satisfied for less restrictive conditions. But a general treatment needs a separate consideration. The \( q \)-Wronskian of (1.5) satisfies the first order \( q \)-difference equation (3.3) whatever the conditions which the functions \( a_j, 0 \leq j \leq n \) satisfy. But, in this case, the \( q \)-Wronskian can not be determined by using Theorem 3.4. An example of this case is the second order \( q \)-difference equation

\[
qx^2(1 - q)^2 D_q^2 y(x) + x(1 - q)^2 D_q y(x) + (x^2 q^{2-\nu} + (1 - q^\nu)(1 - q^{-\nu})) y(xq) = 0,
\]

where \( \nu > -1 \), which has a f.s. \( \{J_{\nu}(x;q^2), J_{-\nu}(xq^{-\nu};q^2)\} \) and it has been treated by R. F. Swarttouw and H. G. Meijer [15]. This class of problems may be considered as singular \( q \)-difference equations, while we are dealing with regular equations.

2. It is worthy to mention here that if equation (1.5) has the form

\[
a_0(x) D_q^n y(x) + a_1(x)(D_q^{n-1}) y(xq) + \cdots + a_n(x)y(qx) = 0, \quad x \in I, \quad (3.11)
\]

then substituting with \( D_q^n y(x) = -\sum_{j=1}^n \frac{a_j(x)}{a_0(x)} D_q^{n-j} y(qx) \) in (3.1) above we could derive a theory similar to that of the present section. In this case the associated \( q \)-Wronskian of solutions \( z_1, \ldots, z_n \) of (3.11) will satisfy the simplified first order \( q \)-difference equation

\[
D_q W_q(x) = -\frac{a_1(x)}{a_0(x)} W_q(qx).
\]

Consequently

\[
W_q(x) = \prod_{k=0}^{\infty} \left( 1 - xq^k(1 - q) \frac{a_1(xq^k)}{a_0(xq^k)} \right) W_q(0), \quad x \in J \setminus \{0\}.
\]

Similar to differential equations if \( a_1 \equiv 0 \), then \( W_q(x) \) is identically a constant. It should be noted that problems involving equation of the form (3.11) plays an important role in defining self adjoint eigenvalue problems, see e.g. [5, 8].

4. Applications

The theory introduced in the previous two sections can be used to obtain a general formula for the solutions of the inhomogeneous equation (1.3). Obviously, if \( \psi_1 \) and \( \psi_2 \) are two solutions of (1.3), then \( \psi_1 - \psi_2 \) is a solution of the corresponding homogeneous equation (1.5). Thus if \( \psi \) is a solution of (1.3) and \( \{\phi_i\}_{i=1}^n \) is a f.s. for (1.5), then there are unique constants \( \{c_i\}_{i=1}^n \) such that

\[
\psi = c_1 \phi_1 + \cdots + c_n \phi_n + \psi_0,
\]
where $\psi_0$ is a particular solution of (1.3). Now, we introduce a $q$-analog of the method of variation of parameters to find a particular solution $\psi_0$ of (1.3). Here also the functions $a_r$ and $b$ are continuous at zero functions defined on $I$ such that $a_0(x) \neq 0$ for all $x \in I$.

**Theorem 4.1.** Let $\{\phi_i\}_{i=1}^n$ be a fundamental set of (1.5) in $J$. Then, any solution $\psi$ of (1.3) is given by

$$
\psi(x) = \sum_{i=1}^{n} \left( c_i + \int_0^x \frac{W_q(t) (\phi_1, \ldots, \phi_n) (qt)}{W_q (\phi_1, \ldots, \phi_n) (qt)} \cdot \frac{b(t)}{a_0(t)} d_q t \right) \phi_i(x),
$$

(4.1)

where the $c_i$'s are constants and $W_q (\phi_1, \ldots, \phi_n) (x)$ is the determinant obtained from $W_q (\phi_1, \ldots, \phi_n) (x)$ by replacing the $r$-th column by $(0, \ldots, 0, 1)$.

**Proof.** Let $\psi$ be a solution of (1.3). If $\psi_0$ is a particular solution of (1.3), then for some constants $c_1, c_2, \ldots, c_n,

$$
\psi = \psi_0 + c_1 \phi_1 + \cdots + c_n \phi_n,
$$

where $c_1, \ldots, c_n$ are constants. Assume that $\psi_0$ has the form

$$
\psi_0(x) = u_1(x) \phi_1(x) + \cdots + u_n(x) \phi_n(x),
$$

where $u_1, \ldots, u_n$ are functions satisfying the system

$$
\begin{align*}
D_q u_1(x) \phi_1(qx) + \cdots + D_q u_n(x) \phi_n(qx) &= 0 \\
D_q u_1(x) D_q x \phi_1(qx) + \cdots + D_q u_n(x) D_q x \phi_n(qx) &= 0 \\
& \quad \vdots \\
D_q u_1(x) D_q^{n-2} \phi_1(qx) + \cdots + D_q u_n(x) D_q^{n-2} \phi_n(qx) &= 0 \\
D_q u_1(x) D_q^{n-1} \phi_1(qx) + \cdots + D_q u_n(x) D_q^{n-1} \phi_n(qx) &= \frac{b(x)}{a_0(x)}.
\end{align*}
$$

(4.2)

System (4.2) is an inhomogeneous linear system of equations in the $n$ unknowns $\{D_q u_i\}_{i=1}^n$. The determinant of the coefficients is $W_q (\phi_1, \ldots, \phi_n) (qx) \neq 0$ since $\phi_r, 1 \leq r \leq n$, is a f.s. for (1.5). Hence (4.2) can be solved for the $D_q u_r$ and

$$
D_q u_r(x) = \frac{W_q (\phi_1, \ldots, \phi_n) (qx)}{W_q (\phi_1, \ldots, \phi_n) (qx)} \cdot \frac{b(x)}{a_0(x)}, \quad r = 1, \ldots, n.
$$

Since $\frac{b(x)}{a_0(x)}$ is continuous at zero, then from Theorem 1.1 $D_q u_r$ is $q$-integrable on $[0, x]$, for all $x \in J$. Thus, a suitable choice for $u_r(x)$ is

$$
\begin{align*}
u_r(x) &= \int_0^x \frac{W_q (\phi_1, \ldots, \phi_n) (qt)}{W_q (\phi_1, \ldots, \phi_n) (qt)} \cdot \frac{b(t)}{a_0(t)} d_q t, \\
&= \int_0^x \frac{W_q (\phi_1, \ldots, \phi_n) (qt)}{W_q (\phi_1, \ldots, \phi_n) (qt)} \cdot \frac{b(t)}{a_0(t)} d_q t,
\end{align*}
$$

(4.2)
and then $\psi_0$ has the form
\[
\psi_0(x) = \sum_{i=1}^n \phi_i(x) \int_0^x \frac{W_{q,i} (\phi_1, \ldots, \phi_n) (qt)}{W_q (\phi_1, \ldots, \phi_n) (qt)} \cdot \frac{b(t)}{a_0(t)} \, dq \, t,
\]
proving formula (4.1).

**Example 4.2.** Consider the equation
\[
-\frac{1}{q} D_{q^{-1}} D_q y(x) + y(x) = b(x),
\]
where $b(\cdot)$ is a continuous function defined in $\mathbb{R}$. The corresponding homogeneous equation is
\[
D_q^2 y(x) - q y(qx) = 0.
\]
A fundamental set of solutions of (4.4) is \{\sin(x; q), \cos(x; q)\}. Substituting in (4.1) and using $W_q (\sin(\cdot; q), \cos(\cdot; q))(x) \equiv -1$, every solution of (4.3) has the form
\[
\psi(x) = c_1 \sin(x; q) + c_2 \cos(x; q)
- q \int_0^x \left( \sin(x; q) \cos(qt; q) - \cos(x; q) \sin(qt; q) \right) b(qt) \, dq \, t,
\]
where $x \in \mathbb{R}$, $c_1$ and $c_2$ are arbitrary constants.

**References**


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