Approximation of Solutions of Stochastic Differential Equations by Discontinuous Galerkin Methods

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Abstract. The generalized solution of a system of Stratonovich equations is approximated by a discontinuous Galerkin method. A piecewise polynomial approximation is introduced. The convergence and error estimates are proved. The solution of Galerkin equations can be approximated by the solution of a system of equations with an inhomogeneous random part and the simulation of a stochastic integral.

Keywords: Approximation of solution of a stochastic differential equation, Stratonovich integral, Galerkin method

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1. Introduction

This paper is concerned with piecewise polynomial approximation of the solution of a system of ordinary stochastic differential equations in the sense of Stratonovich. Under certain conditions it is well known that a solution of a system of Stratonovich equations is a solution of a modified Itô system [3: p. 237]. The approximation of solutions of stochastic differential equations is studied by many authors. For example, a stochastic Taylor formula was developed and applied to the approximation of solutions of ordinary Itô equations (see [4: p. 163] and [5: p. 78]). Especially, stochastic variants of the Euler and Runge-Kutta methods are obtained by application of the stochastic Taylor formula. In [3: Theorem 7.2/p. 394] a Stratonovich equation is approximated by a sequence of stochastic differential equations with piecewise differentiable paths. Therefore the Stratonovich interpretation is often important for the applications.

Here we consider another method known in the deterministic case as completely discontinuous Galerkin method [1]. The investigations differ from the deterministic case since the paths of solutions of stochastic differential equations are not differentiable. Subsequently, the methods of stochastic analysis must be used. In Section 2 we interpret a system of Stratonovich equations on a fixed interval $[0, T]$ as system of stochastic
variational equations and prove for the solution of the last an existence and uniqueness theorem and a regularity property (Theorem 2 and Theorem 3). The approximation is studied in Section 3. For that the interval $[0,T]$ is partitioned into $N$ intervals $I_n = [t_{n-1}, t_n]$ by points $t_n$ ($0 \leq n \leq N$) with $0 = t_0 < t_1 < \ldots < t_N = T$. On each interval $I_n$ a random polynomial (Galerkin approximation) is constructed by solution of random variational equations. If the partitions of the intervals are small enough, then a unique solution exists (Theorem 4). The convergence in mean square and error of approximation estimates are contained in Section 4. A computing possibility is given in Section 5.

2. Variational formulation

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t \in [0,T]} \subset \mathcal{F}$, let $(w(t))_{t \in [0,T]}$ be an $\mathcal{F}_t$-adapted $m$-dimensional Wiener process and

$$
\begin{align*}
a_i : [0,T] \times \mathbb{R}^d &\rightarrow \mathbb{R} \\
\sigma_{ij} : [0,T] \times \mathbb{R}^d &\rightarrow \mathbb{R}
\end{align*}
$$

measurable functions with

$$
|a_i(t, X)| + |\sigma_{ij}(t, X)| \leq C(1 + |X|) \quad (X \in \mathbb{R}^d, t \in [0,T])
$$

for some constant $C > 0$ and

$$
|a_i(t, X) - a_i(t, Y)| \leq D_1 |X - Y| \quad (X, Y \in \mathbb{R}^d, t \in [0,T])
$$

for some constant $D_1 > 0$. Assume that $\frac{\partial \sigma_{ij}(t, X)}{\partial X_k}$ exists and

$$
\left| \frac{\partial \sigma_{ij}(t, X)}{\partial X_k} \right| \leq K \quad (i = 1, \ldots, d; j = 1, \ldots, m)
$$

for some constant $K > 0$. Then the functions $\sigma_{ij}$ are Lipschitz continuous over $\mathbb{R}^d$ with Lipschitz constant $K$. Define

$$
a(t, X(t)) = (a_i(t, X(t)))_{i=1,\ldots,d}
$$

$$
\sigma(t, X(t)) = (\sigma_{ij}(t, X(t)))_{j=1,\ldots,m}
$$

and let $X_0 : \Omega \rightarrow \mathbb{R}^d$ be $\mathcal{F}_0$-measurable.

Further we consider the system of Stratonovich equations

$$
\begin{align*}
dX(t) &= a(t, X(t)) \, dt + \sigma(t, X(t)) \circ dw(t) \\
X(0) &= X_0
\end{align*}
$$

(2.4)
which is defined by

\[ X(t) = X_0 + \int_0^t a(s, X(s)) \, ds + \int_0^t \sigma(s, X(s)) \, dw(s) \]  
(2.5)

for all \( t \in [0, T] \) with probability 1. The stochastic integral is defined in the sense of Stratonovich [3: p. 237]. From [3] we can deduce that (2.4) is equivalent to the modified Ito equation

\[
\begin{align*}
\left\{ 
\begin{array}{l}
dX(t) = b(t, X(t)) \, dt + \sigma(t, X(t)) \, dw(t) \\
X(0) = X_0
\end{array}
\right. 
\end{align*}
\]  
(2.6)

where \( b = b(t, X) \in \mathbb{R}^d \) with components

\[ b_i(t, X) = a_i(t, X) + \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^d \frac{\partial \sigma_{ij}(t, X)}{\partial X_k} \sigma_{kj}(t, X) \quad (i = 1, \ldots, d), \]

equation (2.6) is defined as

\[ X(t) = X_0 + \int_0^t b(s, X(s)) \, ds + \int_0^t \sigma(s, X(s)) \, dw(s) \]  
(2.6)'

\( \mathcal{P} \)-a.s. for all \( t \in [0, T] \) and the stochastic integral is an Ito integral. Further assume

\[ |b_i(t, X) - b_i(t, Y)| \leq D_2 |X - Y| \]  
(2.7)

for fixed \( D_2 > 0 \).

The classical existence and uniqueness results are summarized in the next theorem (see, for example, [2: Satz 1/p. 38]).

**Theorem 1.** Under the above assumptions equation (2.6) has a unique \( \mathcal{F}_t \)-adapted continuous \( \mathbb{R}^d \)-valued solution \( X(t) = (X_1(t), \ldots, X_d(t)) \) with \( \mathbb{E} \sup_{t \in [0, T]} |X_i(t)|^2 < \infty \) \((i = 1, \ldots, d)\).

Now we introduce a mesh-dependent variational formulation of equation (2.6). For a given number \( N \in \mathbb{N} \) we introduce partitions

\[ 0 = t_0 < t_1 < \ldots < t_N = T \]

with

\[ \max \{ t_{n+1} - t_n : n = 0, \ldots, N - 1 \} =: h_N \to 0 \]

as \( N \to \infty \). Assume there is a constant \( c > 0 \) with \( t_n - t_{n-1} \geq ch_N \) for all \( n \). Let \( H^1 = H^1(\Omega \times [0, T]) \) denote the space of all \( \mathcal{F}_t \)-adapted random \( \mathbb{R}^d \)-valued processes \((V(t))_{t \in [0, T]}\) where the paths have generalized derivations \((V'(t))_{t \in [0, T]}\) with

\[ \mathbb{E} \sup_{0 \leq t \leq T} |V'(t)|^2 < \infty. \]
That is, \((V'(t))_{t \in [0,T]}\) is an \(\mathcal{F}_t\)-adapted \(\mathbb{R}^d\)-valued process defined by

\[
\int_0^T (V(t), \varphi'(t)) \, dt = -\int_0^T (V'(t), \varphi) \, dt
\]

\((\mathcal{P}\text{-a.s.})\) for all \(\varphi \in C_0^\infty[0,T]\), where \((\cdot, \cdot)\) is the scalar product over \(\mathbb{R}^d\). Further we introduce the following function spaces:

- \(L^2_n(\Omega)\): Space of all \(\mathcal{F}_{t_n}\)-measurable functions \(U : \Omega \to \mathbb{R}^d\) with \(E|U|^2 < \infty\) \((n = 0, \ldots, N)\).
- \(L^2(\Omega, [t_{n-1}, t_n])\): Space of all \(\mathbb{R}^d\)-valued \(\mathcal{F}_t\)-adapted processes \((Y(t))\) where \(t \in [t_{n-1}, t_n]\) with \(E \int_{t_{n-1}}^{t_n} |Y(t)|^2 \, dt < \infty\) \((n = 1, \ldots, N)\).
- \(\tilde{Y}_N = \prod_{n=1}^N L^2(\Omega, [t_{n-1}, t_n])\).
- \(\tilde{U}_N = \prod_{n=0}^N L^2(\Omega)\).
- \(\tilde{V}_N = \prod_{n=1}^N H^1(\Omega, [t_{n-1}, t_n])\).

The variational problem consists in finding of \((U_0, \ldots, U_N; Y_1, \ldots, Y_N) \in \tilde{U}_N \times \tilde{Y}_N\) such that

\[
\begin{aligned}
U_0 &= X_0, \\
\langle U_n, V_n(t_n) \rangle &= \langle U_{n-1}, V_n(t_{n-1}) \rangle \\
&\quad + \int_{t_{n-1}}^{t_n} (\langle Y_n(t), V_n'(t) \rangle + \langle b(t, Y_n(t)), V_n(t) \rangle) \, dt \\
&\quad + \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, Y_n(t)) \, dw(t) \rangle
\end{aligned}
\tag{2.8}
\]

holds for all \(V_n \in H^1(\Omega, [t_{n-1}, t_n])\) and \(n = 1, \ldots, N\). To solve problem (2.8) we apply the following

**Lemma 1.**

1. The function \(\Phi : H^1(\Omega, [t_{n-1}, t_n]) \to L^2(\Omega, [t_{n-1}, t_n]) \times L^2_n(\Omega)\) defined by \(\Phi(V_n(\cdot)) = (-V_n'(\cdot), V_n(\cdot))\) is an isomorphism.

2. Let \((B_1(t))_{t \in [0,T]}\) and \((B_2(t))_{t \in [0,T]}\) be \(\mathcal{F}_t\)-adapted stochastic processes with values in \(\mathbb{R}^d\) and \(\mathbb{R}^{d \times m}\) so that the Ito differential \(B_1(t) \, dt + B_2(t) \, dw(t)\) exists and let
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Let \( U_{n-1} \in L^2_{n-1}(\Omega) \) be given. Then there are \( U_n \in L^2_n(\Omega) \) and \( Y_n \in \tilde{L}^2(\Omega \times [t_{n-1}, t_n]) \) with

\[
\langle U_n, V_n(t_n) \rangle - \langle U_{n-1}, V_n(t_{n-1}) \rangle - \int_{t_{n-1}}^{t_n} \langle Y_n(t), V'_n(t) \rangle dt = \int_{t_{n-1}}^{t_n} \langle B_1(t), V_n(t) \rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), B_2(t) dw(t) \rangle
\]

(2.9)

\( \mathcal{P} \)-a.s. for all \( V_n \in H^1(\Omega \times [t_{n-1}, t_n]) \) \( (n = 1, \ldots, N) \) and \( (U_n, Y_n) \) is with probability 1 unique.

**Proof.** Assertion (1) is obviously. Assertion (2): Applying the Ito formula to \( (Y(t), V_n(t)) \) where

\[
dY(t) = B_1(t) dt + B_2(t) dw(t) \quad (t \in [t_{n-1}, t_n])
\]

\( Y(t_{n-1}) = U_{n-1} \)

and \( V_n \in H^1(\Omega \times [t_{n-1}, t_n]) \) we find

\[
\langle Y(t_n), V_n(t_n) \rangle - \langle U_{n-1}, V_n(t_{n-1}) \rangle - \int_{t_{n-1}}^{t_n} \langle Y(t), V'_n(t) \rangle dt = \int_{t_{n-1}}^{t_n} \langle B_1(t), V_n(t) \rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), B_2(t) dw(t) \rangle.
\]

Consequently, \( (U_n, Y_n) \) with \( U_n = Y(t_n) \) and \( Y_n = Y \) solve problem (2.9). It is easy to see with an indirect proof that \( (U_n, Y_n) \) is the unique solution of this problem with probability 1.

**Remark 1.** Obviously, statement (2) in Lemma 1 holds also for the Stratonovich integral.

**Theorem 2.** There is a unique solution \( (U_0, \ldots, U_N; Y_1, \ldots, Y_N) \in \tilde{U}_N \times \tilde{Y}_N \) of problem (2.8) for sufficient small \( h_n > 0 \).

**Proof.** On an interval \([t_{n-1}, t_n]\) suppose that an \( \mathcal{F}_{t_{n-1}} \)-measurable \( \mathbb{R}^d \)-valued variable \( U_{n-1} \) with \( E|U_{n-1}|^2 < \infty \) is given. Then \( (U_n, Y_n) \in L^2_n(\Omega) \times \tilde{L}^2(\Omega \times [t_{n-1}, t_n]) \) has to be determined so that

\[
\langle U_n, V_n(t_n) \rangle - \langle U_{n-1}, V_n(t_{n-1}) \rangle - \int_{t_{n-1}}^{t_n} \langle Y_n(t), V'_n(t) \rangle dt = \int_{t_{n-1}}^{t_n} \langle V_n(t), b(t, Y_n(t)) \rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, Y_n(t)) dw(t) \rangle
\]

(2.10)
is valid for all $V_n \in H^1(\Omega \times [t_{n-1}, t_n])$. We define recursively $(U_n^{i+1}, Y_n^{i+1}) \in L^2_t(\Omega) \times L^2(\Omega \times [t_{n-1}, t_n])$ ($i = 0, 1, \ldots$) by the equations

$$
(U_n^{i+1}, V_n(t_n)) - (U_{n-1}, V_n(t_{n-1})) = \int_{t_{n-1}}^{t_n} \langle Y_n^{i+1}(t), V_n'(t) \rangle dt
$$

(2.11)

$$
= \int_{t_{n-1}}^{t_n} \langle V_n(t), b(t, Y_n^i(t)) \rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, Y_n^i(t)) \rangle dw(t)
$$

where (2.11) holds for all $V_n \in H^1(\Omega \times [t_{n-1}, t_n])$ and $Y_n^0 \in \tilde{L}^2(\Omega \times [t_{n-1}, t_n])$ is chosen arbitrarily. Lemma 1 shows that $(U_n^{i+1}, Y_n^{i+1})$ exists uniquely. We consider (2.11) for $i = j$ and $i = j - 1$. Then we subtract these equations and we obtain

$$
(V_n(t), b(t, Y_n^i(t)) - b(t, Y_n^{i-1}(t))) dt
$$

(2.12)

$$
= \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, Y_n^i(t)) - \sigma(t, Y_n^{i-1}(t)) \rangle dw(t)
$$

for all $V_n \in H^1(\Omega \times [t_{n-1}, t_n])$. Obviously, the function

$$
V_n(t) = \begin{cases} 
0 & \text{for } t = t_{n-1} \\
\int_{t_{n-1}}^{t} (Y_n^i(s) - Y_n^{i+1}(s)) ds & \text{for } t \in (t_{n-1}, t_n) \\
0 & \text{for } t = t_n
\end{cases}
$$

(2.13)

is from $H^1(\Omega \times [t_{n-1}, t_n])$. If we choose the above process $V_n$, then we obtain from (2.12)

$$
\int_{t_{n-1}}^{t_n} |Y_n^{j+1}(t) - Y_n^j(t)|^2 dt = \int_{t_{n-1}}^{t_n} \langle V_n(t), b(t, Y_n^j(t)) - b(t, Y_n^{j-1}(t)) \rangle dt
$$

(2.14)

$$
+ \int_{t_{n-1}}^{t_n} \langle V_n(t), (\sigma(t, Y_n^j(t)) - \sigma(t, Y_n^{j-1}(t)) \rangle dw(t)
$$
and

\[
E \int_{t_{n-1}}^{t_n} |Y_{n}^{j+1}(t) - Y_{n}^{j}(t)|^2 \, dt \\
\leq \left( E \int_{t_{n-1}}^{t_n} |b(t, Y_{n}^{j}(t)) - b(t, Y_{n}^{j-1}(t))|^2 \, dt \right)^{\frac{1}{2}} \left( E \int_{t_{n-1}}^{t_n} |V_n(t)|^2 \, dt \right)^{\frac{1}{2}} \\
\leq D_2 \left( E \int_{t_{n-1}}^{t_n} |Y_{n}^{j}(t) - Y_{n}^{j-1}(t)|^2 \, dt \right)^{\frac{1}{2}} \left( E \int_{t_{n-1}}^{t_n} \left( \int_{t_{n-1}}^{t} (Y_{n}^{j}(s) - Y_{n}^{j+1}(s)) \, ds \right)^2 \, dt \right)^{\frac{1}{2}} \\
\leq D_2 \left( E \int_{t_{n-1}}^{t_n} |Y_{n}^{j}(t) - Y_{n}^{j-1}(t)|^2 \, dt \right)^{\frac{1}{2}} h_{n}^{\frac{1}{2}} \left( E \int_{t_{n-1}}^{t_n} |Y_{n}^{j}(t) - Y_{n}^{j+1}(t)|^2 \, dt \right)^{\frac{1}{2}}
\]

where the properties of the Ito integral, the Lipschitz continuity of \( b \) and the Schwarz inequality in \( L^2([t_{n-1}, t_n]) \) were applied. Consequently, it follows

\[
\|Y_{n}^{j+1}(\cdot) - Y_{n}^{j}(\cdot)\|_{L^2(\Omega \times [t_{n-1}, t_n])} \leq D_2 h_{n}^{\frac{1}{2}} \|Y_{n}^{j}(\cdot) - Y_{n}^{j-1}(\cdot)\|_{L^2(\Omega \times [t_{n-1}, t_n])} \\
\vdots \\
\leq (D_2 h_{n}^{\frac{1}{2}})^j \|Y_{n}^{1}(\cdot) - Y_{n}^{0}(\cdot)\|_{L^2(\Omega \times [t_{n-1}, t_n])}
\]

and for \( p > 0 \)

\[
\|Y_{n}^{j+p}(\cdot) - Y_{n}^{j}(\cdot)\|_{L^2(\Omega \times [t_{n-1}, t_n])} \\
\leq \|Y_{n}^{j+p}(\cdot) - Y_{n}^{j+p-1}(\cdot)\|_{L^2(\Omega \times [t_{n-1}, t_n])} \\
+ \|Y_{n}^{j+p-1}(\cdot) - Y_{n}^{j+p-2}(\cdot)\|_{L^2(\Omega \times [t_{n-1}, t_n])} \\
\vdots \\
+ \|Y_{n}^{j+1}(\cdot) - Y_{n}^{j}(\cdot)\|_{L^2(\Omega \times [t_{n-1}, t_n])} \\
\leq \left[ (D_2 h_{n}^{\frac{1}{2}})^{j+p-1} + (D_2 h_{n}^{\frac{1}{2}})^{j+p-2} + \ldots + (D_2 h_{n}^{\frac{1}{2}})^{j} \right] \\
\times \|Y_{n}^{1}(\cdot) - Y_{n}^{0}(\cdot)\|_{L^2(\Omega \times [t_{n-1}, t_n])}.
\]

Therefore \( \{Y_{n}^{j}(\cdot)\}_j \) is a Cauchy sequence in \( L^2(\Omega \times [t_{n-1}, t_n]) \), since the term \([...]\) converges to 0 for sufficient small \( h_N > 0 \) as \( j, p \to \infty \). Thus, the limit \( Y_n(\cdot) = \lim_{j, p \to \infty} Y_{n}^{j}(\cdot) \)
exists in $L^2(\Omega \times [t_{n-1}, t_n])$ and consequently,

$$
\lim_{t \to \infty} \left[ \int_{t_{n-1}}^{t_n} \left( \langle Y_{n+1}^i(t), V_n^i(t) \rangle \right) dt + \int_{t_{n-1}}^{t_n} \left( \langle V_n^i(t), b(t, Y_n^i(t)) \rangle + \int_{t_{n-1}}^{t_n} \langle V_n^i(t), \sigma(t, Y_n^i(t)) \rangle dw(t) \right) \right]
$$

$$
= \int_{t_{n-1}}^{t_n} \langle Y_n(t), V_n^i(t) \rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), b(t, Y_n(t)) \rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, Y_n(t)) \rangle dw(t)
$$

holds in probability. Then (2.11) shows if we choose $V_n \equiv c \in \mathbb{R}^d$ that there exists an $\mathcal{F}_{t_n}$-measurable function $U_n : \Omega \to \mathbb{R}^d$ with

$$
\langle U_n, c \rangle - \langle U_{n-1}, c \rangle = \int_{t_{n-1}}^{t_n} \langle c, b(t, Y_n(t)) \rangle dt + \int_{t_{n-1}}^{t_n} \langle c, \sigma(t, Y_n(t)) \rangle dw(t)
$$

It follows from the properties of $b$ and $\sigma$ that $E|U_n|^2 < \infty$. It is clear that $(U_n, Y_n(\cdot))$ is the unique solution of (2.10). We proceed in this way to the next interval and so on in a finite number of steps. At the end we obtain the unique solution of problem (2.8).

We can prove a regularity property of the solution of problem (2.8).

**Theorem 3.** Let $(X_0; U_1, \ldots, U_N; Y_1, \ldots, Y_N)$ and $(X(t))$ the solutions of problems (2.8) and (2.6), respectively. Then $X(t_1) = U_1, \ldots, X(t_N) = U_N$ and $X(t) = Y_n(t)$ for $t \in [t_{n-1}, t_n)$ $(n = 1, \ldots, N)$.

**Proof.** The solution of equation (2.6) also defines the solution of problem (2.8). This follows from the Ito formula:

$$
\langle X(t_n), V_n(t_n) \rangle = \langle X(t_{n-1}, V_n(t_{n-1})) + \int_{t_{n-1}}^{t_n} \langle X(t), V_n^i(t) \rangle dt + \int_{t_{n-1}}^{t_n} \langle b(t, X(t)), V_n(t) \rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, X(t)) \rangle dw(t) \rangle
$$

where $X(0) = X_0$. The statement of the theorem results from the uniqueness of the solution of problem (2.8).
Lemma 2. A solution of problem (2.8) is also a solution of the problem

\[ U_0 = X_0 \]

\[ (U_n, V_n(t_n)) = (U_{n-1}, V_n(t_{n-1})) \]

\[ + \int_{t_{n-1}}^{t_n} \left[ \langle Y_n(t), V_n'(t) \rangle + \langle a(t, Y_n(t)), V_n(t) \rangle \right] dt \]

\[ + \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, Y_n(t)) \circ dw(t) \rangle \]

for all \( V_n \in H^1(\Omega \times [t_{n-1}, t_n]) \) \( (n = 1, \ldots, N) \) and conversely, where the stochastic integral is the Stratonovich integral.

Proof. It is obviously \( \square \)

3. Approximation

We want to approximate the solution of problem (2.8) by random polynomials. Let \( P^k([t_{n-1}, t_n], \mathbb{R}^d) \) the space of all polynomials \( P_1, \ldots, P_d \) of degree \( k \). If their coefficients are from \( L_2^k(\Omega) \), then we write \( P^k_{n-1}([t_{n-1}, t_n], \mathbb{R}^d) \).

Lemma 3.

1. The function \( \Psi : P^{k+1}([t_{n-1}, t_n], \mathbb{R}^d) \to P^k([t_{n-1}, t_n], \mathbb{R}^d) \times \mathbb{R}^d \) defined by \( \Psi(V_n(u)) = (-V_n(u), V_n(t_n)) \) is an isomorphism.

2. Let \((B_1(t))_{t \in [0, T]}\) and \((B_2(t))_{t \in [0, T]}\) be \( \mathcal{F}_t \)-adapted continuous stochastic processes with values in \( \mathbb{R}^d \) and \( \mathbb{R}^{d \times m} \) so that the Itô differential \( B_1(t) dt + B_2(t) dw(t) \) exists. Then there are \( U_n \in L_2^k(\Omega) \) and \( Y_n \in P^k_{n-1}([t_{n-1}, t_n], \mathbb{R}^d) \) with

\[ (U_n, V_n(t_n)) = \int_{t_{n-1}}^{t_n} \langle Y_n(t), V_n(t) \rangle dt \]

\[ = \int_{t_{n-1}}^{t_n} \langle B_1(t), V_n(t) \rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), B_2(t) dw(t) \rangle \]

\( \mathcal{P} \text{-a.s. for all } V_n \in P^{k+1}_{n-1}([t_{n-1}, t_n], \mathbb{R}^d) \) and \((U_n, Y_n)\) is with probability 1 unique.

Proof. Assertion (1) is clear. Assertion (2): Let \( t_{n-1} = s_0 < s_1 < \ldots < s_r = t_n \) be a partition with

\[ \lim_{r \to \infty} \max_{0 < i \leq r-1} (s_{i+1} - s_i) = 0. \]

If \( V_n \in P^{k+1}([t_{n-1}, t_n], \mathbb{R}^d) \), then

\[ \sum_{i=0}^{r-1} \langle B_1(s_i), V_n(s_i) \rangle (s_{i+1} - s_i) + \sum_{i=0}^{r-1} \langle V_n(s_i), B_2(s_i) (w(s_{i+1}) - w(s_i)) \rangle \]
defines for fixed $\omega \in \Omega$ a linear continuous functional $\rho_r$ on the space of polynomials of degree $k + 1$, and also a linear continuous functional on $P^k([t_{n-1}, t_n], \mathbb{R}^d) \times R^d$ since $P^{k+1}([t_{n-1}, t_n], \mathbb{R}^d)$ is isomorphic to this space. Then because of the definition of the isomorphism $\Psi$ there are $U_n^r \in R^d$ and $Y_n^r(\cdot) \in P^k([t_{n-1}, t_n], R^d)$ so that

$$\rho_r(V_n) = \langle U_n^r, V_n(t_n) \rangle - \int_{t_{n-1}}^{t_n} \langle Y_n^r(t), V_n(t) \rangle dt.$$  

Subsequently we have

$$\langle U_n^r, V_n(t_n) \rangle - \int_{t_{n-1}}^{t_n} \langle Y_n^r(t), V_n(t) \rangle dt = \sum_{i=0}^{r-1} \langle B_1(s_i), V_n(s_i) \rangle (s_{i+1} - s_i) + \sum_{i=0}^{r-1} \langle V_n(s_i), B_2(s_i)(w(s_{i+1}) - w(s_i)) \rangle$$

(3.1)

for all $V_n \in P^{k+1}([t_{n-1}, t_n], \mathbb{R}^d)$ with probability 1 and for all $r$. If we choose especially $V_n \in P^{k+1}_n([t_{n-1}, t_n], \mathbb{R}^d)$ so that $V_n(t_n) = 0$, then

$$-E \int_{t_{n-1}}^{t_n} \langle Y_n^r(t), V_n(t) \rangle dt = E \sum_{i=0}^{r-1} \langle B_1(s_i), V_n(s_i) \rangle (s_{i+1} - s_i)$$

defines a linear continuous functional on $P^{k}_n([t_{n-1}, t_n], \mathbb{R}^d)$ where the values of the polynomials are zero in $t_n$. The space $P^{k+1}_n([t_{n-1}, t_n], \mathbb{R}^d)$ is isomorphic to the Hilbert space $L^2_{n-1}(\Omega; \mathbb{R}^d \times \ldots \times \mathbb{R}^d)$. Consequently $Y_n^r$ is also from $P^{k+1}_n([t_{n-1}, t_n], \mathbb{R}^d)$. Obviously, $U_n^r$ is from $L^2_n(\Omega)$.

The left-hand side of (3.1) is convergent in mean square to the limit

$$\langle U_n, V_n(t_n) \rangle - \int_{t_{n-1}}^{t_n} \langle Y_n(t), V_n(t) \rangle dt$$

since the right-hand side of (3.1) is convergent, namely to

$$\int_{t_{n-1}}^{t_n} \langle B_1(t), V_n(t) \rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), B_2(t) dw(t) \rangle$$

in the mean square. The uniqueness of $U_n$ and $Y_n$ follows with an indirect proof.
We introduce the variational problem to find \((U_0, \ldots, U_N; Y_1, \ldots, Y_N)\) from \(\bar{U}_N \times \prod_{i=0}^{N-1} P_i^k([t_i, t_{i+1}], \mathbb{R}^d)\) so that

\[
\begin{align*}
U_0 &= X_0 \\
(U_n, V_n(t_n)) &= (U_{n-1}, V_n(t_{n-1})) \\
&+ \int_{t_{n-1}}^{t_n} \left( \langle Y_n(t), V_n(t) \rangle + \langle b(t, Y_n(t)), V_n(t) \rangle \right) dt \\
&+ \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, Y_n(t)) \rangle dw(t)
\end{align*}
\]

holds for all \(V_n \in P^{k+1}_i([t_{n-1}, t_n], \mathbb{R}^d)\) and \(n = 1, \ldots, N\).

**Theorem 4.** Assume that the hypotheses of Theorem 2 are verified. Then the variational problem (3.2) has a unique solution for sufficiently small \(h_N > 0\).

The proof is like that for Theorem 2 if we apply Lemma 3 instead of Lemma 1. Hence it is omitted.

### 4. Error estimates

This section contains the theorem which establishes the convergence of \(U_n\) and \(Y_n(\cdot)\) for \(h_N \to 0\).

**Theorem 5.** Let \(X\) and \((U_0, \ldots, U_N; Y_1, \ldots, Y_N)\) be the solutions of problems (2.6) and (3.2) where the assumptions (2.1), (2.2), (2.3) and (2.7) are fulfilled. Then

1. \(\max_n (E|U_n - X(t_n)|^2)^{\frac{1}{2}} \leq C h_N^{\frac{1}{2}}\)

and

2. \(E \int_0^T \left| \sum_{n=1}^{N} Y_n(s)X_{[t_{n-1}, t_n]}(s) - X(s) \right|^2 ds \leq C_1 T h_N\)

where \(C\) and \(C_1\) are positive constants.

**Proof.** Assume \(X\) solves the stochastic equation (2.6). Then from equation (3.2) there follows

\[
\begin{align*}
\langle U_n - X(t_n), V_n(t_n) \rangle - &\langle U_{n-1} - X(t_{n-1}), V_n(t_{n-1}) \rangle \\
= &\int_{t_{n-1}}^{t_n} \left( \langle Y_n(t) - X(t), V_n(t) \rangle + \langle b(t, Y_n(t)) - b(t, X(t)), V_n(t) \rangle \right) dt \\
&+ \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, Y_n(t)) - \sigma(t, X(t)) \rangle dw(t)
\end{align*}
\]
If we choose \( V_n(t) = (1, 0, \ldots, 0), \ldots, V_n(t) = (0, \ldots, 0, 1) \), then we obtain equations for the components of \( U_n - X(t_n) \). Then the following inequalities hold for the norms:

\[
\left( E|U_n - X(t_n)|^2 \right)^{\frac{1}{2}} \leq \left( E|U_{n-1} - X(t_{n-1})|^2 \right)^{\frac{1}{2}} \\
+ \left( E \left[ \int_{t_{n-1}}^{t_n} \left| (b(t, Y_n(t)) - b(t, X(t))) \right|^2 \right] \right)^{\frac{1}{2}} \\
+ \left( E \left[ \int_{t_{n-1}}^{t_n} \left| (\sigma(t, Y_n(t)) - \sigma(t, X(t))) \right|^2 dw(t) \right] \right)^{\frac{1}{2}} \\
\leq \left( E(U_{n-1} - X(t_{n-1}))^2 \right)^{\frac{1}{2}} \\
+ D_2 (1 + 2\sqrt{\lambda N}) \left( \int_{t_{n-1}}^{t_n} E|Y_n(t) - X(t)|^2 \right)^{\frac{1}{2}}
\]

(4.2)

where the Lipschitz continuity of \( b \) and \( \sigma \), the Schwarz inequality in \( L^2([t_{n-1}, t_n]) \) and properties of the Ito integral were applied. Now introduce the \( L^2 \)-projector \( P_L \) of \( L^2([t_{n-1}, t_n]) \) onto \( P^k([t_{n-1}, t_n], \mathbb{R}^d) \) and an arbitrary polynomial \( \bar{Y} \in P^k([t_{n-1}, t_n], \mathbb{R}^d) \). If we substitute for \( V_n \) into (4.1) the solution of the problem

\[
\dot{V}_n(t) = -P_L(Y_n - \bar{Y}_n) \quad (t \in [t_{n-1}, t_n]) \\
V_n(t_{n-1}) = 0
\]

we obtain from equation (4.1) the estimate

\[
\left\langle U_n - X(t_n), -\int_{t_{n-1}}^{t_n} P_L(Y_n - \bar{Y}_n) dt \right\rangle \\
= -\int_{t_{n-1}}^{t_n} \left\langle Y_n - X(t), P_L(Y_n - \bar{Y}_n) \right\rangle dt \\
- \int_{t_{n-1}}^{t} \left\langle b(t, Y_n) - b(t, X(t)), \int_{t_{n-1}}^{t} P_L(Y_n - \bar{Y}_n) ds \right\rangle dt \\
- \int_{t_{n-1}}^{t} \left\langle \sigma(t, Y_n) - \sigma(t, X(t)), \int_{t_{n-1}}^{t} P_L(Y_n - \bar{Y}_n) dsw(t) \right\rangle.
\]
From the last equation we obtain with elementar transformations

\[
\left( E \int_{t_{n-1}}^{t_n} |P(Y_n - \tilde{Y}_n)|^2 dt \right)^{\frac{1}{2}} \\
\leq h_N^{\frac{1}{2}} (E|U_{n-1} - X(t_{n-1})|^2) \frac{1}{2} \\
+ \left[ 1 + \sqrt{2}D_2 + (2\sqrt{2}D_2 + \sqrt{2D_2h_N})h_N \right] \\
\times \left( E \int_{t_{n-1}}^{t_n} |	ilde{Y}_n(t) - X(t)|^2 dt \right)^{\frac{1}{2}} \\
+ \left[ \sqrt{2}D_2 + (2\sqrt{2}D_2 + \sqrt{2D_2h_N})h_N \right] \left( E \int_{t_{n-1}}^{t_n} |Y_n(t) - \tilde{Y}_n(t)|^2 dt \right)^{\frac{1}{2}}
\]

where properties of the projector and the Ito integral, the Lipschitz continuity of \(b\), the Schwarz inequality in \(L^2(\Omega \times [t_{n-1}, t_n] \times [t_{n-1}, t_n])\) and the triangle inequality were applied. The constants \(D_1\) and \(D_2\) are independent from \(h_N\) and \(n\). Inequality (4.3) and the inequality \(\beta_1|Y|^2 \leq |\langle P(Y), Y(t_n) \rangle| \leq \beta_2|Y|^2\) yield

\[
\left[ \beta_1 - \sqrt{2}D_2 - (2\sqrt{2}D_2 + \sqrt{2D_2h_N})h_N \right] \left( E \int_{t_{n-1}}^{t_n} |Y_n(t) - \tilde{Y}_n(t)|^2 dt \right)^{\frac{1}{2}} \\
\leq h_N^{\frac{1}{2}} (E|U_{n-1} - X(t_{n-1})|^2) \frac{1}{2} \\
+ \left[ 1 + \sqrt{2}D_2 + (2\sqrt{2}D_2 + \sqrt{2D_2h_N})h_N \right] \left( E \int_{t_{n-1}}^{t_n} |	ilde{Y}_n(t) - X(t)|^2 dt \right)^{\frac{1}{2}}
\]

The last inequality and inequality (4.2) become

\[
(E|U_n - X(t_n)|^2)^{\frac{1}{2}} \\
\leq (E|U_{n-1} - X(t_{n-1})|^2)^{\frac{1}{2}} + q_c h_N^{\frac{1}{2}} (E|U_{n-1} - X(t_{n-1})|^2)^{\frac{1}{2}} \\
+ c \left[ 1 + q \left( 1 + \sqrt{2}D_2 + (2\sqrt{2}D_2 + \sqrt{2D_2h_N})h_N \right) \right] \left( E \int_{t_{n-1}}^{t_n} |	ilde{Y}_n(t) - X(t)|^2 dt \right)^{\frac{1}{2}}
\]

where \(c\) and \(q\) are constants which depend from the Lipschitz constant \(D_2\) and \(T\).
Through special selection of $\hat{Y}_n(\cdot)$ we obtain for $k_0 > 0$

$$
\left( E \int_{t_{n-1}}^{t_n} |\hat{Y}_n(t) - X(t)|^2 dt \right)^{\frac{1}{2}} \leq \left( \int_{t_{n-1}}^{t_n} E \sup_{t \in [t_{n-1}, t_n]} |\hat{Y}_n(t) - X(t)|^2 dt \right)^{\frac{1}{2}} \leq k_0 h_{N}^{\frac{1}{2}}.
$$

Then there follows

$$
(E|U_n - X(t_n)|^2)^{\frac{1}{2}} \leq \left[ 1 + c h_{N}^{\frac{1}{2}} \right] (E|U_{n-1} - X(t_{n-1})|^2)^{\frac{1}{2}} + c \left[ 1 + q \left\{ 1 + h_{N}^{\frac{1}{2}} (2\sqrt{2}D_2 + \sqrt{2D_2 h_n} + \sqrt{2}D_2) \right\} k_0 h_{N}^{\frac{1}{2}} \right].
$$

(4.6)

Now we apply [1: Lemma A.2.2] of (4.6) with $M = 1$, $U_0 = 0$ and $\delta = 1$ and obtain

$$
(E|U_n - X(t_n)|^2)^{\frac{1}{2}} \leq k_1 h_{N}^{\frac{1}{2}} \quad (n \in \mathbb{N}).
$$

This estimate substituted into the right side of (4.6) yields

$$
\max_{1 \leq n \leq N} (E|U_n - X(t_n)|^2)^{\frac{1}{2}} \leq C h_{N}^{\frac{1}{2}}
$$

for all $n = 1, \ldots, N$ with some constant $C > 0$.

The second statement of Theorem 4 we obtain from (4.4) with the help of elementar transformations. Indeed, we have

$$
\left( E \int_{t_{n-1}}^{t_n} |Y_n(t) - X(t)|^2 dt \right)^{\frac{1}{2}} \leq c h_{N}^{\frac{1}{2}} (E|U_{n-1} - X(t_{n-1})|^2)^{\frac{1}{2}} + c \left[ 1 + \sqrt{2}D_2 + (2\sqrt{2}D_2 + \sqrt{2D_2 h_n}) h_{N}^{\frac{1}{2}} \right] \left( E \int_{t_{n-1}}^{t_n} |Y_n(t) - X(t)|^2 dt \right)^{\frac{1}{2}}.
$$

(4.7)

Through special selection of $\hat{Y}_n$ defined as

$$
\hat{Y}_n(t) = X(t_{n-2}) + \frac{1}{t_n - t_{n-2}} (X(t_{n-2}) - X(t_{n-1}))(t - t_{n-1})
$$

we obtain

$$
E \int_{t_{n-1}}^{t_n} |\hat{Y}_n(t) - X(t)|^2 dt \leq c_1 h_{N}^2
$$
and then $E|X(t) - X(s)|^2 \leq c_1|t - s|$ with some constant $c_1 > 0$. This equation, (4.6) and the first statement of Theorem 4 yield

$$E \int_{t_{n-1}}^{t_n} |Y_n(t) - X(t)|^2 dt \leq c_2 h_n^2$$

and finally

$$E \int_0^T \left| \sum_{n=1}^{N} Y_n(s)X(t_{n-1},t_n) - X(s) \right|^2 ds \leq C_1 T h_N \quad (C_1,T \in \mathbb{R}).$$

Thus the assertion is proved. \]

5. A Computing possibility

We now return to the problem of computing the solution of problem (3.2). We assume $d = 1$. It follows from (3.2) for $V_n = 1$ that

$$U_n = U_{n-1} + \int_{t_{n-1}}^{t_n} b(t,Y_n(t)) dt + \int_{t_{n-1}}^{t_n} \sigma(t,Y_n(t)) dw(t). \quad (5.1)$$

Let $\{\phi_0, \ldots, \phi_k\}$ be a base in $P^k([t_{n-1},t_n], \mathbb{R}^1)$. Then $Y_n$ has the representation

$$Y_n(t) = \sum_{j=0}^{k} Y_{nj}\phi_j(t)$$

with $Y_{nj} \in L^2_{\infty-1}(\Omega)$ and we have to determine $Y_{nj} \ (j = 1, \ldots, k)$. At first we calculate for given $\hat{U}_{n-1} \in L^2_{\infty-1}(\Omega)$ (in the case $n = 1$ we have $\hat{U}_0 = X_0$) random variables $\hat{Y}_{nj} \in L^2_{\infty-1}(\Omega)$ with

$$\hat{Y}_n(t) = \hat{U}_{n-1} + \int_{t_{n-1}}^{t} b(s,\hat{Y}_n(s)) ds \quad (5.2)$$

where $\hat{Y}_n(t) = \sum_{j=0}^{k} \hat{Y}_{nj}\phi_j(t)$. That is, we have to solve for fixed $t \in [t_{n-1},t_n]$ a (nonlinear) equation with random inhomogeneous part. Then we define $\hat{U}_n$ as

$$\hat{U}_n(t) = \hat{U}_{n-1} + \int_{t_{n-1}}^{t} b(t,\hat{Y}_n(t)) dt + \int_{t_{n-1}}^{t} \sigma(t,\hat{Y}_n(t)) dw(t). \quad (5.3)$$

Obviously, $(\hat{U}_n(t_n - 0), \hat{Y}_n(t))$ is a solution of (5.1) and subsequently, it follows by the Ito formula that $(U_n, Y_n)$ with

$$U_n = \hat{U}_n(t_n - 0) \quad \text{and} \quad Y_n(t) = \hat{Y}_n(t)$$

solves problem (3.2).
References


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