On the Limiting Regularity Result of some Nonlinear Elliptic Equations

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Abstract. We shall prove the limiting regularity $W^{1, N(p-1)\left(\frac{N-1}{N}\right)}$ of solutions of some nonlinear elliptic problems with right hand side in $L^{\text{Log}^\alpha L}(\Omega)$ and $\alpha \geq \frac{N-1}{N}$. Also, an improved regularity is given when $\alpha < \frac{N-1}{N}$.

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1. Introduction

We deal with boundary value problems

$$\begin{aligned}
\mathcal{A}(u) := -\text{div}(a(\cdot, u, \nabla u)) &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}$$

(E)

where $\Omega$ is a regular bounded domain of $\mathbb{R}^N, N \geq 2$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, measurable with respect to $x$ in $\Omega$ for every $(s, \xi)$ in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to $(s, \xi)$ in $\mathbb{R} \times \mathbb{R}^N$ for almost every $x$ in $\Omega$). We assume that there exist a real positive constant $\nu > 0$, a nonnegative function $k$ in $L^p(\Omega)$, $p' = \frac{p}{p-1}$, where $2 - \frac{1}{N} < p \leq N$, such that for almost every $x$ in $\Omega$, for every $s$ in $\mathbb{R}$, for every $\xi$ and $\xi^*$ in $\mathbb{R}^N$:

$$\begin{aligned}
a(x, s, \xi)\xi &\geq \nu|\xi|^p \quad (1.1) \\
[a(x, s, \xi) - a(x, s, \xi^*)][\xi - \xi^*] &> 0, \quad \xi \neq \xi^* \quad (1.2) \\
|a(x, s, \xi)| &\leq k(x) + |s|^{p-1} + |\xi|^{p-1}. \quad (1.3)
\end{aligned}$$

The use of the $L^{\text{Log}^\alpha L}(\Omega)$ space to study the problem (E) in the linear case, is
early introduced by G. Stampacchia in [14] (for the case \( \alpha = \frac{N-1}{N} \)), by A. Pas-
sareli di Napoli and C. Sbordonne in [13] (for \( 0 < \alpha \leq 1 \)) and recently by
A. Fiorenza and M. Krebec in [7] for the case \( \alpha \geq \frac{N-1}{N} \). In the nonlinear case,
the particular situations were given in [4]. Another approach to reach the lim-
ting regularity was given in [3].

Our main result consists in reaching the limiting regularity \( W^{1,\bar{p}}_0(\Omega), \bar{p} = \frac{N(p-1)}{N-1} \) with \( f \) belonging to the space \( L\text{Log}^\alpha L(\Omega), \alpha \geq \frac{N-1}{N} \) in the nonlinear

For the sake of simplicity, we restrict our studies to the \( p \)-Laplacian problem
model, i.e., \( a(\cdot, u, \nabla u) = |\nabla u|^{p-2}\nabla u \).

2. Preliminaries

We list some well known results about Orlicz and Orlicz–Sobolev spaces.

2.1. Let \( M : \mathbb{R}^+ \to \mathbb{R}^+ \) be an \( N \)-function, i.e., \( M \) is continuous, convex
with \( M(t) > 0 \) for \( t > 0, \frac{M(t)}{t} \to 0 \) as \( t \to 0 \) and \( \frac{M(t)}{t} \to \infty \) as \( t \to \infty \).
Equivalently, \( M \) admits the representation \( M(t) = \int_0^t a(s) \, ds \), where \( a : \mathbb{R}^+ \to \mathbb{R}^+ \)
is nondecreasing, right continuous, with \( a(0) = 0, a(t) > 0 \) for \( t > 0 \) and \( a(t) \to \infty \) as \( t \to \infty \). The N-function \( \overline{M} \) conjugate to \( M \) is defined by \( \overline{M}(t) = \int_0^t \overline{a}(s) \, ds \), \( \overline{a} : \mathbb{R}^+ \to \mathbb{R}^+ \) is given by \( \overline{a}(t) = \sup \{ s : a(s) \leq t \} \) (see [1, 10]). The
N-function is said to satisfy the \( \Delta_2 \)-condition if, for some \( k > 0 \),

\[
M(2t) \leq kM(t) \quad \forall t \geq 0.
\]  

If (2.1) holds only for \( t \geq t_0 > 0 \), then \( M \) is said to satisfy the \( \Delta_2 \)-condition
near infinity.

We will extend these N-functions into even functions on all \( \mathbb{R} \).

2.2. Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). The \( \text{Orlicz class} \) \( K_M(\Omega) \) (resp. the \( \text{Orlicz}
space} L_M(\Omega) \)) is defined as the set of (equivalences classes of) real valued mea-
surable functions \( u \) on \( \Omega \) such that \( \int_{\Omega} M(u(x)) \, dx < +\infty \) (resp. \( \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) \, dx < +\infty \) for some \( \lambda > 0 \)). \( L_M(\Omega) \) is a Banach space under the norm

\[
\| u \|_{M, \Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) \, dx \leq 1 \right\}
\]

and \( K_M(\Omega) \) is a convex subset of \( L_M(\Omega) \).

The closure in \( L_M(\Omega) \) of the set of bounded measurable functions with
compact support in \( \overline{\Omega} \) is denoted by \( E_M(\Omega) \). The equality \( E_M(\Omega) = L_M(\Omega) \)
holds if and only if \( M \) satisfies the \( \Delta_2 \)-condition, for all \( t \) or for \( t \) large according
to whether \( \Omega \) has infinite measure or not. The dual of \( E_M(\Omega) \) can be identified
with \( L_{\overline{M}}(\Omega) \) by means of pairing \( \int_{\Omega} u(x)v(x) \, dx \) and the dual norm on \( L_{\overline{M}}(\Omega) \)
is equivalently to \( \| u \|_{\overline{M}, \Omega} \).
The space $L_M(\Omega)$ is reflexive if and only if $M$ and $\overline{M}$ satisfy the $\Delta_2$-condition, for all $t$ or for $t$ large according to whether $\Omega$ has infinite measure or not.

2.3. We now turn to the Orlicz-Sobolev space. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm

$$
\|u\|_{1,M,\Omega} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_{M,\Omega}.
$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_M)$ and $\sigma(\prod L_M, \prod L_M)$.

The space $W^1_0E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W^1_0L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_M)$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$.

Let $W^{-1}L_M(\Omega)$ (resp. $W^{-1}E_M(\Omega)$) denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\leq 1$ of functions in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set $\Omega$ has the segment property, then the space $D(\Omega)$ is dense in $W^1_0L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\prod L_M, \prod L_M)$ (see [8, 9]). Consequently, the action of a distribution in $W^{-1}L_M(\Omega)$ on an element of $W^1_0L_M(\Omega)$ is well-defined.

We denote by $LLog^aL(\Omega)$ the Orlicz space $L_M(\Omega)$ where $M(t) \approx t\ln^a(t)$ as $t \to \infty$.

The following abstract lemma will be applied in the following.

**Lemma 2.1** ([2]). Let $F: \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian with $F(0) = 0$. Let $M$ be an N-function and let $u \in W^1_0L_M(\Omega)$ (resp. $W^1_0E_M(\Omega)$). Then $F(u) \in W^1_0L_M(\Omega)$ (resp. $W^1_0E_M(\Omega)$). Moreover, if the set of discontinuity points of $F'$ is finite, then

$$
\frac{\partial}{\partial x_i} F(u) = \begin{cases} 
F'(u) \frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \notin D\} \\
0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}.
\end{cases}
$$

3. Main result

Let $M$ be an N-function such that

(H) \quad \left\{ \begin{align*} 
K(s) &= (M^{-1}(s))^p \quad \text{is convex and} \\
\int_0^1 M \circ B^{-1} \left( \frac{1}{r^{1-p} \log^a(\frac{1}{r})} \right) \, dr &< +\infty, \quad B(t) = t^{p-1}.
\end{align*} \right.


Theorem 3.1. Under the assumptions (1.1)–(1.3), $2 - \frac{1}{N} < p \leq N$ and $f$ in $L \log^{\alpha} L(\Omega)$ with $\alpha \geq \frac{N-1}{N}$, there exists at least a weak solution $u \in W^{1,q}_0(\Omega)$ of problem (E) where $q = \frac{N(p-1)}{N-1}$. Moreover, if $\alpha > \frac{N-1}{N}$, then $u \in W^{1,\tilde{\alpha}}_0LM(\Omega)$ for every $N$-function $M$ satisfying (H).

Remark 3.2. The proof of the last theorem allows us to obtain an improved regularity of the solution $u$ of (E) in the Orlicz–Sobolev spaces. For example,

$$u \in W^{1,\tilde{\alpha}}_0LM(\Omega), M(t) = \frac{\tilde{\alpha}}{\log^{\alpha}(e+t)}, \quad \text{for all } \sigma > 1 - \frac{\alpha N}{N-1} \text{ if } \alpha \in [0, \frac{N-1}{N-1}]$$

$$u \in W^{1,\tilde{\alpha}}_0LM(\Omega), M(t) = \tilde{\alpha} \log^{\alpha}(e+t), \quad \text{for all } \sigma < \frac{\alpha N}{N-1} - 1 \text{ if } \alpha > \frac{N-1}{N}.$$  

For the case $\alpha = \frac{N-1}{N}, p < N$, the regularity $W^{1,\tilde{\alpha}}_0(\Omega)$ is optimal.

Hereafter, we denote by $X_N$ the real number defined by $X_N = NC_N$, $C_N$ is the measure of the unit ball of $\mathbb{R}^N$. 

The following lemma (see [15] for the general case) plays an essential role for estimation of the approximate solutions of the problem.

Lemma 3.3. Let $u \in W^{1,p}_0(\Omega), 1 < p < +\infty$. Then

$$-\mu'(t) \geq X_N \mu(t)^{1-\frac{1}{p}} \left( - \frac{1}{X_N \mu(t)^{1-\frac{1}{p}}} \frac{d}{dt} \int_{\{|u| > t\}} |\nabla u|^p \, dx \right)^{-\frac{1}{p-1}}.$$  

Proof of Theorem 3.1. If $\alpha > \frac{N-1}{N}, 2 - \frac{1}{N} < p \leq N$, then we consider the approximate problem

$$\left\{ \begin{array}{l}
A(u_n) := -\text{div}(a(\cdot, u_n, \nabla u_n)) = f_n \quad \text{in } \Omega \\
u_n \in W^{1,p}_0(\Omega),
\end{array} \right.$$  

(3.1)

where $(f_n)$ is a smooth sequence of functions satisfying $f_n \to f$ in $L^N(\Omega)$ for the modular convergence, $H(t) = t \log^{\alpha}(1+t)$.

Let $\varphi$ be a truncation defined by

$$\varphi(\xi) = \begin{cases} 
0, & 0 \leq \xi \leq t \\
\frac{1}{h}(\xi - t), & t < \xi < t + h \\
1, & \xi \geq t + h \\
-\varphi(-\xi), & \xi < 0,
\end{cases}$$  

(3.2)

for all $t, h > 0$. Without loss of generality, we omit the index $n$. Using $v = \varphi(u)$ as a test function in (3.1), we obtain

$$\int_{\Omega} a(\cdot, u, \nabla u) \nabla u \varphi'(u) \, dx = \int_{\Omega} f \varphi(u) \, dx$$

$$\frac{1}{h} \int_{\{t < |u| < t+h\}} |\nabla u|^p \, dx \leq C \int_{\{u \geq t+h\}} f \, dx.$$  

And letting $h \to 0$, we have
\[ -\frac{d}{dt} \int_{|u|>t} |\nabla u|^p \, dx \leq C \int_{|u|\geq t} f \, dx. \tag{3.3} \]
By using Lemma 3.3 we obtain (supposing $-\mu'(t) > 0$ which does not affected the proof)
\[ \frac{1}{\mu(t)} \frac{d}{dt} \int_{|u|>t} |\nabla u|^p \, dx \leq \left( -\frac{1}{\mathcal{X}_N \mu(t)^{1-\frac{1}{p}}} \frac{d}{dt} \int_{|u|>t} |\nabla u|^p \, dx \right)^{\frac{p}{p-1}} \]
or equivalently
\[ \left( \frac{1}{\mu(t)} \frac{d}{dt} \int_{|u|>t} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \leq \left( -\frac{1}{\mathcal{X}_N \mu(t)^{1-\frac{1}{p}}} \frac{d}{dt} \int_{|u|>t} |\nabla u|^p \, dx \right)^{\frac{p}{p-1}}. \]

Let $M$ be an N-function satisfying (H). Jensen’s inequality involves
\[ K \left( \int_{|u|<|u|<t+h]} M(|\nabla u|) \right) \leq \frac{\int_{|u|<|u|<t+h]} (|\nabla u|)^p}{-\mu(t+h) + \mu(t)}. \]

Then
\[ M^{-1} \left( \frac{1}{\mu'(t)} \frac{d}{dt} \int_{|u|>t} M(|\nabla u|) \, dx \right) \leq \left( \frac{1}{\mu'(t)} \frac{d}{dt} \int_{|u|>t} |\nabla u|^p \, dx \right)^{\frac{1}{2}} \leq \left( -\frac{1}{\mathcal{X}_N \mu(t)^{1-\frac{1}{p}}} \frac{d}{dt} \int_{|u|>t} |\nabla u|^p \, dx \right)^{\frac{p}{p-1}}. \]
Therefore we have
\[ -\frac{d}{dt} \int_{|u|>t} M(|\nabla u|) \, dx \leq (-\mu'(t))M \left( -\frac{1}{\mathcal{X}_N \mu(t)^{1-\frac{1}{p}}} \frac{d}{dt} \int_{|u|>t} |\nabla u|^p \, dx \right)^{\frac{1}{p-1}}. \]
Combining with (3.3) and the fact that the function $t \to \int_{|u|>t} M(|\nabla u|) \, dx$ is absolutely continuous, we obtain
\[
\int_{\Omega} M(|\nabla u|) \, dx = \int_{0}^{+\infty} \left( -\frac{d}{dt} \int_{|u|>t} M(|\nabla u|) \, dx \right) \, dt \\
\leq \int_{0}^{+\infty} (-\mu'(t))M \left( \frac{C \int_{|u|\geq t} f \, dx}{\mathcal{X}_N \mu(t)^{1-\frac{1}{p}}} \right)^{\frac{1}{p-1}} \, dt \\
\leq \frac{1}{C'} \int^{C'}_{0} M \left( \frac{C}{r^{1-\frac{1}{p}} \log^{n} \left( \frac{1}{r} \right)} \right)^{\frac{1}{p-1}} \, dr < \infty,
\]
where $C' = \left(\frac{1}{2}\right)^{N'}$, and the last inequality is obtained by using the Hölder inequality on $\int_{|u| \geq t} f \, dx$.

Finally, we deduce that $(\nabla u_n)_{n \geq 0}$ is bounded in $(L^M(\Omega))^N$ for every N-function satisfying (H). In particular, $(\nabla u_n)_{n \geq 0}$ is bounded in $(L^p(\Omega))^N$. As in [4], the almost everywhere convergence of the gradients can be obtained and the proof of theorem follows with the same way.

We deal now with the case $\alpha = \frac{N-1}{N}$ and $2 - \frac{1}{N} < p < N$. We recall that the authors in [4] have proved some regularity result but by assuming that $\alpha = \frac{N}{N-1}$ and $p = N$. Consider now the following approximate problems:

\[
\begin{aligned}
&\left\{- \text{div} \left( a(x, u_n, \nabla u_n) \right) - \frac{1}{n} \text{div} \left( |\nabla u_n|^{N-2} \nabla u_n \right) = f_n \quad \text{in} \quad \Omega \\
&u_n \in W^{1,N}_0(\Omega) \right. 
\end{aligned}
\]

The solutions $u_n$ exist thanks to the Leray–Lions theorem (see [11]). Taking $v = u_n$ as test function in the problem (3.4), we have

\[
\int_\Omega |\nabla u_n|^p \, dx + \frac{1}{n} \int_\Omega |\nabla u_n|^N \, dx \leq 2\|f_n\|_{A}\|u_n\|_{A} \leq C\|u_n\|_{W^{1,N}_0},
\]

where we have used the continuous and optimal injection $W^{1,N}_0(\Omega) \hookrightarrow L_A(\Omega)$ with $A(t) = e^{t^{N'}} - 1$ (see [5]). Then we deduce $\frac{1}{n}(\int_\Omega |\nabla u_n|^N \, dx)^{\frac{N-1}{N}} \leq C$. Let now $\phi \in W^{1,N}_0(\Omega)$ as test function, one has

\[
\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \phi \, dx + \frac{1}{n} \int_\Omega |\nabla u_n|^{N-2} \nabla u_n \nabla \phi \, dx = \int_\Omega f_n \phi \, dx,
\]

so

\[
\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \phi \, dx \leq \left| \frac{1}{n} \int_\Omega |\nabla u_n|^{N-2} \nabla u_n \nabla \phi \, dx \right| + C\|\phi\|_{W^{1,N}_0} \leq C\|\phi\|_{W^{1,N}_0}
\]

which implies, thanks to [6, Theorem 4.1], that $\int_{\Omega} |\nabla u_n|^q \, dx \leq C$, where here and below $C$ denote positive constants not depending on $n$. Therefore, we can see that there exist a measurable function $u \in W^{1,q}_0(\Omega)$ and a subsequence also denoted $(u_n)_n$,

\[
\begin{aligned}
&u_n \rightharpoonup u \quad \text{weakly in } W^{1,q}_0(\Omega) \\
&T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W^{1,p}_0(\Omega),
\end{aligned}
\]

where $T_k$ is the usual truncation defined by $T_k(s) = \max(-k, \min(k, s))$, for all $s \in \mathbb{R}$, for all $k \geq 0$. Let $v \in D(\Omega)$ and choose the test function $T_k(u_n - v), n > k + \|v\|_\infty$, in the approximate problem, we have

\[
\begin{aligned}
&\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla T_k(u_n - v) \, dx + \frac{1}{n} \int_\Omega |\nabla u_n|^{N-2} \nabla u_n \nabla T_k(u_n - v) \, dx \\
&= \int_\Omega f_n T_k(u_n - v) \, dx
\end{aligned}
\]
which we rewrite as follows:

\[
\begin{align*}
\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v) \nabla T_k(u_n - v) \, dx \\
+ \int_\Omega |\nabla v|^{p-2}\nabla v \nabla T_k(u_n - v) \, dx \\
+ \frac{1}{n} \int_\Omega (|\nabla u_n|^{N-2}\nabla u_n - |\nabla v|^{N-2}\nabla v) \nabla T_k(u_n - v) \, dx \\
+ \frac{1}{n} \int_\Omega |\nabla v|^{N-2}\nabla v \nabla T_k(u_n - v) \, dx \\
= \int f_n T_k(u_n - v) \, dx.
\end{align*}
\]

This obviously gives

\[
\int_\Omega |\nabla v|^{p-2}\nabla v \nabla T_k(u_n - v) \, dx + \frac{1}{n} \int_\Omega |\nabla v|^{N-2}\nabla v \nabla T_k(u_n - v) \, dx \leq \int f_n T_k(u_n - v) \, dx.
\]

By using the fact that \( T_k(u_n - v) \rightarrow T_k(u - v) \) weakly in \( W^{1,p}_0(\Omega) \), we obtain

\[
\int_\Omega |\nabla v|^{p-2}\nabla v \nabla T_k(u - v) \, dx \leq \int_\Omega f T_k(u - v) \, dx, \quad \forall v \in \mathcal{D}(\Omega),
\]

By the density argument the last inequality remains true for all \( v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \).

To prove that \( u \) is a weak solution of the problem (E), we follow the technique used in [12]. Let \( h \) and \( k \) be positive real numbers, let \( t \) belong to \((-1, 1)\) and let \( \psi \) be a function in \( W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \). Choose \( \phi = T_h(u) + tT_k(u - \psi) \) in the previous inequality, we have \( u \) is a so-called entropy solution of (E) which completes the proof of the theorem.

\[\square\]

References


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