Solvability of Boundary Value Problems for the Inclusion
\( u_{tt} - u_{xx} \in g(t, x, u) \) via the Theory of Multi-Valued A-Proper Maps

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The existence of a coincidence point \( x \) for the inclusion
\[
Lx \in \Gamma(x),
\]
is studied where \( L: D(L) \subset E \to F \) is a linear hyperbolic operator and \( \Gamma: E \to 2^F \) is a convex-valued map. It is shown that any such monotone demi-continuous map \( \Gamma \) is weakly A-proper. Some existence theorems for (•) are established and the results are applied to a boundary value problem for the inclusion \( u_{tt} - u_{xx} \in g(t, x, u) \).

0. Let \( E, F \) be Banach spaces, \( L: D(L) \subset E \to F \) a linear operator, and \( \Gamma: E \to 2^F \) a convex-valued map. The aim of this paper is to establish some existence theorems for the coincidence inclusion
\[
Lx \in \Gamma(x),
\]
and to present an application of those results to a boundary value problem for the inclusion
\[
u_{tt} - u_{xx} \in g(t, x, u).
\]

For a Fredholm operator \( L \), the problem (0.1) has been studied by many authors, see [4, 9, 17, 19]. That case can be applied, for example, to differential inclusions of the type \( Lu \in g(x, u, Du) \) where \( L \) is elliptic or, in particular, \( Lu = u'' \) (see [10, 18])
but not to hyperbolic operators, as it is the case with (0.2). Of our interest is to extend to the multi-valued case some of the results presented in [11] where \( L \) was a closed operator with \( \text{dim Ker } L = \text{codim Im } L = \infty \), and where \( \Gamma = f: E \to F \) was a single-valued map. We refer the reader to [11] for the extensive references to that topic.

The main result of Section 1, Theorem 1.3, is an analogue of the Leray-Schauder, Theorem on a priori bounds. The proof is based on

a) the method of Topological Transversality introduced and developed by Granas [6, 8],

b) the coincidence theory of Fredholm operators of index zero with multi-valued maps presented in [9],

c) adopting, to multi-valued maps, the technique of \( A \)-proper maps originally due to Petryshyn ([15]; see also [16]) who was later joined by Browder [2] in the research of that class of maps.

In Section 2, we show that any monotone demi-continuous convex-valued map is weakly \( A \)-proper. For a single-valued map, results of this type appear in most of papers using, explicitly or not, \( A \)-proper mapping techniques, e.g. in [11–14, 20]. Our direct approach, however, seems to be particularly simple.

In Section 3, we study generalized solutions of a periodic-Dirichlet problem for the inclusion (0.2). Its formulation is modelled on a result of Mawhin [12].

In Section 4, we derive the existence of a generalized optimal solution of the periodic-Dirichlet problem for the equation

\[
\frac{du}{dt} - u_{xx} = f(t, x, u),
\]

The growth condition on \( f \) in Corollary 4.2 comes from [12]. Differential inclusions of first order, as a tool for investigating equations with discontinuous right-hand side had been first considered by Filippov [7] and they became a frequent tool in the optimal control theory (see e.g., [1: Chapter II]). For the elliptic boundary value problems, the concept of what we call optimal solution (also called solution in the sense of Filippov) has been used by Chang [5]. To the authors' best knowledge, optimal solutions of hyperbolic equations with discontinuous right-hand side have not been previously studied.

1. In what follows, \( E, F \) are Banach spaces and \( L: D(L) \subset E \to F \) is a densely defined linear (not necessarily bounded) operator. We assume the following conditions on \( L \):

\[
\begin{align*}
(\text{L1}) & \quad L \text{ is closed (i.e. the graph of } L \text{ is closed in } E \times F); \\
(\text{L2}) & \quad \text{Ker } L \text{ and } \text{R}(L) \text{ are closed and topologically complemented, i.e. } E = \text{Ker } L \\
& \quad \oplus E_0, F = F_0 \oplus \text{R}(L), E_0 \text{ and } F_0 \text{ closed}; \\
(\text{L3}) & \quad \text{dim Ker } L = \text{codim } \text{R}(L) = \infty.
\end{align*}
\]

With the decomposition given in (L2) we associate the linear projections \( P: E \to E_0, Q: F \to \text{R}(L) \). By \( K: \text{R}(L) \to E_0 \) we denote the right inverse of \( L \), i.e. the inverse of the operator \( L_{|E_0 \cap D(L)}: E_0 \cap D(L) \to \text{R}(L) \). Since \( L \) is closed, \( K \) is a linear bounded operator.

We associate with \( L \) a Fredholm Factorization \( \Pi = (E_n, P_n, F_n, Q_n)_{n \in \mathbb{N}} \) defined in [11]. We recall what this means. First, by \( \{E_n\} \) and \( \{F_n\} \) we denote the dense finite-dimensional filtrations of \( \text{Ker } L \) and \( F_0 \) respectively, i.e. \( E_n \subset E_{n+1} \) and \( F_n \subset F_{n+1} \) for \( n \in \mathbb{N} \), \( \bigcup E_n = \text{Ker } L, \bigcup F_n = F_0 \) and \( \text{dim } E_n = \text{dim } F_n < \infty \), \( n \in \mathbb{N} \). We suppose that there exist linear projections \( P_n': \text{Ker } L \to F_n' \), \( Q_n': F_0 \to F_n' \) such that

\[
\|P_n' x - x\| = d(x, E_n'), x \in \text{Ker } L, \quad \|Q_n' y - y\| = d(y, F_n'), y \in F_0,
\]

(1.1)
where \( d(., .) \) denotes the distance from a point to a set. We finally define \( E_n := E'_n + E_0, \)
\( F_n := \text{F}_n' \oplus R(L), P_n := P + P_n'(I - P); E \to E_n, Q_n := \text{Q} + Q_n'(I - Q); \text{F} \to \text{F}_n, \)
\( n \in \mathbb{N}. \) Thus \( \Pi = \{ E_n, P_n, F_n, Q_n \}_{n \in \mathbb{N}} \) is defined. From (1.1) it follows that
\[
P_n x \to x \quad \text{for} \quad x \in \text{Ker } L; \quad Q_n y \to y \quad \text{for} \quad y \in \text{F}_0;
\]
\[
P_n x \to x \quad \text{for} \quad x \in E; \quad Q_n y \to y \quad \text{for} \quad y \in \text{F}.
\]

We also note that, for any \( n, \) the operator
\[
L_n := L_{|E_n \cap \text{D}(L)}: E_n \cap \text{D}(L) \to \text{F}_n
\]
is a Fredholm operator of index zero. If \( J_n: E_n' \to F_n' \) is any isomorphism, then
\[
T_n = J_n(I - P): E_n \to F_n \text{ is a Fredholm resolvent of finite rank of } L_n, \text{i.e. an operator of finite rank such that } L_n + T_n \text{ is bijective.}
\]
We note that \( (L_n + T_n)^{-1} = KQ + J_n^{-1}(I - Q); F_n \to E_n \text{ is a bounded operator which is compact whenever } K \text{ is compact.}
\]

Let us recall that a multi-valued map \( F: X \to 2^Y \) is called upper semicontinuous if \( \bigcup \{ F(x) \mid x \in X \} \) is open for any open \( U \) in \( Y. \) We are concerned with maps \( \Gamma: X \to 2^Y, \) where \( X \subseteq E \text{ or } X \subseteq E \times [0, 1] \) has a non-empty intersection with \( E \cap \text{D}(L) \) (respectively with \( (E \cap \text{D}(L)) \times [0, 1] \)) for all but finitely many \( n. \) In general, \( \Gamma \) is not assumed upper semicontinuous but we always assume that the values of \( \Gamma \) are non-empty closed and convex. We use the notation
\[
X_n = X \cap E_n \quad \text{(resp., } X_n = X \cap (E_n \times [0, 1]){)};
\]
\[
\Gamma_n = Q_n \circ \Gamma: X_n \to 2^{F_0}.
\]

A map \( \Gamma: X \to 2^Y \) is called \textit{L-compact} if the map \( \Gamma_n: X_n \to 2^{F_n} \) is \( L_n \)-compact in the sense of \( [9], \) for a.e. \( n. \) This means that \( (L_n + T_n)^{-1} \circ \Gamma_n : X_n \to 2^{F_n} \) is upper semicontinuous and it sends bounded sets to relatively compact sets, where \( T_n \) is defined above. Let now \( A \subseteq X \) be a pair of closed bounded subsets of \( E \) and let \( \Gamma: X \to 2^Y \) be an \textit{L-compact} map. We say that \( \Gamma \in \mathcal{K}(X, A) \) if, for a.e. \( n, \) \( \Gamma_n \) has no coincidence point with \( L_n \) in \( A_n \cap \text{D}(L), \text{i.e. } Lx \notin \Gamma_n(x) \text{ for all } x \in A_n \cap \text{D}(L). \) (We assume about \( A \) that \( A_n \cap \text{D}(L) = A \cap E_n \cap \text{D}(L) = \emptyset \text{ for a.e. } n. \) ) Such \( \Gamma \) is called \textit{L-essential} if \( \Gamma_n \) is \( L_n \)-essential in the sense of \( [9], \) for a.e. \( n. \) This means that every map \( \Phi_n : X_n \to 2^{F_n} \) with \( \Phi_n|_{A_n} = \Gamma_n|_{A_n} \) has a coincidence point with \( L_n \) in \( X_n \cap \text{D}(L). \) We say that \( \mathcal{H}: X \times [0, 1] \to 2^Y \) is a \textit{homotopy} between maps \( \Gamma, \Phi \in \mathcal{K}(X, A) \) if
\[
(i) \quad \mathcal{H}(\cdot, 0) = \Gamma \text{ and } \mathcal{H}(\cdot, 1) = \Phi;
\]
\[
(ii) \quad \mathcal{H}_n \text{ is } L_n \text{-compact, for a.e. } n;
\]
\[
(iii) \quad Lx \notin \mathcal{H}_n(x, t) \text{ for all } x \in A_n \cap \text{D}(L), t \in [0, 1].
\]
We write \( \Gamma \sim \Phi. \) It is easy to show that \( \sim \) is an equivalence relation.

\textbf{Proposition 1.1 (Topological Transversality Theorem):} Suppose that \( \Gamma \sim \Phi \in \mathcal{K}(X, A) \). Then \( \Gamma \) is \textit{L-essential} if and only if \( \Phi \) is \textit{L-essential.}

The proof is an immediate consequence of \([9]: \text{II. 4.5}]\) and the above definitions \( \square \)

Unlike in \([9], \) an \textit{L-essential} map may have no coincidence point with \( L \) on \( X \cap \text{D}(L). \) We must therefore restrict our study to maps defined below. A map \( \Gamma: X \to 2^Y \) is called \textit{A-proper} (respectively \textit{weakly A-proper}) with respect to the Fredholm factorization \( \Pi \) of \( L \) if for any bounded sequence \( \{x_k \in X_n \cap \text{D}(L)\} \) with \( d(Lx_k, \Gamma_n(x_k)) \to 0 \) as \( n_k \to \infty, \) there exists a subsequence \( \{x_{k_l}\} \) of \( \{x_k\} \) converging (resp. weakly converging) to an element \( x \in X \cap \text{D}(L), \) such that \( Lx \in \Gamma(x). \) Although this definition depends on \( \Pi, \) we will normally omit saying "with respect to \( \Pi \" \) having a given Fredholm factorization of \( L \) in mind.
Proposition 1.2: Let $U$ be an open bounded subset of $E$ and let $L: E \rightarrow 2^F$ be a weakly $A$-proper $L$-compact map. If $\Gamma \in \mathcal{K}_L(\overline{U}, \partial U)$, we suppose that it is $L$-essential. Then there exists an $x \in \text{co}U \cap D(L)$ such that $Lx \in \Gamma(x)$. If $\Gamma \not\in \mathcal{K}_L(\overline{U}, \partial U)$, then the same conclusion follows.

Proof: If $\Gamma \in \mathcal{K}_L(\overline{U}, \partial U)$, then, for a.e. $n$, there exists $x_n \in U \cap D(L)$ with $Lx_n \in \Gamma_n(x_n)$. Since $L$ is weakly $A$-proper, there is a subsequence $x_n \rightarrow x \in E \cap D(L)$ such that $Lx \in \Gamma(x)$. To reach the conclusion, we just note that a weak limit of a sequence of points in $U$ belongs to the closed convex hull $\text{co}U$ of $U$.

Theorem 1.3: Let $S: E \rightarrow F$ be an $A$-proper $L$-compact linear operator such that $L - S$ is injective and let $L: E \rightarrow 2^F$ be a weakly $A$-proper $L$-compact map satisfying the following condition: There exists a constant $M > 0$ (called a priori bound) and $n_0 \in \mathbb{N}$ such that every solution $x_n \in E \cap D(L)$ of

$$Lx \in (1 - \lambda)S_nx + \lambda \Gamma_n(x)$$

with $n > n_0$ and $\lambda \in (0, 1)$ must have norm less than $M$. Then there exists an $x \in E \cap D(L)$ with $\|x\| \leq M$ such that $Lx \in \Gamma(x)$.

Proof: Let $U = \{x \in E \mid \|x\| < M\}$. We may assume that $\Gamma \in \mathcal{K}_L(\overline{U}, \partial U)$, since otherwise the conclusion follows directly from Proposition 1.2. Next, we note that $S \in \mathcal{K}_L(\overline{U}, \partial U)$. Indeed, the contrary would imply the existence of $x_k \rightarrow \infty$, $x_k \in E \cap D(L)$, $\|x_k\| = M$, such that $Lx_k = Sx_k$. Since $S$ is $A$-proper, a subsequence of $x_k$ tends to an $x$ with $Lx = Sx$, $\|x\| = M$. This contradicts injectivity of $L - S$. It now follows from [9: II. 4.9] that $S$ is $L$-essential in $\mathcal{K}_L(\overline{U}, \partial U)$. It is easy to verify that the formula

$$\mathcal{H}(x, t) := (1 + t)Sx + t\Gamma(x), \quad x \in \overline{U}, \quad t \in [0, 1],$$

defines a homotopy from $S$ to $\Gamma$ in $\mathcal{K}_L(\overline{U}, \partial U)$. In fact, since we already know that $S$ and $\Gamma$ are in $\mathcal{K}_L(\overline{U}, \partial U)$, the property (iii) of homotopy must be only verified for $t \in (0, 1)$ and that is exactly guaranteed by (1.3). The conclusion now follows from Propositions 1.1 and 1.2.

Remark 1.4: Analogous definitions and results can be given for $L$-condensing multi-valued maps, i.e. such $\Gamma$ that, for a.e. $n$, $\Gamma_n$ is $L_n$-condensing with respect to a given measure of noncompactness on $E_n$ (see [9, 11]).

2. In this section $E = F = H$ is a separable Hilbert space with a scalar product $(\cdot, \cdot)$, and $L: D(L) \subset H \rightarrow H$ is a self-adjoint operator satisfying (L1)–(L3) of the previous section. This implies that $R(L) = \text{Ker} L^\perp$. Let $P$ be the orthogonal projection on $H$ onto $R(L)$, and let $\{v_n\}_{n \in \mathbb{N}}$ be an orthonormal set spanning a dense subspace of $\text{Ker} L$. We define

$$H_n := \text{Lin} \{v_1, \ldots, v_n\},$$

$$P_n := \text{Ker} L \rightarrow H_n', \quad P_n'x = \sum_{k=1}^{n} (x, v_k) v_k,$$

$$H_n = H_n' \oplus R(L), \quad P_n = P + P_n'(I - P): H \rightarrow H_n.$$
If follows that \( \Pi = (H_n, P_n, H_n, P_n) \) is a Fredholm factorization associated with \( L \). A map \( \Gamma: H \to 2^H \) is called \textit{monotone} if for any \( x_1, x_2, z_1, z_2 \in H \) with \( z_1 \in \Gamma(x_1) \) and \( z_2 \in \Gamma(x_2) \), we have \( (z_1 - z_2, x_1 - x_2) \geq 0 \). It is known that a compact-valued map \( \Gamma: H \to 2^H \) is upper semicontinuous if and only if \( \Gamma \) sends relatively compact sets to relatively compact sets and the graph of \( \Gamma \) is closed in \( H \times H \). Above, we mean the strong (norm) topology on \( H \times H \).

\section*{Theorem 2.1:} Any weakly continuous map \( \Gamma: H \to 2^H \) is weakly \( A \)-proper.

\textbf{Proof:} Let \( \{x_k \in H_n \cap D(L)\} \) be a bounded sequence with \( d(Lx_k, \Gamma_n(x_k)) \to 0 \) as \( n \to \infty \) and let \( \{z_k \in \Gamma(x_k)\} \) be a sequence with \( Lz_k - P_n z_k \to 0 \). Since \( \{(x_k, z_k)\} \) is bounded and \( H \times H \) reflexive, we may assume by passing to a subsequence that \( (x_k, z_k) \to (x, z) \). Therefore \( z \in \Gamma(x) \). We need to show that \( Lx = z \). First note that

\[ Lx_k = P Lz_k = P(Lx_k - P_n z_k) + Pz_k \to Pz \]

as \( k \to \infty \). Since \( L \) is closed, it is weakly closed hence \( Lx = Pz \). It remains to show that

\[ (I - P)z = 0. \]

Indeed, since \( z_k \to z \) and \( P_n'(I - P) w \to (I - P) w \) for all \( w \in H \), we have

\[ (P_n'(I - P) z_k - P_n'(I - P) z, w) = (z_k - z, P_n'(I - P) w) \to 0 \]

for all \( w \). Next, it is verified that \( P_n'(I - P) z_k = (P - I) (Lx_k - P_n z_k) \to 0 \), so by (2.2), \( P_n'(I - P) z \to 0 \). On the other hand, \( P_n'(I - P) z \to (I - P) z \), and (2.1) follows. 

\section*{Theorem 2.2:} Let \( \Gamma: H \to 2^H \) be a demi-continuous map such that either \( \Gamma \) or \( -\Gamma \) is monotone. Then \( \Gamma \) is weakly continuous. Consequently, \( \Gamma \) is weakly \( A \)-proper.

\textbf{Proof:} Since the weak continuity of \( -\Gamma \) is equivalent to that of \( \Gamma \), it is enough to give the proof for \( \Gamma \) monotone. Let \( x_k \to x, z_k \to z, z_k \in \Gamma(x_k), k \in \mathbb{N} \). We have to show that \( z \in \Gamma(x) \). Suppose for contradiction that \( z \notin \Gamma(x) \). \( \Gamma(x) \) is closed and convex, so the Hahn-Banach separation theorem implies the existence of \( y \in H \) and \( \alpha \in \mathbb{R} \) such that

\[ (y, z) < \alpha < (y, w) \]

for all \( w \in \Gamma(x) \). We put \( y_m = x - t_m y \), where \( t_m > 0, t_m \to 0 \), and we choose \( z_m \in \Gamma(y_m) \). Without loss of generality, \( z_m \to z_0 \in H \). Since \( y_m \to x \) and \( \Gamma \) is demi-continuous, we have \( z_0 \in \Gamma(x) \). \( \Gamma \) is monotone, therefore \( (z_k - z_m, x_k - x + t_m y) \geq 0 \) for all \( k, m \in \mathbb{N} \) and, consequently,

\[ 0 \leq \lim_{k \to \infty} (z_k - z_m, x_k - x + t_m y) = \lim_{k \to \infty} (z - z_m, t_m y), \]
for all $m \in \mathbb{N}$. Thus 
\[ 0 \leq \lim_{k \to \infty} (z_k - z_m, y) \]
for all $m \in \mathbb{N}$. Since $z_m \to z_0$, passing to the limit as $m \to \infty$ gives
\[ 0 \leq \lim_{k \to \infty} (z_k - z_0, y). \tag{2.4} \]

By comparing (2.3) with (2.4) and using the fact that $z_k \to z$, we get
\[ 0 \leq \lim_{k \to \infty} (z_k, y) - (z_0, y) < \lim_{k \to \infty} (z_k, y) - (z, y) \]
\[ = \lim_{k \to \infty} (z_k - z, y) = \lim_{k \to \infty} (z_k - z, y) = 0 \]
and the contradiction is reached. The second conclusion now follows from Theorem 2.1.

We complete this section by showing a relation between demi-continuous and upper semicontinuous maps.

**Lemma 2.3:** If $A: H \to H$ is a compact linear operator and $F: H \to 2^H$ a demi-continuous map, then $A \circ F$ is upper semicontinuous and compact. Consequently, if the right inverse $K$ of $L$ is compact and $F$ demi-continuous, then $K$ is $L$-compact.

**Proof:** Since $A \circ F$ sends bounded sets to relatively compact sets, we must only show that the graph of this map is closed in $H \times H$. For, let $x_k \to x$ and $A z_k \to y$, where $z_k \in \Gamma(x_k)$, $k \in \mathbb{N}$. We have to show that $y \in A(\Gamma(x))$. Since $\{z_k\}$ is bounded, there is a subsequence $z_{k_l} \to z$. $\Gamma$ is demi-continuous, so $z \in \Gamma(x)$. Any continuous linear operator is weakly continuous, hence $A z_{k_l} \to Az$. Consequently $y = Az \in A(\Gamma(x))$. The second conclusion follows from the first one by the comment on the Fredholm resolvent $T_n$ in Section 1.

**Remark 2.4:** For simplicity of the presentation we have restricted the study to $E = F = H$ a separable Hilbert space. However, the above definitions and results can be easily extended to self-adjoint operators $L: D(L): E \to E^*$, where $E$ is a separable reflexive Banach space of type $\mathfrak{f}$, $E^*$ its dual and the scalar product is replaced by the duality product, see [11].

3. Let us recall that a multifunction $g: D \subset \mathbb{R}^m \to 2^{\mathbb{R}^m}$ is called measurable if 
\[ \{x \in D \mid g(x) \subset U\} \]
is Lebesgue measurable for any open $U$ in $\mathbb{R}^m$. By a single-valued selection of $g$ we mean a function $s: D \to \mathbb{R}^m$ such that $s(x) \in g(x)$ for all $x$. It is known that any measurable multifunction has a single-valued selection, c.f. [3]. A multifunction $g: \mathbb{R}^m \to 2^{\mathbb{R}^m}$ will be called monotone if it is monotone in the sense of Section 2 with respect to the Euclidean scalar product $\langle \cdot, \cdot \rangle$. The Euclidean norm is denoted by $|\cdot|$. We use the notation $|g(x)| := \sup \{|z| \mid z \in g(x)\}$. Let $J = (0, 2\pi) \times (0, \pi)$ and let $g: J \times \mathbb{R} \to 2^{\mathbb{R}^m}$ be a multifunction with non-empty closed convex values. Such $g$ is called a Carathéodory multifunction if
\[ (a) \quad \langle t, x \rangle \to g(t, x, u) \]
is measurable for all $u \in \mathbb{R}^m$;
\[ (b) \quad u \to g(t, x, u) \]
is upper semicontinuous for all $(t, x) \in J$.

We let $H = L^2(J, \mathbb{R})$. A Nemitskii (multivalued) operator for a Carathéodory multifunction $g$, $\Gamma_g: H \to 2^H$, is defined by
\[ \Gamma_g(u) = \{v \in H \mid v(t, x) \in g(t, x, u(t, x)) \mid \text{for a.e. } (t, x) \in J\}. \]

**Lemma 3.1:** Let $g$ be a Carathéodory multifunction (with non-empty closed convex values) and suppose that there exist an $h \in L^2(J, \mathbb{R})$ and a constant $c > 0$ such that
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for all $u \in \mathbb{R}^q$ and a.e. $(x, t) \in J$

$$|g(t, x, u)| \leq h(t, x) + c|u|.$$  \hspace{1cm} (3.1)

Then $\Gamma_g': H \to 2^H$ is a demi-continuous map.

**Proof:** We refer the reader to Lemma 4.2 in [7]. This result was proved there for functions with values in $\mathbb{R}$ but it immediately follows in this formulation since $H = L^2(J, \mathbb{R}) \times \cdots \times L^2(J, \mathbb{R})$ (q copies)

We are concerned with the existence of solutions $u$ of the inclusion

$$u_{tt} - u_{xx} \in g(t, x, u).$$  \hspace{1cm} (3.2)

We say that $u \in H$ is a generalized solution of the periodic-Dirichlet problem for (3.2) if there exists a selection $s \in H$ of the multifunction $(t, x) \to g(t, x, u(t, x))$ such that

$$\int_J \langle u, v_{tt} - v_{xx} \rangle = \int_J \langle s, v \rangle$$  \hspace{1cm} (3.3)

for all $v \in C^2(J, \mathbb{R}^q)$ satisfying the boundary conditions

$$v(0, x) = v(2\pi, x), \quad v_t(0, x) = v_t(2\pi, x), \quad x \in [0, \pi],$$ \hspace{1cm} (3.4)

It is verified that the following set of functions is orthonormal in $H$:

$$v_{m,n,k}(t, x) = \frac{1}{\sqrt{\pi}} e^{imn_1} \sin(nx) e_k, \quad m \in \mathbb{Z}, \quad n \in \mathbb{N}, \quad k = 1, 2, \ldots, q,$$

where $\{e_k\}$ is the standard basis of $\mathbb{R}^q$. We define $L: D(L) \subset H \to H$ by

$$D(L) = \left\{ u \in H \left| \sum_{k=1}^{q} \sum_{(m,n) \in \mathbb{Z} \times \mathbb{N}} \left| (n^2 - m^2) (u, v_{m,n,k}) \right|^2 < \infty \right. \right\},$$

$$Lu = \sum_{k=1}^{q} \sum_{(m,n) \in \mathbb{Z} \times \mathbb{N}} (n^2 - m^2) (u, v_{m,n,m}) v_{m,n,k}.$$

By standard arguments (see [15]), it follows that: $D(L)$ is dense in $H$, $L$ verifies assumptions which were made in Section 2, the spectrum of $L$ is $\sigma(L) = \{ n^2 - m^2 : m \in \mathbb{Z}, n \in \mathbb{N} \}$ and the right inverse $K: R(L) \to H$ of $L$ is compact. Moreover, $u \in H$ is a generalized solution of the periodic-Dirichlet problem for $u_{tt} - u_{xx} = h(t, x)$, $h \in H$, if and only if $u \in D(L)$ and $Lu \in \Gamma_g(u)$.

**Theorem 3.2:** Let $g: J \times \mathbb{R}^q \to 2^{\mathbb{R}^q}$ be a Carathéodory multifunction satisfying the following conditions:

(i) Either $u \to g(t, x, u)$ or $u \to -g(t, x, u)$ is monotone for a.e. $(t, x) \in J$.

(ii) There exist $r \in \mathbb{R} \setminus \sigma(L)$, $0 \leq \delta < d(r, \sigma(L))$, and $h \in L^2(J, \mathbb{R})$ such that

$$|g(t, x, u) - ru| \leq \delta |u| + h(t, x),$$

for all $u \in \mathbb{R}^q$ and a.e. $(t, x) \in J$.

Then there exists $u \in H \cap D(L)$ with $Lu \in \Gamma_g(u)$.

**Proof:** We shall use Theorem 1.3 for the map $\Gamma_g$ and the operator $S = rl$. From (i) it follows by integration that $\Gamma_g'$ or $-\Gamma_g'$ is monotone and, from (ii), (3.1) follows. Hence, Lemma 3.1 and Theorem 2.2 imply that $I$ is weakly $A$-proper. Since $K$ is compact, both $\Gamma$ and $S$ are $L$-compact, by Lemma 2.3. By the choice of $r$, $L - S$ is bijective with the bounded inverse of norm $\| (L' - rI)^{-1} \| = (d(r, \sigma(L)))^{-1} = \alpha$. In
particular, it easily follows that $S$ is $A$-proper. It remains to determine an a priori bound $M$ on solutions $u \in H \cap \mathcal{D}(L)$ of (1.3). For, let $n \in \mathbb{N}$, $\lambda \in (0, 1)$ and $u \in H \cap \mathcal{D}(L)$ satisfy $Lu \in (1 - \lambda) rP_n u + \lambda P_n \{P_n u\}$. Since $P_n u = u$, we obtain $u \in \lambda \times (L - rI)^{-1} P_n \{P_n u - ru\}$. Hence, by using the condition (ii), we get $\|u\| \leq ||(L - rI)^{-1}|| (\delta||u|| + \|h\|) = \alpha \delta||u|| + \alpha \|h\|$, therefore $||u|| \leq \alpha \|h\|/(1 - \alpha \delta)$. It remains to choose for $M$ any number greater than the right-hand side of the last inequality.

Remark 3.3: In the definition of Carathéodory multifunction, the condition (a) can be replaced by the following weaker condition:

$$(a'): (t, x) \rightarrow g(t, x, u)$$ is measurable for all $u$ from a dense subset of $\mathbb{R}$.

The conclusion of Lemma 3.1 will remain true, see [10].

4. In what follows, $f: J \times \mathbb{R} \rightarrow \mathbb{R}$ is a function which does not satisfy, a priori, any continuity condition. In this case, there is no hope of solving any boundary value problem for

$$u_{tt} - u_{xx} = f(t, x, u) \quad (4.1)$$

in the usual sense but we may look for optimal solutions in the following sense: Let $f, f: J \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by:

$$f(t, x, u) = \lim_{v \to u} f(t, x, v), \quad f(t, x, u) = \lim_{v \to u} f(t, x, v).$$

An optimal solution of (4.1) is a function $u$ verifying

$$u_{tt} - u_{xx} \leq f(t, x, u) \leq \bar{f}(t, x, u) \quad (4.2)$$

for a.e. $(t, x) \in J$. A generalized solution of the periodic-Dirichlet problem for (4.2) is such $u \in L^2(J, \mathbb{R})$ that

$$\int_J f(t, x, u) \cdot v \leq \int_J u \cdot (v_{tt} - v_{xx}) \leq \int_J \bar{f}(t, x, u) \cdot v \quad (4.3)$$

for all $v \in C^2(J, \mathbb{R})$ satisfying the boundary conditions (3.4).

Theorem 4.1: Suppose $f: J \times \mathbb{R} \rightarrow \mathbb{R}$ verifies the following conditions:

(i) The set of those $(t, x) \in J$ that $a < f(t, x, u) \leq \bar{f}(t, x, u) < b$ is measurable for all $a, b, u \in \mathbb{R}$.

(ii) $u \rightarrow f(t, x, u)$ is either non-decreasing or non-increasing for a.e. $(t, x) \in J$.

(iii) There exist $r \in \mathbb{R} \setminus \sigma(L)$, $0 \leq \delta < d[r, \sigma(L)]$ and $h \in L^2(J, \mathbb{R})$ such that, for all $u \in \mathbb{R}$ and a.e. $(t, x) \in J$, $|f(t, x, u) - ru| \leq \delta |u| + h(t, x)$.

Then there exists a generalized solution $u \in L^2(J, \mathbb{R})$ of the periodic-Dirichlet problem for (4.2).

Proof: Let $g(t, x, u) = [f(t, x, u), \bar{f}(t, x, u)]$. The problem (4.2) is equivalent to (3.2), and (4.3) to (3.3) with $g = 1$. It follows from Proposition 4.4 in [10], from (i) and (iii) that $g$ is Carathéodory multifunction. It instantly follows that $g$ satisfies the hypotheses of Theorem 3.2, hence the conclusion.

Corollary 4.2: Let $f$ be as in Theorem 4.1 with the condition (iii) replaced by the following two:

a) For any $M > 0$ there exists $h \in L^2(J, \mathbb{R})$ such that

$$|f(t, x, u)| \leq h(t, x) \text{ for a.e. } (t, x) \in J \text{ and all } u \in \mathbb{R} \text{ with } |u| < M.$$
b) There exists an $a < b$ with $(a, b) \cap \sigma(L) = \emptyset$ such that

$$a < \lim_{|x| \to \infty} \frac{f(t, x, w)}{u} \leq \lim_{|x| \to \infty} \frac{f(t, x, w)}{u} < b$$

for a.e. $(t, x) \in J$.

Then the conclusion of Theorem 4.1 remains true.

Proof: For verification that a) and b) imply the condition (iii) of Theorem 4.1, we refer the reader to [11].

REFERENCES


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