On Lower Semicontinuity and Metric Upper Semicontinuity of Nemytskii Set-Valued Operators

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Abstract. Sufficient conditions of lower semicontinuity and metric upper semicontinuity of Nemytskii set-valued operators \( N_F \) generated by a set-valued function \( F : \Omega \times X \to 2^Y \), where \( X \) and \( Y \) are Orlicz-Musielak F-spaces are presented.

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In years 1933 - 1934 V. Nemytskii (see [10] and [11]) considered the operator \( F : L^2[a,b] \to L^2[a,b] \) defined by \( y(\cdot) = N_F(x(\cdot)) \), where \( y(t) = F(t, x(t)) \). From that time the operator \( N_F \) was generalized in several way and there is a lot of papers devoted to such subjects. Operators of this type are now called Nemytskii operators.

In last years new important applications of Nemytskii set-valued operators in the theory of differential and integral inclusions appear (see [1 - 4]). For these applications lower semicontinuity and metric upper semicontinuity of Nemytskii set-valued operators are important.

In [1] Appell, Ngyen and Zabrejko give sufficient conditions of lower semicontinuity of Nemytskii set-valued operators acting in so-called ideal spaces. We shall not give the definition here. We want only to mention that ideal spaces are some spaces of functions defined on a measure space \( \Omega \) admitting values in finite-dimensional spaces and it can be shown that each Orlicz space admitting values in a finite-dimensional space is an ideal space.

Thus the natural problem arose to give sufficient conditions of lower semicontinuity and metric upper semicontinuity of Nemytskii set-valued operators for spaces consisting of functions admitting values in infinite-dimensional spaces.

In this paper sufficient conditions of lower semicontinuity and metric upper semicontinuity of Nemytskii set-valued operators acting in Orlicz-Musielak F-spaces are
Let $X$ be a separable $F$-space (i.e. a complete linear metric space; basic properties can be found in [12]) with an $F$-norm $\| \cdot \|_X$ and $(\Omega, \Sigma, \mu)$ a measure space with complete and $\sigma$-finite measure $\mu$. A function $x = x(t)$ mapping $\Omega$ into $X$ is called measurable if for every open set $Q \subseteq X$ the inverse image $x^{-1}(Q) = \{ t \in \Omega : x(t) \in Q \}$ is measurable, i.e. if $x^{-1}(Q) \in \Sigma$. The set of all measurable functions defined on $\Omega$ with values in $X$ we shall denote by $S(\Omega, X)$.

A closed-valued set-valued function $F = F(t)$ mapping $\Omega$ into subsets of $X$ is called measurable if for every open set $Q \subseteq X$ the inverse image $F^{-1}(Q) = \{ t \in \Omega : F(t) \cap Q \neq \emptyset \}$ is measurable, i.e. if $F^{-1}(Q) \in \Sigma$. By a measurable selection of the set-valued function $F$ we mean a (single-valued) function $x_F$ such that $x_F(t) \in F(t)$ for all $t \in \Omega$.

Let $F = F(t, x)$ be a closed-valued set-valued function mapping $\Omega \times X$ into subsets of an $F$-space $Y$, i.e. into $2^Y$. We say that $F$ is sup-measurable if for any measurable function $x(\cdot) : \Omega \rightarrow X$ the set-valued function $s \rightarrow F(s, x(s)) : \Omega \rightarrow 2^Y$ is measurable.

Given a sup-measurable closed-valued set-valued function $F : \Omega \times X \rightarrow 2^Y$. This set-valued function induces a set-valued operator $N_F : S(\Omega, X) \rightarrow S(\Omega, Y)$ defined by $y(t) = F(t, x(t))$. The set-valued operator $N_F$ is called superposition operator (or Nemytskii operator) generated by the set-valued function $F$.

Let $N = N(t, u)$ be a real-valued measurable function defined on $\Omega \times R$ such that for every $t \in \Omega$ the function $N(t, \cdot)$ is increasing and moreover $N(t, 0) = 0$ for all $t \in \Omega$. Then we can define on $S(\Omega, X)$ a metrizing modular

$$\rho_{N, \mu}(x(\cdot)) = \int \rho(t, \| x(t) \|_X) \, d\mu$$

(see, e.g., Nakano ([7], [8; p. 153] and [9; p. 204]), Musielak [5; p. 1] and Rolewicz [12; p. 6]). The set of those measurable functions $x(\cdot) \in S(\Omega, X)$ that there is a positive $k$ such that $\rho_{N, \mu}(kx(\cdot)) < \infty$ we shall denote by $N(L(\Omega, \Sigma, \mu; X))$. Recall that a metrizing modular on a linear space $X$ is a function $\rho : X \rightarrow [0, \infty]$ having the following properties:

(\textmd{md1}) $\rho(x) = 0$ if and only if $x = 0$

(\textmd{md2}) $\rho(ax) = \rho(x)$ provided $|a| = 1$

(\textmd{md3}) $\rho(ax + by) \leq \rho(x) + \rho(y)$ provided $a, b > 0$ and $a + b = 1$

(\textmd{md4}) $\rho(a_n x) \rightarrow 0$ provided $a_n \rightarrow 0$ and $\rho(x) < +\infty$

(\textmd{md5}) $\rho(ax_n) \rightarrow \rho(x_n) \rightarrow 0$.

We shall denote by $(X, \rho)$ a linear space with a modular $\rho$ and we shall call it modular space. Let $(X, \rho)$ be a modular space with metrizing modular $\rho$. It is known that $\rho$ induces in the space $X$ an $F$-norm $\| \cdot \|_X$ by

$$\| x \|_X = \inf \left\{ \varepsilon > 0 \mid \rho \left( \frac{x}{\varepsilon} \right) < \varepsilon \right\}.$$
The norm $\| \cdot \|_X$ is equivalent to the module $\rho$ in the sense that, for any sequence $\{x_n\}$, $\|x_n\|_X \to 0$ if and only if $\rho(x_n) \to 0$ (see Musielak and Orlicz [6] and Musielak [5: p. 2]; see also Rolewicz [12: p. 8]). Observe that if the functions $x_n(\cdot) \in N(L(\Omega, \Sigma, \mu; X))$ \ ($n \in \mathbb{N}$) have disjoint supports, then

$$\sum_{n=1}^{\infty} \rho_{N,M}(x_n(\cdot)) = \rho_{N,M} \left( \sum_{n=1}^{\infty} x_n(\cdot) \right).$$

(3)

Let $(X, \rho_X)$ and $(Y, \rho_Y)$ be two modular spaces. A set-valued operator $\Gamma = \Gamma(x)$ mapping $(X, \rho_X)$ into subsets of $(Y, \rho_Y)$ will be called lower semicontinuous at a point $(x_0, y_0) \in X \times Y$ if it is lower semicontinuous at this point in the metric induced by the $F$-norms introduced above. In other words $\Gamma$ is lower semicontinuous at $(x_0, y_0)$ if for every $r > 0$ there is a number $q(x_0, y_0)(r) > 0$ with the following property: for every $x \in X$ such that $\rho_{N,M}(x_0(\cdot) - x(\cdot)) < q(x_0, y_0)(r)$ there is any $y$ such that $\rho_{M,Y}(y_0(\cdot) - y(\cdot)) < r$. This is equivalent with the property that $\rho_{N,M}(x_0(\cdot) - x(\cdot)) < q(x_0, y_0)(r)$ implies

$$\Gamma(x) \cap Q \neq \emptyset$$

where $Q = \{ y : \rho_{M,Y}(y_0(\cdot) - y(\cdot)) < r \}$.

A set-valued operator $\Gamma$ mapping $(X, \rho_X)$ into subsets of $(Y, \rho_Y)$ will be called lower semicontinuous at a point $x_0 \in X$ if it is lower semicontinuous at all the points $(x_0, y_0)$ \ ($y_0 \in \Gamma(x_0)$).

Now we shall introduce a notion of global lower semicontinuity at $x_0$, which is nothing else than an uniformization of lower semicontinuity on $\Gamma(x_0)$. More precisely, we say that a set-valued mapping $\Gamma$ is globally lower semicontinuous at the point $x_0$, if for every $r > 0$ there is a $q(x_0)(r) > 0$ with the following property: for every $y_0 \in \Gamma(x_0)$ and for every $x \in X$ such that $\rho_{N,M}(x_0(\cdot) - x(\cdot)) < q(x_0)(r)$ there is an $y \in \Gamma(x)$ such that $\rho_{M,Y}(y_0(\cdot) - y(\cdot)) < r$. This is equivalent with the property that $\rho_{N,M}(x_0(\cdot) - x(\cdot)) < q(x_0)(r)$ implies

$$\Gamma(x_0) \subseteq B(\Gamma(x), r)$$

(5)

where

$$B(A, s) = \left\{ y(\cdot) : \inf_{y_1(\cdot) \in A} \rho_{M,Y}(y_1(\cdot) - y(\cdot)) < s \right\}.$$  

The essential difference between lower semicontinuity and global lower semicontinuity is that in the first case $q(x_0, y_0) > 0$ depends on $x_0$ and $y_0 \in \Gamma(x_0)$, while in the second case it depends on $x_0$ only.

Changing the role of $x$ and $x_0$ in formula (5) we obtain a notion of metric upper semicontinuity. We say that the set-valued mapping $\Gamma$ is metric upper semicontinuous (Hausdorff upper semicontinuous) at a point $x_0$ if for every $r > 0$ there is a $q(x_0)(r) > 0$ such that $\rho_{N,M}(x_0(\cdot) - x(\cdot)) < q(x_0)(r)$ implies

$$\Gamma(x) \subseteq B(\Gamma(x_0), r).$$

(6)

The both notions are not equivalent. Indeed, it is easy to give an example of a set-valued mapping $\Gamma$ which is globally lower semicontinuous at a point $x_0$, but which is
not metric upper semicontinuous at this point. Conversely, an example of a set-valued mapping \( \Gamma \) which is metric upper semicontinuous at a point \( x_0 \) but which is not globally lower semicontinuous at this point can be easily given too.

In the sequel we shall add some assumptions about the functions \( N(t, u) \) and \( M(t, u) \). Namely, we assume that the function \( N = N(t, u) \) satisfies the following condition

\[
(A) \quad \text{For every } \varepsilon > 0 \text{ there are numbers } \alpha > 0 \text{ and } \delta > 0 \text{ such that for every measurable set } E \subset \Omega \text{ with } \mu(E) > \delta \text{ we have } \int_{E} N(t, \varepsilon) \, dt \geq \alpha.
\]

Observe that in Orlicz spaces, i.e. in the case when \( N(t, u) = N_0(u) \) depends only on \( u \), the condition (A) is satisfied. Further, we assume that the function \( M(t, u) \) satisfies the following condition

\[
(\Delta_2) \quad \text{There are a constant } k \geq 1 \text{ and a non-negative function } \delta \text{ such that } \int_{E} M(t, \delta(t)) \, d\mu < +\infty \text{ and for almost all } t, \text{ if } u \geq \delta(t), \text{ then } M(t, 2u) \leq k M(t, u).
\]

The condition \((\Delta_2)\) plays an essential role in the theory of Orlicz-Musielak spaces. Observe that if the function \( M(t, u) \) satisfies condition \((\Delta_2)\), then from the fact that \( \rho_{M,\mu}(x) < +\infty \) and \( \rho_{M,\mu}(y) < +\infty \) it follows that \( \rho_{M,\mu}(ax + by) < +\infty \) for all real \( a \) and \( b \). We conclude that \( M \left( L(\Omega, \Sigma; \mu; Y) \right) \) is the set of those measurable functions \( y(\cdot) \) with values in the space \( X \), that \( \rho_{M,\mu}(y) < +\infty \) (see [5: p. 52] and [6]).

**Theorem 1.** Let \( X, Y \) be two separable \( F \)-spaces, \( (\Omega, \Sigma, \mu) \) a measure space with complete non-atomic \( \sigma \)-finite measure \( \mu \), and \( N_F \) a Nemytskii set-valued operator mapping of a modular space \((N(L(\Omega, \Sigma; \mu; X)), \rho_{N,\mu})\) into subsets of a modular space \((M(L(\Omega, \Sigma; \mu; Y)), \rho_{M,\mu})\) induced by a sup-measurable set-valued function \( F(t, u) : \Omega \times X \to 2^Y \). Suppose that

(i) the function \( N = N(t, u) \) satisfies condition \((A)\)

(ii) the function \( M = M(t, u) \) satisfies condition \((\Delta_2)\).

If the function \( = F(t, u) \) is globally lower semicontinuous with respect to \( u \) for almost all \( t \in \Omega \), then the operator \( N_F \) is globally lower semicontinuous.

**Proof.** Suppose that the operator \( N_F \) is not globally lower semicontinuous. This implies that there are a number \( r > 0 \) and sequences \( \{x_n(\cdot)\}, \{y_n(\cdot)\} \) with \( y_n(\cdot) \in N_F(x_0(\cdot)) \) such that

\[
\rho_{N,\mu}(x_n(\cdot) - x_0(\cdot)) \to 0
\]

\[
\inf_{z(\cdot) \in N_F(x_n(\cdot))} \rho_{M,\mu}(z(\cdot) - y_n(\cdot)) > r \quad (n \in \mathbb{N}).
\]

By [2: Theorem 8.24 and Corollary 8.23], for arbitrary \( \eta > 0 \) there is a sequence \( \{z_n(\cdot)\} \) of measurable selections \( z_n(\cdot) \in N_F(x_n(\cdot)) \) such that, for almost all \( t \in \Omega \),

\[
\|y_n(t) - z_n(t)\|_Y < (1 + \eta)d_Y(y_n(t), F(t, x_n(t)))
\]
where \( d_Y(y, A) = \inf_{z \in A} \| z - y \|_Y \) denotes the distance of a point \( y \) to a set \( A \) in the norm \( \| \cdot \|_Y \). We denote \( u_n(t) = \| y_n(t) - z_n(t) \|_Y \). Since \( z_n(\cdot) \in N_F(x_n(\cdot)) \), then

\[
\int_\Omega M(t, u_n(t)) \, d\mu \geq \inf_{z(\cdot) \in N_F(x_n(\cdot))} \rho_{M, \mu}(z(\cdot) - y_n(\cdot)) > r \quad (n \in \mathbb{N}).
\]

The convergence \( \rho_{N, \mu}(z_n(\cdot) - x_0(\cdot)) \to 0 \) implies that the sequence \( \{x_n(\cdot) - x_0(\cdot)\} \) contains a subsequence \( \{x_{n_k}(\cdot) - x_0(\cdot)\} \) such that

\[
\sum_{k=1}^{\infty} \int_{\Omega} N(t, \| x_{n_k}(t) - x_0(t) \|_X) \, d\mu < +\infty.
\]

Thus replacing the sequence \( \{x_n(\cdot) - x_0(\cdot)\} \) by the subsequence \( \{x_{n_k}(\cdot) - x_0(\cdot)\} \) we can assume without loss of generality that

\[
\sum_{n=1}^{\infty} \int_{\Omega} N(t, \| x_n(t) - x_0(t) \|_X) \, d\mu < +\infty.
\]

Now we have the following two possibilities: either (1) \( \mu(\Omega) \) is finite or (2) \( \mu(\Omega) \) is infinite.

**Case (1):** \( \mu(\Omega) \) is finite. We shall construct by induction a sequence of positive numbers \( \{\varepsilon_k\} \), a sequence of measurable sets \( \{\Omega_k\} \) \( (\Omega_k \subset \Omega) \) and a subsequence \( \{x_{n_k} - x_0\} \) such that the following conditions are satisfied:

(a) \( \varepsilon_{k+1} < \frac{\varepsilon_k}{2} \)
(b) \( \mu(\Omega_k) \leq \varepsilon_k \)
(c) \( \int_{\Omega_k} M(t, u_{n_k}(t)) \, d\mu > \frac{2}{3} r \)
(d) \( \int_D M(t, u_{n_k}(t)) \, d\mu < \frac{1}{3} r \) for any set \( D \subset \Omega_k \) such that \( \mu(D) \leq 2\varepsilon_{k+1} \).

We put \( \varepsilon_1 = \mu(\Omega), x_n - x_0 = x_1 - x_0 \) and \( \Omega_1 = \Omega \). Suppose that \( \varepsilon_k, x_{n_k} - x_0 \) and \( \Omega_k \) have been constructed. Since \( N_F \) is a Nemytskii set-valued operator mapping of the modular space \( (N(L_1(\Omega, \Sigma, \mu; X)), \rho_{N, \mu}) \) into subsets of the modular space \( (M(L_1(\Omega, \Sigma, \mu; Y)), \rho_{M, \mu}) \), by property \( \Delta_2 \),

\[
\int_{\Omega_k} M(t, u_{n_k}(t)) \, d\mu < +\infty.
\]

Thus the function \( M(t, u_{n_k}(t)) \) is absolutely continuous. Hence it is easy to find \( \varepsilon_{k+1} \) satisfying conditions (a) and (d). Since the function \( N = N(t, u) \) satisfies condition (A) and \( \rho(x_n(\cdot) - x_0(\cdot)) \to 0 \), the functions \( x_n \) tend to \( x_0 \) in measure. Replacing eventually the sequence \( \{x_{n_k} - x_0\} \) by its subsequence, we can assume without loss of generality that \( x_{n_k}(t) \) tends to \( x_0(t) \) almost everywhere. By the global lower semicontinuity of \( F(t, x) \), we obtain that \( d_Y(y_n(t), F(t, x_n(t))) \) tends to 0 almost everywhere.
Since $\mu(\Omega) < +\infty$, for the sequence $\{z_n(\cdot)\}$ of measurable selections chosen at the beginning of this proof such that (7) holds, there are an index $n_{k+1}$ and a set $E_{k+1} \subset \Omega$ such that, for $t \in E_{k+1}$,
\[ M(t, u_{n_k}(t)) < \frac{r}{3\mu(\Omega)}. \quad (11) \]
and
\[ \mu(\Omega \setminus E_{k+1}) < \epsilon_{k+1}. \quad (12) \]
Let $\Omega_{k+1} = \Omega \setminus E_{k+1}$. Observe that (12) implies condition (b). By (8) and (11), we obtain
\[ \int_{\Omega_{k+1}} M(t, u_{n_{k+1}}(t)) \, d\mu = \int_{\Omega} M(t, u_{n_{k+1}}(t)) \, d\mu - \int_{E_{k+1}} M(t, u_{n_{k+1}}(t)) \, d\mu > \frac{2}{3}r. \quad (13) \]
By properties (a) and (b), we get
\[ \mu \left( \bigcup_{j=k+1}^{\infty} \Omega_j \right) \leq \sum_{j=k+1}^{\infty} \mu(\Omega_j) \leq \sum_{j=k+1}^{\infty} \epsilon_j < 2\epsilon_{k+1}. \]
Thus we have constructed a sequence of positive numbers $\{\epsilon_k\}$, a sequence of measurable sets $\{\Omega_k\}, \Omega_k \subset \Omega$, and a subsequence $\{x_{n_k} - x_0\}$ such that conditions (a) - (d) hold.

Now we shall continue the proof. Let
\[ D_k = \Omega_k \setminus \left( \bigcup_{j=k+1}^{\infty} \Omega_j \right) \quad (k \in \mathbb{N}). \]
Define functions $\psi, \psi_0, y_0, z$ in the following way:
\[ \psi(s) = \begin{cases} x_{n_k}(s) & \text{if } s \in D_k \quad (k \in \mathbb{N}) \\ 0 & \text{if } s \notin \bigcup_{j=1}^{\infty} D_j \end{cases} \quad (14) \]
\[ \psi_0(s) = \begin{cases} x_0(s) & \text{if } s \in D_k \quad (k \in \mathbb{N}) \\ 0 & \text{if } s \notin \bigcup_{j=1}^{\infty} D_j \end{cases} \quad (15) \]
\[ y_0(s) = \begin{cases} y_{n_k}(s) & \text{if } s \in D_k \quad (k \in \mathbb{N}) \\ w(s) & \text{if } s \notin \bigcup_{j=1}^{\infty} D_j \end{cases} \quad (16) \]
\[ z(s) = \begin{cases} z_{n_k}(s) & \text{if } s \in D_k \quad (k \in \mathbb{N}) \\ v(s) & \text{if } s \notin \bigcup_{j=1}^{\infty} D_j \end{cases} \quad (17) \]
where $w(\cdot)$ and $v(\cdot)$ belong to $N_F(0)$ (i.e. these are measurable selections of $F(s,0)$).
From conditions (c), (d) and inequalities (11), (13), it follows that
\[ \int_{D_k} M(t, \|y_0(t) - z(t)\|) \, d\mu = \int_{D_k} M(t, u_{n_k}(t)) \, d\mu \]
\[ = \int_{\Omega_k} M(t, u_{n_k}(t)) \, d\mu - \int_{\Omega_k \setminus D_k} M(t, u_{n_k}(t)) \, d\mu > \frac{1}{3}r. \quad (18) \]
Observe that $\psi_0 \in N(L(\Omega, \Sigma, \mu; X))$ and, by (10), also $\psi \in N(L(\Omega, \Sigma, \mu; X))$. It is easy to see that $y_0(\cdot) \in N_F(\psi_0(\cdot))$ and $z(\cdot) \in N_F(\psi(\cdot))$.

On the other hand,

$$\int N(t, \|y_0(t) - z(t)\|) \, d\mu \geq \sum_{k=1}^{\infty} \int N(t, \|y_0(t) - z(t)\|) \, d\mu = +\infty \tag{19}$$

which contradicts the fact that $N_F$ is a set-valued operator mapping of the modular space $N(L(\Omega, \Sigma, \mu; X))$, $\rho_{N, \mu}$ into subsets of the space $M(L(\Omega, \Sigma, \mu; Y))$, $\rho_{M, \mu}$. This finishes the proof of the case when $\mu(\Omega)$ is finite.

**Case (2):** $\mu(\Omega)$ is infinite. We will consider the following two subcases:

1. **(2a)** There are a subset $\Omega_0 \subset \Omega$ with finite measure $\mu(\Omega_0)$ and a number $\beta \in (0, r)$ such that $\int_{\Omega_0} M(t, u_n(t)) \, d\mu \geq \beta$ ($n \in \mathbb{N}$).

2. **(2b)** There are a subsequence $\{u_{n_k}(t)\}$ and a sequence of measurable sets $\{D_k\}$ such that

   - (e) $\mu(D_k) < +\infty$ and $D_i \cap D_j = \emptyset$ for $i \neq j$.
   - (f) $\int_{D_k} M(t, u_{n_k}(t)) \, d\mu \geq \frac{1}{2} r$ ($k \in \mathbb{N}$).

In the subcase (2a) the consideration can be reduced to that of Case (1) with replacing $r$ by $\beta$. In the subcase (2b) we define functions $\psi, \psi_0, y, z$ by formulae (14) – (17). As in Case (1) we obtain that $\psi, \psi_0 \in N(L(\Omega, \Sigma, \mu; X))$, and that $z(\cdot) \in N_F(\psi(\cdot))$ and simultaneously $y_0(\cdot) \in N_F(\psi_0(\cdot))$. On the other hand, by properties (e) and (f) we obtain that $z(t) - y_0(t) \notin (M(L(\Omega, \Sigma, \mu; Y)), \rho_{M, \mu})$, which leads to a contradiction.

**Theorem 2.** Let $X, Y$ be two separable $F$-spaces, $(\Omega, \Sigma, \mu)$ a measure space with complete non-atomic and $\sigma$-finite measure $\mu$, and $N_F$ a Nemitskii set-valued operator mapping of a modular space $N(L(\Omega, \Sigma, \mu; X))$, $\rho_{N, \mu}$ into subsets of a modular space $M(L(\Omega, \Sigma, \mu; Y))$, $\rho_{M, \mu}$ induced by a sup-measurable set-valued function $F = F(t, u) : \Omega \times X \to 2^Y$. Suppose that

- (i) the function $N = N(t, u)$ satisfies condition (A)
- (ii) the function $M = M(t, u)$ satisfies condition $(\Delta_2)$.

If the function $F = F(t, u)$ is lower semicontinuous with respect to $u$ for almost all $t \in \Omega$, then the operator $N_F$ is lower semicontinuous.

**Proof.** Suppose that the operator $N_F$ is not lower semicontinuous at some point $(x_0(\cdot), y_0(\cdot))$ with $y_0(\cdot) \in N_F(x_0(\cdot))$. This implies that there are a number $r > 0$ and a sequence $\{x_n(\cdot)\}$ such that

$$\rho_{N, \mu}(x_n(\cdot)) - x_0(\cdot) \to 0$$

$$\inf_{z(\cdot) \in N_F(x_n(\cdot))} \rho_{M, \mu}(z(\cdot) - y_0(\cdot)) > r \quad (n \in \mathbb{N}).$$
By [2, Theorem 8.24 and Corollary 8.23], for arbitrary \( \eta > 0 \) there is a sequence \( \{z_n(\cdot)\} \) of measurable selections \( z_n(\cdot) \in F_N(x_n(\cdot)) \) such that, for almost all \( t \in \Omega \),

\[
\|y_0(t) - z_n(t)\|_Y < (1 + \eta)d_Y(y_0(t), F(t, x_n(t)))
\]

where as before \( d_Y(y, A) \) denotes the distance of a point \( y \) to a set \( A \) in the norm \( \| \cdot \|_Y \). We denote \( u_n(t) = \|y_0(t) - z_n(t)\|_Y \). Since \( z_n(\cdot) \in F_N(x_n(\cdot)) \), then

\[
\int_{\Omega} M(t, u_n(t)) \, d\mu \geq \inf_{z(\cdot) \in F_N(x_n(\cdot))} \rho_{M, f}(z(\cdot) - y_0(\cdot)) > r \quad (n \in \mathbb{N}).
\]

Then we continue the proof step by step in the same way as in the proof of Theorem 1 replacing \( y_n(\cdot) \) by \( y_0(\cdot) \). The only difference is to show the existence of a subsequence \( \{u_{n_k}\} \) such that the inequalities

\[
M(t, u_{n_{k+1}}(t)) < \frac{r}{3\mu(\Omega)}
\]

and

\[
\mu(\Omega \setminus E_{k+1}) < \varepsilon_{k+1}
\]

hold. In order to do this we replace the global lower semicontinuity of \( \Gamma \) by its lower semicontinuity. Since the function \( N = N(t, u) \) satisfies condition (A) and \( \rho_{N, \mu}(x_n(\cdot) - x_0(\cdot)) \to 0 \), the sequence \( \{x_n(\cdot)\} \) tends to \( x_0(\cdot) \) in measure. Thus by the lower semicontinuity of \( F(t, x) \) with respect to \( x \), \( d_Y(y_0, F(t, x_n(t))) \) tends to 0 in measure.

Since \( \mu(\Omega) < +\infty \), for the sequence of measurable selections \( \{z_n(\cdot)\} \) there are an index \( n_{k+1} \) and a subset \( E_{k+1} \subset \Omega \) such that, for \( t \in E_{k+1} \), the inequalities

\[
M(t, u_{n_{k+1}}(t)) < \frac{r}{3\mu(\Omega)}
\]

and

\[
\mu(\Omega \setminus E_{k+1}) < \varepsilon_{k+1}
\]

hold. The remained part of the proof can be continued in the same way as presented in the proof of Theorem 1.

Theorem 3. Let \( X, Y \) be two separable \( F \)-spaces, \((\Omega, \Sigma, \mu)\) a measure space with complete non-atomic and \( \sigma \)-finite measure \( \mu \), and \( N_F \) a Nemytski set-valued operator mapping of a modular space \((N(L(\Omega, \Sigma, \mu; X)), \rho_{N, \mu})\) into subsets of a modular space \((M(L(\Omega, \Sigma, \mu; Y)), \rho_{M, f})\) induced by a sup-measurable set-valued function \( F = F(t, u) : \Omega \times X \to 2^Y \). Suppose that

(i) the function \( N = N(t, u) \) satisfies condition (A)

(ii) the function \( M = M(t, u) \) satisfies condition (A). If the function \( F = F(t, u) \) is metric upper semicontinuous with respect to \( u \) for almost all \( t \in \Omega \), then the operator \( N_F \) is metric upper semicontinuous.
Proof. Suppose that the operator $N_F$ is not metric upper semicontinuous. This implies that there are a number $r > 0$ and sequences \( \{x_n(\cdot)\} \) and \( \{y_n(\cdot)\} \) with $y_n(\cdot) \in N_F(x_n(\cdot))$ such that
\[
\rho_{N,F}(x_n(\cdot) - x_0(\cdot)) \to 0 \\
\inf_{z(\cdot) \in N_F(x_0(\cdot))} \rho_{M,F}(z(\cdot) - y_n(\cdot)) > r \quad (n \in \mathbb{N}).
\]
By [2: Theorem 8.24 and Corollary 8.23], for arbitrary $\eta > 0$ there is a sequence \( \{z_n(\cdot)\} \) of measurable selections $z_n(\cdot) \in N_F(x_0(\cdot))$ such that, for almost all $t \in \Omega$,
\[
\|y_n(t) - z_n(t)\|_Y < (1 + \eta)dy(y_n(t), F(t, x_0(t))) \quad (7)
\]
where as before $dy(y,A)$ denotes the distance of a point $y$ to a set $A$ in the norm $\| \cdot \|_Y$. We denote $u_n(t) = \|y_n(t) - z_n(t)\|_Y$. Since $z_n(\cdot) \in N_F(x_0(\cdot))$, then
\[
\int_{\Omega} M(t, u_n(t)) \, d\mu \geq \inf_{z(\cdot) \in N_F(x_0(\cdot))} \rho_{M,F}(z(\cdot) - y_n(\cdot)) > r \quad (n \in \mathbb{N}). \quad (8)
\]
Then we continue the proof step by step in the same way as in the proof of Theorem 1. The only difference is to show the existence of a subsequence \( \{u_{n_k}\} \) such that the inequalities
\[
M(t, u_{n_{k+1}}(t)) < \frac{r}{3\mu(\Omega)} \quad (11)
\]
and
\[
\mu(\Omega \setminus E_{k+1}) < \varepsilon_{k+1} \quad (12)
\]
hold. In order to do this we replace the global lower semicontinuity of $\Gamma$ by its metric upper semicontinuity. Since the function $N = N(t, u)$ satisfies condition (A) and $\rho_{N,F}(x_n(\cdot) - x_0(\cdot)) \to 0$, the sequence $\{x_n(\cdot)\}$ tends to $x_0(\cdot)$ in measure. Thus by the metric upper semicontinuity of $F(t, x)$ with respect to $x$, $dy(y_n, F(t, x_0(t)))$ tends to 0 in measure.

Since $\mu(\Omega) < +\infty$, for the sequence of measurable selections $\{z_n(\cdot)\}$ there are an index $n_{k+1}$ and a subset $E_{k+1} \subset \Omega$ such that, for $t \in E_{k+1}$, the inequalities
\[
M(t, u_{n_{k+1}}(t)) < \frac{r}{3\mu(\Omega)} \quad (11)
\]
and
\[
\mu(\Omega \setminus E_{k+1}) < \varepsilon_{k+1} \quad (12)
\]
hold. The remained part of the proof can be continued in the same way as presented in the proof of Theorem 1.

Theorem 3 generalizes Theorem 1 of [3], where it is proved for Banach spaces $X, Y$ and for functions $N(t, u) = u^p, M(t, u) = u^q$ with $1 \leq p \leq q < +\infty$ under some estimation assumptions warranting that a Nemytskii operator $N_F$ induced by a sup-measurable set-valued function $F = F(t, u)$ maps the space $L^p(\Omega, \Sigma, \mu; X)$ into the space $L^q(\Omega, \Sigma, \mu; Y)$. 

References


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