Finite Chainability, 
Locally Lipschitzian and Uniformly Continuous 
Functions

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Abstract. We present a notion of a finitely chainable subset of a metric space X. We show that Y is a finitely chainable subset of X if and only if \( f(Y) \) is a bounded subset of \( \mathbb{R} \) for any uniformly locally Lipschitzian or uniformly continuous real-valued function \( f \) on \( X \). As a corollary we reprove the Atsuji theorem in a slightly stronger form.

Keywords: Metric spaces, finite chainable subsets, uniformly continuous functions, uniformly locally Lipschitzian functions

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0. Introduction

In infinite dimensional metric spaces not all continuous images of bounded sets are bounded. Indeed, in 1948 Hewitt [1: p. 69] showed that in a metric space \( X \) each continuous, real-valued function is bounded if and only if \( X \) is compact.

What happens for uniformly continuous functions? To explain better this problem we begin with

Example 0.1. Let \( \{e_n\}_{n \in \mathbb{N}} \) be the canonical basis of \( l_2 \) and let \( \| \cdot \| \) denote the Euclidean norm. Let \( X_n \) be the segment joining \( e_n \) with \( e_{n+1} \), i.e. \( X_n = \{e_n + t(e_{n+1} - e_n) : 0 \leq t \leq 1 \} \). Let \( X = \bigcup_{n=1}^{\infty} X_n \). Equip \( X \) with two different metrics \( \rho \) and \( d \) defined by

\[
d(x, y) = \|x - y\|
\]

and

\[
\rho(x, y) = \begin{cases} 
2^{-n}d(x, y) & \text{if } x, y \in X_n \\
2^{-n}d(x, e_{n+1}) + D_{n,m} + 2^{-m}d(e_m, y) & \text{if } x, y \in X_n, X_m \ (n < m) \\
2^{-n}d(y, e_{n+1}) + D_{n,m} + 2^{-m}d(e_m, x) & \text{if } y \in X_n, x \in X_m \ (n < m)
\end{cases}
\]

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where $D_{n,m} = \sum_{j=n+1}^{m-1} 2^{-j}d(e_j,e_{j+1})$. Finally, consider a function $f : X \to \mathbb{R}$ defined by

$$f(x) = n + t \quad \text{if } x = e_n + t(e_{n+1} - e_n).$$

Note the following:

a) $(X, d)$ and $(X, \rho)$ are two bounded metric spaces.

b) $d$ and $\rho$ are equivalent but not uniformly equivalent metrics on $X$ (i.e. for every $x \in X$ and $\epsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$, depending not only on $\epsilon$ but also on $x$, such that $\rho(x,y) < \epsilon$ whenever $d(x,y) < \delta_1$ and $d(x,y) < \epsilon$ whenever $\rho(x,y) < \delta_2$.

c) $f$ is a real-valued unbounded function on $X$.

d) $f$ is a uniformly continuous function on the metric space $(X, d)$.

e) $f$ is a continuous but not uniformly continuous function on the metric space $(X, \rho)$.

The situation pointed out in Example 0.1 is not unexpected. Indeed, in 1956 Atsuji [2: Theorem 2] showed that each uniformly continuous real-valued function on a metric space $(X, d)$ is bounded if and only if $X$ is a finite chainable space, i.e. for every $\epsilon > 0$ there are finitely many points $p_1, \ldots, p_l$ and a positive integer $m$ such that any point of $X$ can be bound with some $p_j$ by a finite sequence of $m + 1$ points $x = x_0, \ldots, x_m = p_j$ of $X$ satisfying $d(x_{k-1},x_k) < \epsilon$ $(k = 1, \ldots, m)$.

In this note we introduce and study a notion of a finitely chainable subset of a metric space $X$. The main result of it is Theorem 2.1, which gives a characterization of finitely chainable subsets of $X$. Also, we reprove the Atsuji theorem [2: Theorem 2] in a slightly stronger form.

1. Finite chainability property

In the sequel, $X$ denotes a metric space with a metric $d$, $B(x, r)$ the open ball of a centre $x$ and radius $r$ and $A^\epsilon = \{y \in X : \text{dist}(y, A) < \epsilon\}$ the $\epsilon$-neighbourhood of a set $A \subset X$. Let $x, y \in X$ and $\epsilon > 0$.

**Definition 1.1.** An $\epsilon$-chain of length $m$ joining $x$ with $y$ is a finite sequence of $m + 1$ points (not necessarily distinct) of $X$, $x_0 = x, \ldots, x_m = y$ satisfying $d(x_k, x_{k-1}) < \epsilon$ $(k = 1, \ldots, m)$.

**Definition 1.2** (Compare with [2: Definition 3], where the case $Y = X$ has been considered). A subset $Y$ of $X$ is said to be $X$-finitely chainable if for each $\epsilon > 0$ there are a finite set $q_1, \ldots, q_{l(\epsilon)}$ of points of $X$ and a positive integer $m_Y = m_Y(\epsilon)$ such that any point of $Y$ can be bound with some $q_j$ $(1 \leq j \leq l(\epsilon))$ by an $\epsilon$-chain with length $m_Y(\epsilon)$. The function $m_Y : [0, \infty) \to \mathbb{N}$, $\epsilon \to m_Y(\epsilon)$ is said link's number function. It is a non-increasing function.

**Example 1.3.** We can equip $\mathbb{R}$ with many metrics. For example, the functions

$$d_1(x,y) = |x - y| \quad \text{(1.1)}$$
are three equivalent metrics on $\mathbb{R}$ but only $d_1$ and $d_2$ are uniformly equivalent. The following is easy to see:

a) $(\mathbb{R}, d_1)$ is an unbounded and not finite chainable space.

b) $(\mathbb{R}, d_2)$ is a bounded but not finite chainable space.

c) $(\mathbb{R}, d_3)$ is a bounded finite chainable space.

Now we summarize a few properties of $X$-finite chainable subsets.

**Proposition 1.4.** Let $(X, d)$ be a metric space. Then:

1) The property to be $X$-finite chainable subset is an immersion property, i.e. if $Y$ is $X$-finitely chainable, then $Y$ is $Z$-finitely chainable for every metric space $Z$ which contains metrically $X$.

2) The property to be $X$-finitely chainable is hereditary, i.e. if $Y$ is $X$-finitely chainable, then each subset $Z$ of $Y$ is $X$-finitely chainable.

3) Let $\{(X_j, d_j), j = 1, \ldots, n\}$ be a finite family of metric spaces. Then a subset $A = A_1 \times \ldots \times A_n$ in the metric product space $X = \prod_{j=1}^{n} X_j$ is $X$-finitely chainable if and only if $A_j$ is $X_j$-finitely chainable for $j = 1, \ldots, n$.

4) Let $\{(X_n, d_n) : n \in \mathbb{N}\}$ be a sequence of metric spaces and let $X = \prod_{n=1}^{\infty} X_n$ be the Cartesian product of $X_n$ endowed with the metric

$$d(x, y) = \frac{d_n(x, y)}{1 + d_n(x, y)}.$$ 

For $A_n \subset X_n$ ($n \in \mathbb{N}$) consider the set $A = \prod_{n=1}^{\infty} A_n$. Then $A$ is $X$-finite chainable if and only if $A_n$ is $X_n$-finitely chainable for every $n \in \mathbb{N}$. (This is a version of the Tychonoff Theorem for finite chainability.)

5) The property to be $X$-finitely chainable is a metric property but not a topological one, i.e. equivalent but not uniformly equivalent metrics can induce different $X$-chainable subsets. For uniformly equivalent metrics the classes of $X$-finitely chainable subsets with respect to them are the same.

6) The family of $X$-finitely chainable subsets of $X$ contains the family of bounded metrically convex subsets of $X$, whenever $X$ is a complete metric space.

7) The family of $X$-finitely chainable subsets of $X$ is contained (properly in general) in the family of the bounded subsets of $X$.

8) If $E$ is a normed space, then a subset $Y$ of $E$ is $E$-finitely chainable if and only if $Y$ is bounded.

9) Let $Y$ be a subset of a complete metric space $X$. Then $Y$ is relatively compact if and only if $Y$ is $X$-finite chainable and the link's number function admits a maximum.

10) Let $(X, d_X)$ and $(Z, d_Z)$ be two metric spaces. Let $f : X \to Z$ be a uniformly continuous function. Then $f$ maps $X$-finitely chainable subsets of $X$ into $Z$-finitely chainable subsets of $Z$. 

$$d_2(x, y) = \frac{|x - y|}{1 + |x - y|}$$

(1.2)

$$d_3(x, y) = |\arctan(x) - \arctan(y)|$$

(1.3)
Proof. We only prove statements 4 - 10.

Statement 4: Necessity. Let \( A = \prod_{n=1}^{\infty} A_n \) be \( X \)-finitely chainable. We show that \( A \) is \( X \)-finitely chainable for every \( n \). Fix \( \epsilon > 0 \) and consider \( \eta = \frac{\epsilon}{2^{n(1+\epsilon)}} \). By the \( X \)-finite chainability of \( A \), there exists a number \( j(\eta) \) of elements \( p^1, \ldots, p^{j(\eta)} \in X \) and \( m = m(\eta) \in \mathbb{N} \) such that any \( z = \{x_n\} \in A \) can be bound with some \( p^i \ (1 \leq i \leq j(\eta)) \) by an \( \eta \)-chain in \( X \) \( x^0 = x, \ldots, x^m = p^i \) satisfying \( d(x^{i-1}, x^i) < \epsilon \ (l = 1, \ldots, m) \). Then the \( n \)-th coordinate \( x_n \) of \( x \) can be bound with the \( n \)-th coordinate \( p^i_n \) for some \( i \in \{1, \ldots, j(\eta)\} \) with an \( \epsilon \)-chain in \( X \) of length \( m(\eta) \) since

\[
2^{-n} \frac{d_n(x^{i-1}_n, y^i_n)}{1 + d_n(x^{i-1}_n, y^i_n)} \leq d(x^{i-1}_n, x^i) < \eta \implies d_n(x^{i-1}_n, x^i) < 2^n \frac{\eta}{1 - 2^n \eta} = \epsilon.
\]

Sufficiency. Let \( A_n \) be \( X_n \)-finitely chainable for every \( n \). Take \( \epsilon > 0 \) and fix \( n \) such that \( \sum_{k=n+1}^{\infty} 2^{-k} < \frac{\epsilon}{2} \). Then the thesis follows from property 3) applied to \( A_1 \times \ldots \times A_n \) and from the fact that

\[
d(\{x_k\}, \{y_k\}) = \sum_{k=1}^{\infty} 2^{-k} \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)} = \sum_{k=1}^{n} 2^{-k} \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)} + \frac{\epsilon}{2}.
\]

Statement 5: Examples 0.1 and 1.3 show that the property to be \( X \)-finitely chainable is not a topological one. Now let \( d_1 \) and \( d_2 \) be two uniformly equivalent metrics on \( X \). Let \( A \) be a subset \( (X, d_1) \)-finitely chainable and let \( \epsilon > 0 \) be fixed. Then there exists \( \eta > 0 \) such that \( d_1(x, y) < \eta \) implies \( d_2(x, y) < \epsilon \). On the other hand, there are \( p_1, \ldots, p_\eta \in X \) and \( m(\eta) \in \mathbb{N} \) such that every \( x \in X \) can be bound with some \( p_j \) by an \( \eta \)-chain \( x = x_0, \ldots, x_{m(\eta)} = p_j \) such that \( d_1(x_i, x_{i+1}) < \eta \ (l = 0, \ldots, m(\eta) - 1) \). Note that \( d_2(x_i, x_{i+1}) < \epsilon \), and consequently \( A \) is \( (X, d_2) \)-finitely chainable.

Statement 6: First of all, a bounded set \( A = [0, 1] \cup \{2\} \) is \( \mathbb{R} \)-finitely chainable but not metrically convex. Now, let \( A \) be a bounded, metrically convex subset of \( X \), i.e. for any \( x, y \in A \) there is a point \( z \in A \) such that \( d(x, y) = d(x, z) + d(y, z) \). A theorem of Menger [3: p. 41] states that a convex and complete metric space contain together with \( x \) and \( y \) a metric segment whose extremities are \( x \) and \( y \), that is a subset isometric to an interval of length \( d(x, y) \). Hence we see that if \( x, y \in A \), there exist \( x = x_0, \ldots, x_m = y \) such that

\[
d(x, y) = \sum_{i=1}^{m} d(x_{i-1}, x_i) \quad \text{and} \quad d(x_{i-1}, x_i) < \epsilon. \quad (1.4)
\]

In addition, we can assume that (1.4) holds with

\[
d(x_{i+1}, x_i) + d(x_{i+2}, x_{i+1}) \geq \epsilon.
\]

Indeed, since

\[
d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) = d(x_i, x_{i+2})
\]

if

\[
d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) < \epsilon
\]

we can exclude \( x_{i+1} \) from the chain. Hence by (1.4) it follows that \( \frac{\eta \epsilon}{2} \leq d(x, y) < (n + 1)\epsilon \). Hence, any pair can be bound with an \( \epsilon \)-chain of length \( m(\epsilon) < 2 \text{ diam}(\mathbb{Y}) \).
Statement 7: Note that Examples 0.1 and 1.3 furnish bounded but not $X$-finitely chainable subsets. The boundedness of an $X$-finite chainable subset $A$ follows from the fact that $A \subset \bigcup_{j=1}^{m(\varepsilon)} B(p_j, m_A(\varepsilon)\varepsilon)$ for fixed $\varepsilon > 0$.

Statement 8: Every element $x$ of a bounded set $A \subset E$ can be bound with zero by an $\varepsilon$-chain with knots on the segment $[0, x]$ of length $m(\varepsilon) < \frac{\sup \|x \|_{E}}{\varepsilon}$.

Statement 9: For any $\varepsilon > 0$ we have $m_Y(\varepsilon) = 1$. On the other side, let $M = \max \{m_Y(\varepsilon) : \varepsilon > 0\}$. Fix $\varepsilon > 0$. Then there are finite number of points $p_1, \ldots, p_{(\varepsilon/M)} \in X$ such that every point of $Y$ can be bound with some $p_j$ by an $\frac{\varepsilon}{M}$-chain with length $M$. Thus $Y \subset \bigcup_{n=1}^{(\varepsilon/M)} B(p_j, \varepsilon)$

Statement 10: Fix $\varepsilon > 0$. Let $\delta = \delta(\varepsilon)$ be such that $d_1(x, y) < \delta$ implies $d_2(f(x), f(y)) < \varepsilon$. Let $Y$ be $X$-finitely chainable subset of $X$. Then there are finite number of points $p_1, \ldots, p_{(\varepsilon/M)} \in X$ such that any $y \in Y$ can be bound with some $p_j$ by a $\delta$-chain of length $m_Y(\delta)$. Then any point of $f(Y)$ can be bound with some $f(p_j)$ by an $\varepsilon$-chain of length $m_Y(\delta)$

Now we want to examine some properties (frame, amount, length and so on) of the chains with start knots fixed. In this way we will be able to define a non finite chainability measure that will be useful to prove the connexion between $X$-finite chainability, uniform continuity and uniformly local Lipschitz continuity of functions.

Let $(X, d)$ be a metric space and let $\varepsilon > 0$ be fixed. We denote by $P(x, \varepsilon, n)$ the set of all points in $X$ which can be bound with $x$ by an $\varepsilon$-chain of length $n$, i.e.

$$P(x, \varepsilon, n) = \left\{ y \in X \left| \begin{array}{l}
\text{There exist } \{z_1, \ldots, z_{n-1}\} \subset X \text{ such that } \\
d(x, z_1) < \varepsilon, d(z_1, z_2) < \varepsilon, \ldots, d(z_{n-1}, y) < \varepsilon 
\end{array} \right. \right\}. \quad (1.5)$$

Moreover, we denote by $P(x, \varepsilon)$ the set of all points in $X$ which can be bound with $x$ by an $\varepsilon$-chain with an arbitrary finite length, i.e.

$$P(x, \varepsilon) = \bigcup_{n \in \mathbb{N}} P(x, \varepsilon, n). \quad (1.6)$$

With this notation, step by step, it is easy to verify the following

**Proposition 1.5.**

a) $P(x, \varepsilon, 1) = B(x, \varepsilon)$.

b) $P(x, \varepsilon, n+1) = (P(x, \varepsilon, n))^\varepsilon$ (so any $P(x, \varepsilon, n)$ is an open set).

c) $P(x, \varepsilon, n+1) = P(x, \varepsilon, n)$ for some $n$ implies $P(x, \varepsilon, m) = P(x, \varepsilon, n)$ for any $m \geq n$.

d) $(P(x, \varepsilon))^\varepsilon = P(x, \varepsilon)$, i.e. $P(x, \varepsilon)$ is an isolated set, so if $X$ is a connected metric space, then $P(x, \varepsilon) = X$ for any $x \in X$ and $\varepsilon > 0$.

e) A relation $R$ on $X \times X$ defined by $(x, y) \in R$ if and only if $x \in P(y, \varepsilon)$ is an equivalence relation on $X \times X$.

f) The family $\{P(x, \varepsilon) : x \in X\}$ is an uniformly isolated partition, i.e. $(P(x, \varepsilon))^\varepsilon \cap (P(y, \varepsilon))^\varepsilon = \emptyset$ if $P(x, \varepsilon) \neq P(y, \varepsilon)$.
g) \((U_{i \in I} P(x_i, \varepsilon))^c = \bigcup_{i \in I} P(x_i, \varepsilon)\) for any index set \(I\).

h) If there is infinite number of distinct sets \(P(x_n, \varepsilon)\) \((n \in \mathbb{N})\) and \((Z, d)\) is an unbounded metric space, then a function \(f : X \to Z\) defined by

\[
f(x) = \begin{cases} 
0 & \text{if } x \notin \bigcup_{n \in \mathbb{N}} P(x_n, \varepsilon) \\
\bar{z}_n & \text{if } x \in P(x_n, \varepsilon), \text{n even} \\
\bar{w}_n & \text{if } x \in P(x_n, \varepsilon), \text{n odd}
\end{cases}
\]

where \(\bar{w}_n, \bar{z}_n \in Z\) are fixed points such that \(d(\bar{w}_n, \bar{z}_n) > n\) is an unbounded uniformly locally Lipschitz function on \(X\).

Now, let \(Y\) be a bounded subset of \(X\). Denote by \(N(Y)\) the set of all numbers \(\varepsilon > 0\) for which \(Y\) is chainable by \(\varepsilon\)-chains with fixed finite length, i.e

\[
N(Y) = \left\{ \varepsilon > 0 \mid \text{There exist } p_1, \ldots, p_{n(\varepsilon)} \in X, m_Y(\varepsilon) \in \mathbb{N} \text{ such that } Y \subseteq \bigcup_{j=1}^{n(\varepsilon)} P(p_j, \varepsilon, m_Y(\varepsilon)) \right\}.
\]  

(1.7)

Of course, if \(\varepsilon \in N(Y)\), then the real interval \([\varepsilon, \infty)\) is contained in \(N(Y)\). Put

\[
c(Y) = \inf N(Y).
\]

(1.8)

This is a measure of non finite chainability of \(Y\) and \(Y\) is \(X\)-finitely chainable if and only if \(c(Y) = 0\).

Moreover, the following is easy to see:

a) \(c(Y) \leq \text{diam}(Y)\).

b) \(c(A \cup B) \leq \max\{c(A), c(B)\}\).

c) \(c(A) = c(A)\).

d) \(A \subseteq B\) implies \(c(A) \leq c(B)\).

e) \(X\) is complete if for any decreasing sequence of \(X\)-finitely chainable closed sub-
sets \(\{F_n\}\) one has \(\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset\).

2. The main results

In the sequel \(\mathbb{R}\) is endowed with the Euclidean metric. For sake of completeness, we recall that a function \(f : X \to \mathbb{R}\) is said to be uniformly locally Lipschitzian if there are \(\rho > 0\) and \(L > 0\) such that

\[
\sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \right\} \leq L, \quad x, y \in X \text{ with } 0 < d(x, y) < \rho
\]

(2.1)

Theorem 2.1. Let \((X, d)\) be a metric space and let \(Y \subseteq X\). Then the following conditions are equivalent:

(i) \(Y\) is \(X\)-finitely chainable.
(ii) For any uniformly continuous function \( f : X \to \mathbb{R} \), \( f(Y) \) is a bounded subset of \( \mathbb{R} \).

(iii) For any uniformly locally Lipschitzian function \( f : X \to \mathbb{R} \), \( f(Y) \) is a bounded subset of \( \mathbb{R} \).

**Proof.** By Proposition 1.4(8) and 10), (i) implies (ii). Of course, (ii) implies (iii). So we show that (iii) implies (i). Suppose, on the contrary, that \( Y \subset X \) is not \( X \)-finitely chainable. We will construct a real-valued, uniformly locally Lipschitzian function on \( X \), unbounded on \( Y \). Fix a positive number \( \varepsilon_0 < c(Y) \). Then for any finite set of points \( p_1, ..., p_l \) and for any \( n \in \mathbb{N} \),

\[
Y \setminus \bigcup_{j=1}^{l} P(p_j, \varepsilon_0, n) \neq \emptyset. \tag{2.2}
\]

It can happen or not that there are finitely many points \( p_1, ..., p_l \in X \) such that

\[
Y \subset \bigcup_{j=1}^{l} P(p_j, \varepsilon_0).
\]

We examine both cases separately.

First case: There exist \( p_1, ..., p_l \) such that \( Y \subset \bigcup_{j=1}^{l} P(p_j, \varepsilon_0) \). Then by (2.2) for some \( p_j \) we have \( P(p_j, \varepsilon_0, n) \neq P(p_j, \varepsilon_0) \) for any \( n \in \mathbb{N} \). Hence, by Proposition 1.5(c), \( P(p_j, \varepsilon_0, n) \neq P(p_j, \varepsilon_0, m) \) for any \( n \neq m \). We define \( f : X \to \mathbb{R} \) by

\[
f(x) =
\begin{cases}
0 & \text{if } x \notin P(p_j, \varepsilon_0) \\
d(x, p_j) & \text{if } x \in P(p_j, \varepsilon_0, 1) \\
(n-1)\varepsilon_0 + \text{dist}(x, P(p_j, \varepsilon_0, n-1)) & \text{if } x \in P(p_j, \varepsilon_0, n) \setminus P(p_j, \varepsilon_0, n-1)
\end{cases} \tag{2.3}
\]

The function \( f \) is unbounded on \( Y \) and uniformly locally Lipschitzian on \( X \setminus P(p_j, \varepsilon_0) \). We show that \( f \) is uniformly locally Lipschitzian on \( X \). Put \( \rho = \varepsilon_0 \) and fix \( x_1, x_2 \in X \) which satisfy

\[
d(x_1, x_2) < \rho. \tag{2.4}
\]

We show that

\[
|f(x_1) - f(x_2)| < 2d(x_1, x_2). \tag{2.5}
\]

Since \( p_j \) and \( \varepsilon_0 \) are fixed, to shorten notation, we will write \( P \) instead of \( P(p_j, \varepsilon_0) \), \( P_n \) instead of \( P(p_j, \varepsilon_0, n) \) and \( P_0 = \{p_j\} \). Note that if there is \( l \in \{1, 2\} \) such that \( x_l \in P \), then \( x_l \in P_n \) for some \( n \in \mathbb{N} \). Put

\[
n_0 = \min \{n \in \mathbb{N} : \{x_1, x_2\} \cap P_n \neq \emptyset\}. \tag{2.6}
\]

Without loss, we can assume that \( x_1 \in P_{n_0} \). By (2.4),

\[
d(x_2, P_{n_0}) \leq d(x_1, x_2) < \varepsilon_0.
\]
Hence \( \{x_1, x_2\} \subseteq P_{n_0+1} \) by Proposition 1.5/b. Moreover, by (2.6), \( \{x_1, x_2\} \subseteq P_{n_0+1} \setminus P_{n_0-1} \) (if \( n_0 = 0 \), \( \{x_1, x_2\} \subseteq P_1 \)). Note that, if \( x_1, x_2 \in P_{n_0} \setminus P_{n_0-1} \) then, by the definition of \( f \),

\[
|f(x_1) - f(x_2)| \leq |d(x_1, P_{n_0-1}) - d(x_2, P_{n_0-1})| \leq d(x_1, x_2). \tag{2.7}
\]

Now suppose \( x_2 \in P_{n_0+1} \setminus P_{n_0} \) and \( x_1 \in P_{n_0} \setminus P_{n_0-1} \) (hence \( n_0 \geq 1 \)). By Proposition 1.5/b

\[
d(x_2, P_{n_0-1}) \geq \varepsilon_0. \tag{2.8}
\]

We show that

\[
\varepsilon_0 - d(x_1, x_2) \leq d(x_1, P_{n_0-1}) \leq \varepsilon_0. \tag{2.9}
\]

Since \( x_1 \in P_{n_0} \), \( d(x_1, P_{n_0-1}) < \varepsilon_0 \).

Now, suppose on the contrary that

\[
d(x_1, P_{n_0-1}) < \varepsilon_0 - d(x_1, x_2).
\]

Take \( y \in P_{n_0-1} \) such that \( d(x_1, y) < \varepsilon_0 - d(x_1, x_2) \). Then

\[
d(x_2, P_{n_0-1}) \leq d(x_2, y) \leq d(x_1, y) + d(x_2, x_1) < d(x_2, x_1) + \varepsilon_0 - d(x_2, x_1) = \varepsilon_0,
\]

which is a contradiction with (2.8). Note that in our case

\[
|f(x_2) - f(x_1)| = |\varepsilon_0 + d(x_2, P_{n_0}) - d(x_1, P_{n_0-1})|
= \varepsilon_0 + d(x_2, P_{n_0}) - d(x_1, P_{n_0-1})
\leq \varepsilon_0 + d(x_2, P_{n_0}) - (\varepsilon_0 - d(x_1, x_2))
= d(x_2, P_{n_0}) + d(x_1, x_2)
\leq 2d(x_1, x_2)
\]

which proves (2.5) if \( \{x_1, x_2\} \cap P \neq \emptyset \). Since \( f \) is constant on \( X \setminus P \), the result is proved.

Second case: For every \( p_1, \ldots, p_t \in X \setminus \bigcup_{j=1}^{t} P(p_j, \varepsilon_0) \neq \emptyset \). By Proposition 1.5/f), there is a sequence \( \{y_k\} \subseteq Y \) such that \( P(y_k, \varepsilon_0) \neq P(y_h, \varepsilon_0) \) for \( k \neq h \). Let us define \( f : X \rightarrow \mathbb{R} \) by

\[
f(x) = \begin{cases} 0 & \text{if } x \notin \bigcup_{n \in \mathbb{N}} P(y_n, \varepsilon_0) \\ n & \text{if } x \in P(y_n, \varepsilon_0) \end{cases}. \tag{2.11}
\]

Reasoning as in the case of the previous function, we can show that or \( \{x_1, x_2\} \) satisfying (2.4) is contained in \( X \setminus \bigcup_{n \in \mathbb{N}} P(y_n, \varepsilon_0) \) or there exists a fixed \( n \in \mathbb{N} \) such that \( \{x_1, x_2\} \subseteq P(y_n, \varepsilon_0) \). Since \( f \) is constant on each \( P(y_n, \varepsilon_0) \) and on \( X \setminus \bigcup_{n \in \mathbb{N}} P(y_n, \varepsilon_0) \), (2.5) holds true. The proof is complete.

\begin{remark}
We want to give two examples which show that it is necessary to consider two cases examined in the proof of Theorem 2.1:
\begin{enumerate}
\item[a)] The space \( (X, d) \) from Example 0.1 satisfies the first case.
\item[b)] \( X = \{\lambda e_n : n \in \mathbb{N} \text{ and } \lambda \in [1, 2]\} \), where \( e_n \) is the canonical basis of \( l_2 \) and \( d(x, y) = \|x - y\|_2 \), satisfies the second case.
\end{enumerate}
\end{remark}
It is worth saying that in Theorem 2.1 we can replace \( \mathbb{R} \) with the Euclidean norm by any normed space \((E, \| \cdot \|), E \neq \{0\}\). Indeed, we can define \( g : X \rightarrow E \) by \( g(x) = f(x)y \) where \( y \neq 0 \) is a fixed element from \( E \) and \( f \) is as in the proof of Theorem 2.1.

**Corollary 2.3.** Let \((X, d)\) be a metric space. Then \( X \) is compact if and only if \( X \) is finite chainable and each continuous, real-valued function on \( X \) is uniformly continuous.

**Proof.** By Theorem 2.1, any real-valued continuous function is bounded on \( X \). Thus the result follows by Hewitt's theorem.

**Remark 2.4.** Metric spaces \((X, d)\) for which any real-valued, continuous function on \( X \) is uniformly continuous are widely studied in literature and are known as UC spaces (for references see [4]).

From Theorem 2.1 it is easy to reprove the Atsuji theorem [2: Theorem 2] in a slightly stronger form.

**Corollary 2.5** (compare with [2: Theorem 2]). Let \( X \) be a metric space. Then \( X \) is finitely chainable if and only if \( f(X) \) is a bounded subset of \( \mathbb{R} \) for any uniformly continuous or uniformly locally Lipschitzian function \( f \) on \( X \).

**References**


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