A Real Inversion Formula for the Laplace Transform

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Let \( f \) be the Laplace transform of a square integrable function \( F \) and set

\[
F_N(t) = \int_0^\infty f(s)e^{-st} P_N(st) \, ds \quad (N = 0, 1, 2, \ldots)
\]

for the polynomials

\[
P_N(\xi) = \sum_{0 \leq \nu \leq n \leq N} \frac{(-1)^{\nu+1}(2n)!}{(n+1)!\nu!(n-\nu)!(n+\nu)!} \xi^{n+\nu} \times \left\{ \frac{2n+1}{n+\nu+1} \xi^2 - \left( \frac{2n+1}{n+\nu+1} + 3n+1 \right) \xi + n(n+\nu+1) \right\}.
\]

Then it is proved that the sequence \( \{F_N\}_{N=0}^\infty \) converges to \( F \) in the sense that

\[
\lim_{N \to \infty} \int_0^\infty |F(t) - F_N(t)|^2 \, dt = 0.
\]

Furthermore, a general formula for this result is established.

Key words: Bergman-Selberg spaces, real inversion formulas, Laplace transform, reproducing kernels, reproducing kernel Hilbert spaces

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1. Introduction and result

For any \( q > 0 \), we let \( L_q^2 \) be the class of all square integrable functions with respect to the measure \( t^{1-2q} \, dt \) on the half line \((0, \infty)\). Then we consider the Laplace transform

\[
LCF(z) = \int_0^\infty F(t)e^{-zt} \, dt \quad (z \in R^+ = \{ R \geq 0 \})
\]

for \( F \in L_q^2 \). In [2, §7], it was shown that the image of \( L_q^2 \) under the Laplace transform \( L \) coincides with the reproducing kernel Hilbert space \( H_q \) (Bergman-Selberg space) admitting the reproducing kernel \( K_q(z, \bar{u}) = \Gamma(2q)/(z + \bar{u})^{2q} \) and \( L \) is an isometry of
of the space $L^2_q$ onto the space $H_q$. For $q > \frac{1}{2}$, the Hilbert space $H_q$ consists of all functions $f$ analytic in $\mathbb{R}^+$ with finite norms
\[
\|f\|_q^2 = \frac{4^{q-1}}{\pi (2q-1)} \int \int_{\mathbb{R}^+} |f(z)|^2 x^{2q-2} dx dy \quad (z = x + iy)
\]
and
\[
H_\frac{1}{2} = \left\{ f : f \text{ analytic in } \mathbb{R}^+, \|f\|_{\frac{1}{2}}^2 = \frac{1}{2\pi} \sup_{z>0} \int_{-\infty}^{+\infty} |f(x + iy)|^2 dy < \infty \right\}.
\]

The inverse of the Laplace transform $\mathcal{L}$ is, in general, given by complex forms. The observation in many fields of sciences however gives us, intuitively, real data $[\mathcal{L}F](x)$ only, and so it is important to establish its inversion formula in terms of real data $[\mathcal{L}F](x)$. Such a formula was given for $L^1[(0, \infty), dt]$-functions $F$ by R. P. Boas and D. V. Widder about fifty years ago (see [7, p. 386]). By use of the representations of $H_q$-norms on the positive real line in [5], we shall establish in the next theorem the natural inversion formula of the Laplace transform $\mathcal{L}$ on the space $L^2_q$ in terms of real data $[\mathcal{L}F](x)$ in the framework of the Hilbert space $L^2_q$.

**Theorem.** For any fixed number $q > 0$ and for any function $F \in L^2_q$, put $f = \mathcal{L}F$. Then the inversion formula
\[
F(t) = s - \lim_{N \to \infty} \int_{0}^{\infty} f(x)e^{-xt} P_{N,q}(xt) dx \quad (t > 0)
\]
is valid, where the limit is taken in the space $L^2_q$ and the polynomials $P_{N,q}$ are given by the formula
\[
P_{N,q}(\xi) = \sum_{0 \leq \nu \leq n \leq N} \frac{(-1)^{\nu+1} \Gamma(2n + 2q)}{\nu!(n-\nu)! \Gamma(n + 2q + 1) \Gamma(n + \nu + 2q)} \xi^{n+\nu + 2q - 1}
\times \left\{ 2(n+q) \xi^2 - \left( \frac{2(n+q)}{n+\nu+2q} + 3n+2q \right) \xi + n(n+\nu+2q) \right\}.
\]
Moreover, the series
\[
\sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + 2q + 1)} \int_{0}^{\infty} |\partial_x^n [xf'(x)]|^2 x^{2n+2q-1} dx
\]
converges and the truncation error is estimated by the inequality
\[
\left\| F(t) - \int_{0}^{\infty} f(x)e^{-xt} P_{N,q}(xt) dx \right\|_{L^2_q}^2 \leq \sum_{n=N+1}^{\infty} \frac{1}{n! \Gamma(n + 2q + 1)} \int_{0}^{\infty} |\partial_x^n [xf'(x)]|^2 x^{2n+2q-1} dx.
\]

Note that, even if $q = \frac{1}{2}$, our polynomial $P_{N,\frac{1}{2}}$ is different from the one of R. P. Boas and D. V. Widder.
2. Preliminaries

In order to prove Theorem, we prepare three lemmas.

**Lemma 1.** For any fixed \( q > 0 \), let the function \( f \) be a member of the space \( H_q \) and set, for any non-negative integer \( N \),

\[
f_N(z) = \sum_{n=0}^{N} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \partial^n_\xi [\xi f'(\xi)] \partial^n_\xi [\xi \partial_\xi K_q(\xi, z)] \xi^{2n+2q-1} \, d\xi
\]

for \( z \in \mathbb{R}^+ \). Then, the function \( f_N \) belongs to the space \( H_q \), and the sequence \( \{f_N\}_{N=0}^\infty \) converges to \( f \) in \( H_q \).

**Proof.** Recall first the following representation of the norm in the space \( H_q \):

\[
\|f\|_q^2 = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial^n_\xi [\xi f'(\xi)]|^2 \xi^{2n+2q-1} \, d\xi
\]  \hspace{1cm} (1)

(see [5]). From the reproducing property of \( K_q(\cdot, \xi) \), we have the expressions

\[
K_q(z, \overline{z}) = (K_q(\cdot, \overline{u}), K_q(\cdot, \overline{z}))_q
= \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \partial^n_\xi [\xi \partial_\xi K_q(\xi, \overline{u})] \partial^n_\xi [\xi \partial_\xi K_q(\xi, \overline{z})] \xi^{2n+2q-1} \, d\xi
\]

and

\[
f(z) = (f(\cdot), K_q(\cdot, \overline{z}))_q
= \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \partial^n_\xi [\xi f'(\xi)] \partial^n_\xi [\xi \partial_\xi K_q(\xi, \overline{z})] \xi^{2n+2q-1} \, d\xi
\]

where \((\cdot, \cdot)_q\) denotes the inner product in \( H_q \). Hence, we see by [6, p. 170] (see also [4, p. 96]) that \( f_N \) is a member in \( H_q \) and

\[
\|f - f_N\|_q^2
= \left\| \sum_{n=N+1}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty \partial^n_\xi [\xi f'(\xi)] \partial^n_\xi [\xi \partial_\xi K_q(\xi, \overline{z})] \xi^{2n+2q-1} \, d\xi \right\|_q^2
\leq \sum_{n=N+1}^{\infty} \frac{1}{n!\Gamma(n+2q+1)} \int_0^\infty |\partial^n_\xi [\xi f'(\xi)]|^2 \xi^{2n+2q-1} \, d\xi.
\]  \hspace{1cm} (2)

Therefore, our claim is true. \( \blacksquare \)

**Lemma 2.** For any fixed \( q > 0 \), let the function \( f \) be a member of the space \( H_q \) and set, for any non-negative integer \( N \),

\[
F_N(t) = \sum_{n=0}^{N} \frac{t^{2q-1}}{n!\Gamma(n+2q+1)} \int_0^\infty \partial^n_\xi [\xi f'(\xi)] \partial^n_\xi [\xi \partial_\xi (e^{-tx})] \xi^{2n+2q-1} \, dx
\]
for $t \in (0, \infty)$. Then, the function $F_N$ belongs to the space $L^2_q$, and furthermore, for the functions $f_N$ defined in Lemma 1, $\mathcal{L}F_N = f_N$.

**Proof.** We first prove that, for any $n$, the function $g_n$ defined by

$$g_n(t) = t^{q-\frac{1}{2}} \int_{-\infty}^{\infty} \partial^n_x f'(x) \partial^n_x (e^{-tx}) x^{2n+2q-1} \, dx$$

belongs to the space $L^2[(0, \infty), dt]$. By the Leibniz rule,

$$\partial^n_x [x \partial_x (e^{-tx})] t^{q-\frac{1}{2}} = (-1)^n t^{n+q-\frac{1}{2}} (n - tx) e^{-tx},$$

and we have

$$g_n(t) = (-1)^n t^{n+q-\frac{1}{2}} \int_{-\infty}^{\infty} \partial^n_x [x f'(x)] e^{-tx} x^{2n+2q-1} \, dx$$

$$- (-1)^n t^{n+q+\frac{1}{2}} \int_{-\infty}^{\infty} \partial^n_x [x f'(x)] e^{-tx} x^{2n+2q} \, dx.$$  

Moreover, the expression (1) implies that the functions defined by

$$\partial^n_x [x f'(x)] x^{2n+2q-1} \quad \text{and} \quad \partial^n_x [x f'(x)] x^{2n+2q}$$

are contained in the spaces $L^2_{n+q}$ and $L^2_{n+q+1}$, respectively. Hence the function $g_n$ is the restriction of a member in the set

$$\left\{ \tau^{n+q-\frac{1}{2}} h_1(\tau) + \tau^{n+q+\frac{1}{2}} h_2(\tau) : h_1 \in H_{n+q} \text{ and } h_2 \in H_{n+q+1} \right\}$$

to the half-axis $(0, \infty)$, and it is represented by

$$g_n(t) = t^{n+q-\frac{1}{2}} \hat{h}_1(t) + t^{n+q+\frac{1}{2}} \hat{h}_2(t)$$

for some functions $\hat{h}_1 \in H_{n+q}$ and $\hat{h}_2 \in H_{n+q+1}$. If $n = 0$, we have $g_n(t) = t^{q+\frac{1}{2}} h_3(t)$ for some function $h_3 \in H_q+1$. Furthermore, for $n \neq 0$ we have the representation

$$g_n(t) = t^{n+q-\frac{1}{2}} k_1'(t) + t^{n+q+\frac{1}{2}} k_2'(t)$$

for some functions $k_1 \in H_{n+q-1}$ and $k_2 \in H_{n+q}$ (see [3]). Hence, from (1) we get the relations

$$\int_{0}^{\infty} |t^{n+q-\frac{1}{2}} k_1'(t)|^2 dt = \int_{0}^{\infty} |t k_1'(t)|^2 t^{2n+2q-3} dt < \infty$$

and

$$\int_{0}^{\infty} |t^{n+q+\frac{1}{2}} k_2'(t)|^2 dt = \int_{0}^{\infty} |t k_2'(t)|^2 t^{2n+2q+1} dt < \infty,$$

and so the function $g_n \ (n \neq 0)$ belongs to the space $L^2[(0, \infty), dt]$. Likewise, the function $g_0$ is also a member of the space $L^2[(0, \infty), dt]$. By virtue of the isometry $s(t)$ →
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$s(t)t^{q-1/2}$ of the space $L^2[(0,\infty),dt]$ onto the space $L^2_t$, we conclude that the function $F_N$ belongs to the space $L^2_t$.

Next, in order to prove that $\mathcal{L}F_N = f_N$, we examine, for a fixed number $\xi > 0$, the integrability of the functions

$$\varphi(x,t;n,\xi) = \partial^n_x \{x f'(x)\} \partial^n_x \{x \partial_x (e^{-t x})\} e^{-\xi t} t^{2q-1} x^{2n+2q-1} \quad (n = 0, 1, 2, \ldots)$$

with respect to the Lebesgue measure on the set $(0,\infty) \times (0,\infty)$. We first have the estimate

$$|\varphi(x,t;n,\xi)| \leq |\partial^n_x \{x f'(x)\}| \left| \partial^n_x \{x \partial_x (e^{-t x})\} \right| e^{-\xi t} t^{2q-1} x^{2n+2q-1}$$

$$= |\partial^n_x \{x f'(x)\}| \left| (-t)^n x e^{-t x} + n(-t)^{n-1} e^{-t x} \right| t^{2q-1} e^{-\xi t} x^{2n+2q-1}$$

Therefore, since the functions defined by

$$x \int_0^\infty t^{n+2q} e^{-(x+\xi)t} dt = \Gamma(n+2q+1) x(x+\xi)^{-(n+2q+1)}$$

and

$$n \int_0^\infty t^{n+2q-1} e^{-(x+\xi)t} dt = n \Gamma(n+2q)(x+\xi)^{-(n+2q)}$$

belong to the space $L^2[(0,\infty),x^{2n+2q-1} dx]$, we see by the Schwarz inequality that the function $\varphi(x,t;n,\xi)$ is integrable for all $n$. By the Fubini theorem, the following sequence of equalities is therefore valid:

$$\int_0^\infty F_N(t) e^{-\xi t} dt$$

$$= \sum_{n=0}^N \frac{1}{n! \Gamma(n+2q+1)}$$

$$\times \int_0^\infty \left[ \int_0^\infty \partial^n_x \{x f'(x)\} \partial^n_x \{x \partial_x (e^{-t x})\} x^{2n+2q-1} dx \right] t^{2q-1} e^{-\xi t} dt$$

$$= \sum_{n=0}^N \frac{1}{n! \Gamma(n+2q+1)}$$

$$\times \int_0^\infty \partial^n_x \{x f'(x)\} \left[ \int_0^\infty \partial^n_x \{x \partial_x (e^{-t x})\} e^{-\xi t} t^{2q-1} dt \right] x^{2n+2q-1} dx$$

$$= \sum_{n=0}^N \frac{1}{n! \Gamma(n+2q+1)}$$

$$\times \int_0^\infty \partial^n_x \{x f'(x)\} \partial^n_x \left[ x \partial_x \int_0^\infty e^{-t x} e^{-\xi t} t^{2q-1} dt \right] x^{2n+2q-1} dx$$

$$= \sum_{n=0}^N \frac{1}{n! \Gamma(n+2q+1)}$$

$$\times \int_0^\infty \partial^n_x \{x f'(x)\} \partial^n_x \{x \partial_x K_q(x,\xi)\} x^{2n+2q-1} dx = f_N(\xi).$$

Thus the assertions of the lemma are proved.
Lemma 3. For any fixed $q > 0$, let the function $f$ be a member of the space $H_q$. Then the following statements are true.

(i) If $n \geq 1$ and $0 \leq m \leq n - 1$, then $\partial^m_x [xf'(x)]x^{n+m+2q} = o(1)$ as $x \to 0+$.

(ii) $f(x)x^q = O(1)$ as $x \to 0+$.

Proof. By the Leibniz rule, we have the equality

$$\partial^m_x [xf'(x)] = x\partial^{m+1}_x f(x) + m\partial^m_x f(x).$$

We also see that the function $\partial^{m+1}_x$ belongs to the space $H_{q+m+1}$ (see [3]), and from the Schwarz inequality the following estimate is valid:

$$|\partial^{m+1}_x f(x)| = \left| \left( \partial^{m+1}_x f(\xi), K_{q+m+1}(\xi, x) \right) \right|_{q+m+1} \leq \|\partial^{m+1}_x f\|_{q+m+1} K_{q+m+1}(x, x)^\frac{1}{2} = \|\partial^{m+1}_x f\|_{q+m+1} \Gamma(2q + 2m + 2)^\frac{1}{2} 2^{-(q+m+1)}x^{-(q+m+1)}.$$  

Likewise, the estimates

$$|\partial^m_x f(x)| \leq \|\partial^m_x f\|_{q+m} \Gamma(2q + 2m)^\frac{1}{2} 2^{-(q+m)}x^{-(q+m)}$$

and

$$|f(x)| \leq \|f\|_q \Gamma(2q)^\frac{1}{2} 2^{-q}x^{-q}$$

are valid. Therefore, our lemma is obtained.

3. Proof of Theorem

From Lemma 3, and by integration by parts we have, for any non-negative integer $n$,

$$\int_0^\infty \partial^n_x [xf'(x)]\partial^n_x [x\partial_x (e^{-tx})]x^{2n+2q-1}dx = t^n \int_0^\infty x f'(x)\partial^n_x [(n - tx)e^{-tx}x^{2n+2q-1}]dx$$

$$= -t^n \int_0^\infty f(x)\partial_x [x\partial^n_x ((n - tx)e^{-tx}x^{2n+2q-1})]dx.$$  

Meanwhile, for $n \geq 1$ we also have
\[-t^n\partial_z \left[ x\partial^n \left\{ (n - tx)e^{tx}z^{2n+2q-1} \right\} \right] \]
\[-e^{-tx}t^n \sum_{\nu=0}^{n} \binom{n}{\nu} (-t)^{\nu} \left\{ n\partial_x^{-\nu}x^{2n+2q-1} - t\partial_x^{-\nu}x^{2n+2q} \right\} \]
- \[tx \sum_{\nu=0}^{n} \binom{n}{\nu} (-t)^{\nu} \left\{ n\partial_x^{-\nu}x^{2n+2q-1} - t\partial_x^{-\nu}x^{2n+2q} \right\} \]
+ \[x \sum_{\nu=0}^{n} \binom{n}{\nu} (-t)^{\nu} \left\{ n\partial_x^{-\nu+1}x^{2n+2q-1} - t\partial_x^{-\nu+1}x^{2n+2q} \right\} \]
\[= e^{-tx} \sum_{\nu=0}^{n} (-1)^{\nu+1} \binom{n}{\nu} (xt)^{n+\nu} \frac{\Gamma(2n+2q)}{\Gamma(n+\nu+2q)} \]
\[\times \left\{ \frac{2n+2q}{n+\nu+2q} t^2x^{2\nu+1} - \left( \frac{2n+2q}{n+\nu+2q} + 3n+2q \right) tx^{2q} + n(n+\nu+2q)x^{2q-1} \right\} . \]

Applying Lemma 1 and Lemma 2 to the isometry \( L \), we therefore obtain the inversion formula of \( L \). Also, the inequality (2) gives the estimate of the truncation error \( \|L\|_q \).

**Remark.** For any \( q > 0 \), let the functions \( F \) and \( f \) be as in Theorem. In [3], we see that the function \( f' \) is a member of the space \( H_{q+1} \), and \( \|f'\|_{q+1} = \|f\|_q \). Hence, by the inversion formula in [4, p. 85], we have the inversion formula of the Laplace transform \( L \) in the complex form as follows:

\[ F(t) = s - \lim_{n \to -\infty} \frac{-4^q t^{2q}}{\pi \Gamma(2q+1)} \int_{E_n} f'(z)e^{-z^2} \, dx \, dy, \]

where the limit \( s = \lim_{n \to -\infty} \) is taken in the space \( L^2_q \) and the sequence \( \{E_n\}_{n=0}^\infty \) is a compact exhaustion of \( R^+ \).

**References**


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