Ground states for pseudo-relativistic Hartree equations of critical type

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Abstract. We study the existence of ground state solutions for a class of nonlinear pseudo-relativistic Schrödinger equations with critical two-body interactions. Such equations are characterized by a nonlocal pseudo-differential operator closely related to the square root of the Laplacian. We investigate this problem using variational methods after transforming the problem to an elliptic equation with a nonlinear Neumann boundary conditions.

1. Introduction

The relativistic Hamiltonian for $N$ identical particles of mass $m$, position $x_i$ and momentum $p_i$ interacting through the two-body potential $\alpha W(|x_i - x_j|)$ is given by

$$\mathcal{H} = \sum_{i=1}^{N} \left( \sqrt{p_i^2 c^2 + m^2 c^4} - mc^2 \right) - \alpha \sum_{i \neq j} W(|x_i - x_j|).$$

where $c$ is the speed of light and $\alpha > 0$ is a coupling constant.

According to the usual quantization rules the dynamics of the corresponding system of $N$-identical quantum spinless particles (a Bose gas) is described by the complex wave function $\Psi_N = \Psi_N(t, x_1, \ldots, x_N)$ governed by the Schrödinger equation

$$i\hbar \partial_t \Psi_N = \mathcal{H}_N \Psi_N$$

where $\hbar$ is the Planck’s constant. Here $\mathcal{H}_N : D \subset L^2(\mathbb{R}^3)^{\otimes N} \to L^2(\mathbb{R}^3)^{\otimes N}$ is the quantum mechanics Hamiltonian operator, obtained from the classical Hamiltonian via the usual quantization rule $p \mapsto -i\hbar \nabla$, and defined in a suitable dense domain.

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domain $D$. In the case of interest here, $\mathcal{H}_N$ is

$$\mathcal{H}_N = \left( \sum_{j=1}^{N} \sqrt{-\hbar^2 c^2 \Delta_j + m^2 c^4} - m c^2 \right) - \alpha \sum_{i \neq j} W(|x_i - x_j|),$$

where $W$ is the multiplication operator corresponding to the two-body interaction potential, (e.g., $W(|x|) = |x|^{-1}$ for gravitational interactions).

The operator (from now on we will take $\hbar = 1$ and $c = 1$)

$$\sqrt{-\Delta + m^2}$$

can be defined for all $f \in H^1(\mathbb{R}^N)$ as the inverse Fourier transform of the $L^2$ function $\sqrt{|k|^2 + m^2} \mathcal{F}[f](k)$ (here $\mathcal{F}[f]$ denotes the Fourier transform of $f$) and it is also associated to the quadratic form

$$Q(f; g) = \int_{\mathbb{R}^N} \sqrt{|k|^2 + m^2} \mathcal{F}[f] \mathcal{F}[g] \, dk$$

which can be extended to the space

$$H^{1/2}(\mathbb{R}^N) = \left\{ f \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |k| |\mathcal{F}[f](k)|^2 \, dk < +\infty \right\}$$

(see, e.g., [10] for more details).

In the mean field limit approximation (i.e., $\alpha N \simeq O(1)$ as $N \to +\infty$) of a quantum relativistic Bose gas, one is lead to study the nonlinear mean field equation – called the pseudo-relativistic Hartree equation – given by

$$i \partial_t \psi = \left( \sqrt{-\Delta + m^2} - m \right) \psi - (W * |\psi|^2) \psi.$$

where $*$ denotes convolution. We will consider attractive two-body interaction, and hence $W$ will always be a nonnegative function.

See [11] for the study of this equation when $W$ is the gravitational interaction, and [4] for a rigorous derivation of the mean field equation (1.2) as an $N \to +\infty$ limit of the Schrödinger equation for $N$ quantum particles, and [3] for more recent developments for models involving the pseudo-relativistic operator $\sqrt{-\Delta + m^2}$.

It has recently been proved that for Newton or Yukawa type two-body interactions (i.e., $W(|x|) = |x|^{-1}$ or $|x|^{-1} e^{-|x|}$ in $\mathbb{R}^3$) such an equation is locally well posed in $H^s$, $s \geq 1/2$, and that the solution is global in time for small initial data in $L^2$ (see [8]). Blowup has been proved in [6] and [7].

Due to the focusing nature of the nonlinearity (attractive two-body interaction) there exist solitary waves solutions given by

$$\psi(t, x) = e^{i\mu t} \varphi(x),$$

where $\varphi$ satisfies the nonlinear eigenvalue equation

$$\sqrt{-\Delta + m^2} \varphi - m \varphi - (W * |\varphi|^2) \varphi = -\mu \varphi.$$

In [11] the existence of such solutions (in the case $W(x) = |x|^{-1}$) was proved provided that $M < M_c$, $M_c$ being the Chandrasekhar limit mass.
More precisely, the authors have shown the existence in $H^{1/2}(\mathbb{R}^3)$ of a radial, real-valued, nonnegative minimizer (ground state) of

$$E[\psi] = \frac{1}{2} \int_{\mathbb{R}^3} \bar{\psi} (\sqrt{-\Delta + m^2} - m) \psi \, dx - \frac{1}{4} \int_{\mathbb{R}^3} (|x|^{-1} + |\psi|^2) |\psi|^2 \, dx,$$

with fixed “mass-charge” $M = \int_{\mathbb{R}^3} |\psi|^2 \, dx < M_c$. We call mass-critical the potentials $W$ whose associated functional $E$ exhibits this kind of phenomenon.

More recently, in [5] it has been proved that the ground state solution is regular ($H^{s}(\mathbb{R}^3)$, for all $s \geq 1/2$), strictly positive, and exponentially decaying. Moreover the solution is unique, at least for small $L^2$ norm ([9]).

Let us remark that these last results are heavily based on the specific form (Newton or Yukawa type) of the two-body interactions in the Hartree nonlinearity. Indeed in these cases the estimates of the nonlinearity rely on the following facts:

- for this class of potentials one has that
  $$\frac{e^{-\mu|x|}}{4\pi |x|} * f = (\mu^2 - \Delta)^{-1} f$$
  for $f \in \mathcal{S}(\mathbb{R}^3)$, $\mu \geq 0$;

- the use of a generalized Leibnitz rule for Riesz and Bessel potentials;

- there holds the estimate
  $$\left\| \frac{1}{|x|} * |u|^2 \right\|_{L^\infty} \leq \frac{\pi}{2} \left\| (\Delta)^{1/4} u \right\|_{L^2}^2.$$

In [2] there has been proved an existence and regularity result for the solutions of (1.3) for a wider class of nonlinearities by exploiting the relation of equation (1.3) with an elliptic equation on $\mathbb{R}^{N+1}_+$ with a nonlinear Neumann boundary condition. Such a relation has been recently used to study several problems involving fractional powers of the Laplacian (see e.g. [1] and references therein) and it is based on an alternative definition of the operator (1.1) that can be described as follows. Given any function $u \in \mathcal{S}(\mathbb{R}^N)$ there is a unique function $v \in \mathcal{S}(\mathbb{R}^{N+1}_+)$ (here $\mathbb{R}^{N+1}_+ = \{ (x,y) \in \mathbb{R} \times \mathbb{R}^N \ | \ x > 0 \}$) such that

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
 v(0,y) = u(y) & \text{for } y \in \mathbb{R}^N = \partial \mathbb{R}^{N+1}_+. \end{cases}$$

Setting

$$Tu(y) = -\frac{\partial u}{\partial x}(0,y),$$

we have that the equation

$$\begin{cases} -\Delta w + m^2 w = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
 w(0,y) = Tu(y) = -\frac{\partial u}{\partial x}(0,y) & \text{for } y \in \mathbb{R}^N, \end{cases}$$
has the solution \( w(x, y) = -\frac{\partial u}{\partial x}(x, y) \). From this we have that

\[
T(Tu)(y) = -\frac{\partial w}{\partial x}(0, y) = \frac{\partial^2 v}{\partial x^2}(0, y) = \left(-\Delta_x v + m^2 v\right)(0, y)
\]

and hence \( T^2 = (-\Delta + m^2) \).

In [2] we studied the equation

\[
\sqrt{-\Delta + m^2} v = \mu v + \nu |v|^{p-2} v + \sigma (W * |v|^2) v \quad \text{in } \mathbb{R}^N,
\]

where \( p \in (2, 2N/(N-1)) \), \( \mu < m \) is fixed, \( \nu, \sigma \geq 0 \) (but not both equal to 0), \( W \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \), \( r > N/2 \), \( W(0) = W(|x|) \to 0 \) as \( |x| \to +\infty \).

The results are obtained, following the approach outlined above, by studying the equivalent elliptic problem with nonlinear boundary condition

\[
\begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+,
-\frac{\partial v}{\partial x} = \mu v + \nu |v|^{p-2} v + \sigma (W * |v|^2) v & \text{on } \mathbb{R}^N = \partial \mathbb{R}^{N+1}_+,
\end{cases}
\]

and the associated functional on \( H^1(\mathbb{R}^{N+1}_+) \).

Let us point out that in [2] the \( L^2 \) norm of the solution is not prescribed. In such a case existence of a (positive, radially symmetric) solution can be proved for a class of potentials \( W \) and exponents \( p \) which is larger than the one we deal with here.

When the \( L^2 \) norm is prescribed to be \( M \) (the most relevant problem from a physical point of view), as in [11], then the Newtonian potential \( (|x|^{-1} \text{ in } \mathbb{R}^3) \) is critical, in the sense that minimization of \( E \) given by (1.4) is possible only when \( M < M_c \) (see Theorem 1.1).

The main purpose of this paper is to exploit this approach also for the problem of finding minimizer of the static energy

\[
E[u] = \frac{1}{2} \int_{\mathbb{R}^N} u(\sqrt{-\Delta + m^2} - m) u \, dx + \frac{\eta}{p} \int_{\mathbb{R}^N} |u|^p \, dx - \frac{\sigma}{4} \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 \, dx
\]

with prescribed \( L^2 \) norm, for a wider class of attractive two-body potential including the critical case.

To be more precise, we consider a class of two-body potentials \( W \in L^q_w(\mathbb{R}^N) \), with \( q \geq N \). We recall that \( L^q_w(\mathbb{R}^N) \), the weak \( L^q \) space, is the space of all measurable functions \( f \) such that

\[
\sup_{\alpha > 0} \alpha \left| \left\{ x : |f(x)| > \alpha \right\} \right|^{1/q} < +\infty,
\]

where \( |E| \) denotes the Lebesgue measure of a set \( E \subset \mathbb{R}^N \). Note that \( W(x) = |x|^{-1} \) does not belong to any \( L^q \)-space but it belongs to \( L^q_w(\mathbb{R}^N) \). We say that a potential \( W \) is critical if \( W \in L^N(\mathbb{R}^N) \).

Our main result is the following.
Theorem 1.1. Let $W \in L^q_0(\mathbb{R}^N)$, where $q \geq N \geq 2$, and $W(y) \geq 0$ for all $y \in \mathbb{R}^N$, and suppose that

(1.8) \quad W(\lambda^{-1}y) \geq \lambda^\alpha W(y), \quad \text{for all } \lambda \in (0, 1) \text{ and for some } \alpha > 0.

We also assume that $W(x) = W(|x|)$ is rotationally symmetric and that $W(r) \to 0$ as $r \to +\infty$.

Take $\eta \geq 0$, $\sigma > 0$ and $p \in (2 + 2/q, 2 + 2/(N - 1) = 2N/(N - 1)]$. Then:

- if $\eta > 0$ or $\eta = 0$ and $q > N$, then for all $M > 0$ there is a strictly positive minimizer $u \in H^{1/2}(\mathbb{R}^N)$ of $\mathcal{E}[u]$ such that $\int_{\mathbb{R}^N} u^2 = M$;

- (mass-critical case) if $\eta = 0$ and $q = N$, there is a critical value $M_0 > 0$ such that for all $0 < M < M_0$ there is a strictly positive minimizer $u \in H^{1/2}(\mathbb{R}^N)$ of $\mathcal{E}[u]$ such that $\int_{\mathbb{R}^N} u^2 = M$.

Moreover there exists $\mu > 0$ such that $u$ is a smooth, exponentially decaying at infinity, solution of

$$
(\sqrt{-\Delta + m^2 - m}) u = -\mu u - \eta |u|^{p-2} u + \sigma (W * |u|^2) u \quad \text{in } \mathbb{R}^N,
$$

and $u$ is radial if $W = W(r)$ is a decreasing function of $r > 0$.

Remark 1.2. The nonlinear term $|u|^{p-2} u$ is a defocusing nonlinearity, the convolution term is a focusing nonlinearity. An open problem is to understand if solitons exist also for other ranges of $p$, in particular for $2 < p \leq 2 + 2/q$ and $W \in L^q_0$.

Remark 1.3. If $W \in L^q_0$ and (1.8) holds for some $\alpha > 0$, then necessarily $\alpha \in (0, N/q]$. If $W(x) = |x|^{-\alpha}$, then $W \in L^q_0$ if and only if $\alpha = N/q$.

Remark 1.4. $\mu$ is a Lagrange multiplier.

2. Preliminaries

Let $(x, y) \in \mathbb{R} \times \mathbb{R}^N$. We have already introduced $\mathbb{R}^{N+1}_+ = \{(x, y) \in \mathbb{R}^{N+1} \mid x > 0\}$. We will always denote the norm of $u \in L^p(\mathbb{R}^{N+1}_+)$ by $\|u\|_p$, the norm of $u \in H^1(\mathbb{R}^{N+1}_+)$ by $\|u\|$, and the norm of $v \in L^p(\mathbb{R}^N)$ by $|v|_p$.

We recall that, for all $v \in H^1(\mathbb{R}^{N+1}_+) \cap C^0(\mathbb{R}^{N+1}_+)$,

$$
\int_{\mathbb{R}^N} |v(0, y)|^p \, dy = \int_{\mathbb{R}^N} dy \int_{+\infty}^0 \frac{\partial}{\partial x} |v(x, y)|^p \, dx
\leq p \int_{\mathbb{R}^N} |v(x, y)|^{p-1} \left| \frac{\partial v}{\partial x}(x, y) \right| dx \, dy
\leq p \left( \int_{\mathbb{R}^N} |v(x, y)|^{2(p-1)} \, dx \, dy \right)^{1/2} \left( \int_{\mathbb{R}^N} \left| \frac{\partial v}{\partial x}(x, y) \right|^2 \, dx \, dy \right)^{1/2}.
$$

That is,

(2.1) \quad |v(0, \cdot)|_p^p \leq p \|v\|_p^{p-1} \left\| \frac{\partial v}{\partial x} \right\|_2^2.
that is $\leq 2$ and $1$

\[ (2.3) \]

\[ (2.2) \]

\[ \int_{\mathbb{R}^N} \gamma(v)^p \leq p \|\nabla v\|^p_{2(p-1)} \left\| \frac{\partial v}{\partial x} \right\|^p_2, \]

which, by Sobolev embedding, is finite for all $2 \leq p \leq 2^*$, where we have set $2^* = 2N/(N-1)$. By density of $H^1(\mathbb{R}^{N+1}) \cap C^\infty(\mathbb{R}^{N+1})$ in $H^1(\mathbb{R}^{N+1})$ such an estimate allows us to define the trace $\gamma(v)$ of $v$ for all $v \in H^1(\mathbb{R}^{N+1})$. The inequality

\[ (2.2) \]

holds then for all $v \in H^1(\mathbb{R}^{N+1})$.

It is known that the traces of functions in $H^1(\mathbb{R}^{N+1})$ belong to $H^{1/2}(\mathbb{R}^N)$ and that every function in $H^{1/2}(\mathbb{R}^N)$ is the trace of a function in $H^1(\mathbb{R}_+^{N+1})$. Then (2.2) is in fact equivalent to the well-known fact that $\gamma(v) \in H^{1/2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ provided $q \in [2, 2^*]$. Here we also recall that

\[ \|w\|^2_{H^{1/2}} = \inf \left\{ \|u\|^2 \mid u \in H^1(\mathbb{R}^{N+1}), \gamma(u) = w \right\} = \int_{\mathbb{R}^N} (1 + |\xi|)^{-2} \left| \mathcal{F}w(\xi) \right|^2 d\xi. \]

Let us also introduce the norm of the weak $L^q$-space as follows:

\[ \|f\|_{q,w} = \sup_A |A|^{-1/r} \int_A |f(x)| \, dx \]

where $1/q + 1/r = 1$ and $A$ denotes any measurable set of finite measure $|A|$ (see, e.g., [10] for more details). Using this norm we can state the weak Young inequality. If $g \in L^q_w(\mathbb{R}^N)$, $f \in L^p(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$ where $1 < q, p, r < +\infty$ and $1/q + 1/p + 1/r = 2$, then

\[ (2.3) \]

\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(y) g(y - z) h(y) dy \, dz \leq C_{p,q,r} \|g\|_{q,w} \|f\|_p \|h\|_r. \]

We consider the class of two-body interactions $W \in L^q_w(\mathbb{R}^N)$ for $q \geq N$. By the weak Young inequality and the Hölder inequality we have for $r = 4q/(2q - 1)$ ($\in (2, 2^*)$ since $q \geq N$) and for all $p \in (4q/(2q - 1), 2^*)$,

\[ (2.4) \]

\[ \int_{\mathbb{R}^N} (W * |u|^2) |w|^2 \, dy \leq C \|W\|_{q,w} \|w\|^4 \leq C \|W\|_{q,w} \|w\|^{4 - \frac{2p}{q} - \frac{2p}{q} - \frac{2p}{q}} |w|^{2p/(q)} \]

For $p = 2^*$ we get

\[ (2.5) \]

\[ \int_{\mathbb{R}^N} (W * |w|^2) |w|^2 \, dy \leq C \|W\|_{q,w} \|w\|^{2N/q} |w|^{2N/q}. \]

In the (critical) case $q = N$ this gives

\[ (2.6) \]

\[ \int_{\mathbb{R}^N} (W * |w|^2) |w|^2 \, dy \leq C \|W\|_{N,w} \|w\|^2 |w|^2. \]

We point out that one cannot deduce (2.6) from the weak Young’s inequality (2.3) directly, and that it is not true, in general, that the $L^\infty$ norm of $W * |u|^2$ can be bounded by the $L^2$ norm of $u$ if $W \in L^N_w$. 

For all $v \in H^1(\mathbb{R}^{N+1})$, we consider the functional given by
\begin{align*}
I(v) &= \frac{1}{2} \left( \iint_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 |v|^2) \, dx \, dy - \int_{\mathbb{R}^N} m |\gamma(v)|^2 \, dy \right) \\
& \quad + \frac{\eta}{p} \int_{\mathbb{R}^N} |\gamma(v)|^p \, dy - \frac{\sigma}{4} \int_{\mathbb{R}^N} (W * |\gamma(v)|^2) |\gamma(v)|^2 \, dy.
\end{align*}

In view of (2.2) and (2.4), all the terms in the functional $I$ are well defined if $p \in (2, 2^\#)$ and $W \in L^q_w(\mathbb{R}^N)$ with $q \geq N$.

Note that from (2.1), with $p = 2$, it follows that
\begin{equation}
(2.7) \quad m \int_{\mathbb{R}^N} |\gamma(v)|^2 \, dy \leq 2(m\|v\|_2)\|\nabla v\|_2 \leq \iint_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 |v|^2) \, dx \, dy,
\end{equation}
showing that the quadratic part of the functional $I$ is nonnegative.

Moreover the following property can be checked easily:

Lemma 2.1. For $u \in H^1(\mathbb{R}^{N+1})$, let $w = \gamma(u) \in H^{1/2}(\mathbb{R}^N)$, $\hat{w} = \mathcal{F}(w)$ and
\begin{equation*}
v(x, y) = \mathcal{F}^{-1}(e^{-x\sqrt{m^2 + |\cdot|^2}} \hat{w}) = \int_{\mathbb{R}^N} e^{-x\sqrt{m^2 + |\xi|^2}} \hat{w}(\xi)e^{iy\xi} \, d\xi.
\end{equation*}
Then $v \in H^1(\mathbb{R}^{N+1})$, $\|v\| = \|w\|_{H^{1/2}}$, $I(v) \leq I(u)$ and $I(v) = E[w]$.

3. Minimization problem

We consider the minimization problem
\begin{equation}
(3.1) \quad I(M) = \inf \{I(v) : v \in \mathcal{M}_M\},
\end{equation}
where the manifold $\mathcal{M}_M$ is given by
\begin{equation*}
\mathcal{M}_M = \left\{ v \in H^1(\mathbb{R}^{N+1}) : \int_{\mathbb{R}^N} |\gamma(v)|^2 = M \right\}.
\end{equation*}

Remark 3.1. The term $m \int_{\mathbb{R}^N} |\gamma(v)|^2$ in the functional $I(v)$ is constant for all $v \in \mathcal{M}_M$. The presence of such a term will allow us to show that the infimum of the functional $I$ on $\mathcal{M}_M$ is negative.

Concerning the existence of a minimizer for problem (3.1) we start by proving, in the following lemmas, boundedness from below on $\mathcal{M}_M$ of the functional $I$, and some properties of the infimum $I(M)$.

Lemma 3.2. The functional $I$ is bounded from below and coercive on $\mathcal{M}_M \subset H^1(\mathbb{R}^{N+1})$ for all $M > 0$ if $\eta > 0$ or $q > N$ and for all $M$ small enough if $\eta = 0$ and $q = N$.
Proof. First we examine first the convolution term. If \( \eta > 0 \), from (2.4) and \( |\gamma(u)|_2^2 = M \) we have

\[
0 \leq \int_{\mathbb{R}^N} (W \ast |\gamma(u)|^2) |\gamma(u)|^2 \leq C \|W\|_{q,w} |\gamma(u)|_2^4 \frac{2p}{q(p-2)} |\gamma(u)|_{p(p-2)}^{2p}
\]

\[
= C \|W\|_{q,w} M^{2-\frac{p}{q(p-2)}} |\gamma(u)|_{p(p-2)}^{2p}.
\]

Since by assumption \( \frac{2p}{q(p-2)} < p \), this is enough to prove coercivity if \( \eta > 0 \).

Indeed in such a case we have that

\[
I(u) \geq \frac{1}{2} \|u\|^2 - \frac{1}{2} m M + C_1 \|\gamma(u)\|_p^p - C_2 |\gamma(u)|_{p(p-2)}^{2p} \geq \frac{1}{2} \|u\|^2 - C_3.
\]

In the case \( \eta = 0 \) we deduce from (2.6) and \( |\gamma(u)|_2 \leq C \|u\| \) that

\[
I(u) \geq \|u\|^2 - m M - C \|W\|_{q,w} M^{2-N/q} \|u\|^{2N/q}.
\]

It is then clear that the functional is bounded from below and coercive whenever \( q > N \) and, when \( q = N \), if \( \|W\|_{N,w} \) is small enough.

Lemma 3.3. \( I(M) < 0 \) for all \( M > 0 \).

Proof. Take any function \( u \in C_0^\infty(\mathbb{R}^N) \) such that \( |u|_2^2 = M \), and let \( w(x,y) = e^{-mx}u(y) \). Then,

\[
I(M) = \inf_{v \in \mathcal{M}_M} I(v) \leq I(w)
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^N} \left( (\partial_x w)^2 + |\nabla_y w|^2 + m^2 |w|^2 \right) dx dy - \frac{m}{2} \int_{\mathbb{R}^N} |u|^2 dy + G(u)
\]

\[
= \frac{m}{4} \int_{\mathbb{R}^N} |u|^2 dy + \frac{1}{4m} \int_{\mathbb{R}^N} |\nabla_y u|^2 dy + \frac{m}{4} \int_{\mathbb{R}^N} |u|^2 dy - \frac{m}{2} \int_{\mathbb{R}^N} |u|^2 dy + G(u)
\]

\[
= \frac{1}{4m} \int_{\mathbb{R}^N} |\nabla_y u|^2 dy + G(u),
\]

where

\[
G(u) = \frac{\eta}{p} \int_{\mathbb{R}^N} |u|^p dy - \frac{\sigma}{4} \int_{\mathbb{R}^N} (W \ast |u|^2) |u|^2 dy.
\]

For \( \lambda > 0 \) take \( u_\lambda(y) = \lambda^{N/2}u(\lambda y) \) and \( w_\lambda(x,y) = e^{-mx}u_\lambda(y) \in \mathcal{M}_M \). We find that

\[
I(M) \leq \inf_{\lambda > 0} I(w_\lambda)
\]

\[
\leq \inf_{\lambda \in (0,1)} \left[ \frac{1}{4m} \int_{\mathbb{R}^N} |\nabla_y u_\lambda|^2 + \frac{\eta}{p} \int_{\mathbb{R}^N} |u_\lambda|^p - \frac{\sigma}{4} \int_{\mathbb{R}^N} (W \ast |u_\lambda|^2) |u_\lambda|^2 \right]
\]

\[
\leq \inf_{\lambda \in (0,1)} \left[ \frac{\lambda^2}{4m} \int_{\mathbb{R}^N} |\nabla_y u|^2 + \frac{\eta \lambda^{N(p-2)}}{p} \int_{\mathbb{R}^N} |u|^p - \frac{\sigma \lambda^\alpha}{4} \int_{\mathbb{R}^N} (W \ast |u|^2) |u|^2 \right],
\]

and since \( \alpha < N(p^2 - 1) < 2 \), the infimum is negative. 

\( \square \)
Lemma 3.4. For all $M > 0$ and $\beta \in (0, M)$ we have that $I(M) < I(M - \beta) + I(\beta)$. Moreover, $I(M)/M$ is a concave function of $M$ and hence $I(M)$ is a continuous function of $M$.

Proof. The subadditivity is a consequence of the fact that, for all $\theta > 1$,

\begin{equation}
(3.3) \quad I(\theta M) < \theta I(M), \quad \text{which implies} \quad \frac{1}{\theta} I(M) < I(M/\theta).
\end{equation}

Indeed, taking $\theta_1 = M/\beta$ and $\theta_2 = M/(M - \beta)$, we have that

\[ I(M) = \frac{\beta}{M} I(M) + \frac{M - \beta}{M} I(M) < I(\beta) + I(M - \beta). \]

To prove that (3.3) holds, we remark that for all $v \in \mathcal{M}_\lambda$ and $\lambda = \theta^{1/2} > 1$ we have, thanks to (2.7),

\[ I(v) = \frac{\lambda^2}{2} \left[ \int_\mathbb{R}^N \left( |\nabla v|^2 + m^2 |v|^2 \right) dx + m \int_\mathbb{R}^N |\gamma(v)|^2 dy \right] + \frac{\eta \lambda^p}{p} \int_\mathbb{R}^N |\gamma(v)|^p dy - \frac{\sigma \lambda^4}{4} \iint_\mathbb{R}^N (W * |\gamma(v)|^2) |\gamma(v)|^2 dy \leq \lambda^4 I(v). \]

Hence, since $I(M) < 0$,

\[ I(\theta M) = \inf_{|\gamma(v)|_2^2 = \theta M} I(v) \leq \inf_{|\gamma(v)|_2^2 = M} I(\theta^{1/2} v) \leq \theta^2 \inf_{|\gamma(v)|_2^2 = M} I(v) = \theta^2 I(M) < \theta I(M) < I(M). \]

To prove the concavity of $I(M)/M$, we remark that

\[ \frac{I(M)}{M} = \frac{1}{M} \inf_{u \in \mathcal{M}_1} I(u) = \inf_{u \in \mathcal{M}_1} \frac{I(\sqrt{M} u)}{M}. \]

We now show that, for all $u \in \mathcal{M}_1$, $M \mapsto I(\sqrt{M} u)/M$ is a concave function of $M$. This will immediately prove that $I(M)/M$ is a concave function. Since

\[ \frac{I(\sqrt{M} v)}{M} = \frac{1}{2} \left( \iint_\mathbb{R}^{N+1} \left( |\nabla v|^2 + m^2 |v|^2 \right) dx dy - \int_{\mathbb{R}^N} m |\gamma(v)|^2 dy \right) + \frac{\eta M^{p/2 - 1}}{p} \int_\mathbb{R}^N |\gamma(v)|^p dy - \frac{\sigma M}{4} \iint_\mathbb{R}^N (W * |\gamma(v)|^2) |\gamma(v)|^2 dy, \]

it is immediate to check that the second derivative with respect to the variable $M$ is negative for all $M > 0$ when $p/2 < 2$ and that the function is linear when $p = 4$ (namely the critical exponent for $N = 2$).

We are now ready to prove the existence of a minimizer for the functional $I$ on $\mathcal{M}_M$. \qed
Proposition 3.5. For every $M > 0$ there is a function $u \in H^1(\mathbb{R}^{N+1})$ such that

\[
\begin{cases}
    I(u) = I(M), \\
    \int_{\mathbb{R}^N} |\gamma(u)|^2 \, dy = M,
\end{cases}
\]

i.e., a minimizer for $\mathcal{I}$ in $\mathcal{M}_M$.

Proof. Let $\{u_n\} \subset \mathcal{M}_M$ be a minimizing sequence. It follows from Lemma 2.1 that

\[
v_n(x, y) = F^{-1}(e^{x} - \sqrt{m^2 + |\cdot|^2}F(\gamma(u_n)))
\]

is also minimizing. From Lemma 3.2 we deduce that $v_n$ is bounded in $H^1(\mathbb{R}^{N+1})$ and that $w_n = \gamma(v_n)$ is bounded in $H^{1/2}(\mathbb{R}^{N})$ and $\int_{\mathbb{R}^N} |w_n|^2 \, dy = M$.

We will now use the concentration-compactness method of P. L. Lions \[12\]. Namely, one of the following cases must occur:

- **vanishing** for all $R > 0$,
  \[
  \lim_{n \to +\infty} \sup_{z \in \mathbb{R}^N} \int_{z + B_R} |w_{n_k}|^2 \, dy = 0;
  \]

- **dichotomy** for a subsequence $\{n_k\}$,
  \[
  \lim_{R \to +\infty} \lim_{k \to +\infty} \sup_{z \in \mathbb{R}^N} \int_{z + B_R} |w_{n_k}|^2 \, dy = \alpha \in (0, M);
  \]

- **compactness** for all $\epsilon > 0$ there is $R > 0$, a sequence $\{y_k\}$ and a subsequence $\{w_{n_k}\}$ such that
  \[
  \int_{y_k + B_R} |w_{n_k}|^2 \, dy \geq M - \epsilon.
  \]

Following the usual strategy we will show that the vanishing and dichotomy cases cannot occur.

Lemma 3.6. If vanishing occurs, then

\[
\int_{\mathbb{R}^N} (W * |w_n|^2) |w_n|^2 \, dy \to 0.
\]

Proof. Take any $\delta > 0$ and $R > 0$. Define $W_\delta = WI_{\{|W| \geq \delta\}}$ and

\[
W^R_\delta(|y|) = \left(W_\delta(|y|) - R\right)_+ I_{\{|y| < R\}} + W_\delta(|y|)I_{\{|y| \geq R\}},
\]

where $I_A$ is the characteristic function of the set $A$. Then it easy to check that $W \in L^s_\delta(\mathbb{R}^N)$ implies that $W_\delta \in L^s(\mathbb{R}^N)$ for any $s \in [1, q)$ and moreover that $|W^R_\delta|_s \to 0$ as $R \to +\infty$ for any $\delta > 0$. Also define $\Gamma^R_\delta = W_\delta - W^R_\delta$. It is clear that

\[
0 \leq (W - W_\delta)(|y|) \leq \delta, \quad 0 \leq \Gamma^R_\delta(|y|) \leq R \quad \forall y \in \mathbb{R}^N
\]

Then, for any given $\delta > 0$ and $R > 0$ and for some $s \geq N/2$ (which implies that $2 < 4s/(2s - 1) \leq 2N/(N - 1)$), we get from the Young inequality (also taking into
Proof. If dichotomy occurs, then there is a sequence follows.

\[ \int_{\mathbb{R}^N} (W \ast |w_n|^2) |w_n|^2 \]

\[ \leq \int_{\mathbb{R}^N} \left( (W - W_\delta) \ast |w_n|^2 \right) |w_n|^2 + \int_{\mathbb{R}^N} (W_\delta^R \ast |w_n|^2) |w_n|^2 + \int_{\mathbb{R}^N} (\Gamma_\delta^R \ast |w_n|^2) |w_n|^2 \]

\[ \leq \delta |w_n|^2 + |W_\delta^R| |w_n|^2 + R \int_{\mathbb{R}^N} |w_n(y)|^2 |w_n(z)|^2 \leq R \, dy \, dz \]

\[ \leq \delta M^2 + C |W_\delta^R| + RM \sup_{z \in \mathbb{R}^N} \int_{\mathbb{R}^N} |w_n|^2 \, dy. \]

Now, first letting \( n \to +\infty \), then letting \( R \to +\infty \), and finally letting \( \delta \to 0^+ \), we conclude the proof of the lemma.

\( \square \)

Lemma 3.7. If dichotomy occurs, then for any \( \alpha \in (0, M) \) we have

\[ I(M) \geq I(\alpha) + I(M - \alpha). \]

Proof. If dichotomy occurs, then there is a sequence \( \{u_k\} \subset \mathbb{N} \) such that, for any \( \epsilon > 0 \), there exists \( R > 0 \) and a sequence \( \{z_k\} \subset \mathbb{R}^N \) such that

\[ \lim_{k \to +\infty} \int_{z_k + B_R} |w_{n_k}|^2 \, dy \in (\alpha - \epsilon, \alpha + \epsilon). \]

Define \( \tilde{u}_k = w_{n_k} \, (\cdot + z_k) \) and

\[ I(\tilde{u}_k) = \int_{z_k + B_R} |\gamma(\tilde{u}_k)|^2 \, dy \in (\alpha - \epsilon, \alpha + \epsilon). \]

Since \( \{\tilde{u}_k\} \) is a bounded sequence in \( H^1(\mathbb{R}_+^{N+1}) \), \( \tilde{u}_k \to u \) weakly in \( H^1(\mathbb{R}_+^{N+1}) \) and \( \tilde{u}_k = \gamma(\tilde{u}_k) \to w = \gamma(u) \) weakly in \( H^{1/2} \) and strongly in \( L^p_{\text{loc}}(\mathbb{R}_+^N) \) for \( p \in [2, 2N/(N - 1)] \). Hence, for all \( \epsilon > 0 \) there is \( R > 0 \) such that

\[ \int_{B_R} |\gamma(u)|^2 \, dy = \lim_{k \to +\infty} \int_{B_R} |\gamma(\tilde{u}_k)|^2 \, dy \in (\alpha - \epsilon, \alpha + \epsilon) \]

and

\[ \int_{\mathbb{R}^N} |\gamma(u)|^2 \, dy = \lim_{R \to +\infty} \int_{B_R} |\gamma(u)|^2 \, dy = \alpha. \]

We set \( v_k = \tilde{u}_k - u \) and \( \beta_k = \int_{\mathbb{R}^N} |\gamma(v_k)|^2 \, dy \). By weak convergence of the sequence \( \{\gamma(\tilde{u}_k)\} \) in \( L^2 \) we get

\[ \lim_{k \to +\infty} \beta_k = M - \alpha. \]

Now we claim that

\[ I(M) = \lim_{k \to +\infty} \mathcal{I}(\tilde{u}_k) = \mathcal{I}(u) + \lim_{k \to +\infty} \mathcal{I}(v_k) \leq I(\alpha) + \lim_{k \to +\infty} I(\beta_k). \]

Then, by the continuity of the function \( I \), as stated in Lemma 3.4, the lemma follows.
Now we prove the claim. We will show that
\[ \lim_{k \to +\infty} (\mathcal{I}(\tilde{u}_k) - \mathcal{I}(v_k)) = \mathcal{I}(u) \]
Indeed, by weak convergence in \( H^1(\mathbb{R}^{N+1}_+) \), we immediately get
\[
\lim_{k \to +\infty} \left( \int_{\mathbb{R}^{N+1}_+} |\nabla \tilde{u}_k|^2 - \int_{\mathbb{R}^{N+1}_+} |\nabla v_k|^2 \right) = \int_{\mathbb{R}^{N+1}_+} |\nabla u|^2
\]
and by the Brezis–Lieb lemma
\[
\lim_{k \to +\infty} \left( \int_{\mathbb{R}^N} |\gamma(\tilde{u}_k)|^p - \int_{\mathbb{R}^N} |\gamma(v_k)|^p \right) = \int_{\mathbb{R}^N} |\gamma(u)|^p
\]
for \( 2 \leq p \leq 2N/(N-1) \). Hence we have to investigate the last nonlinear term. We will show in Appendix A that
\[
\lim_{k \to +\infty} \left( \int_{\mathbb{R}^N} (W * |\tilde{u}_k|^2) |\tilde{u}_k|^2 - \int_{\mathbb{R}^N} (W * |\gamma(v_k)|^2) |\gamma(v_k)|^2 \right) = \int_{\mathbb{R}^N} (W * |w|^2) |w|^2,
\]
from which the claim follows. 

Finally, since we have ruled out both vanishing and dichotomy, then we may conclude that indeed compactness occurs, namely that for all \( \epsilon > 0 \) there is \( R > 0 \), a sequence \( \{y_k\} \) and a subsequence \( \{w_{n_k}\} \) such that
\[
\int_{y_k + B_R} |w_{n_k}|^2 \, dy \geq M - \epsilon.
\]
Define as before \( \tilde{w}_k = w_{n_k}(\cdot + y_k) \) and \( \tilde{u}_k(x, y) = F^{-1}(e^{-\frac{2}{\sqrt{m+1}}} |w(\tilde{u}_k)|) \). Then \( \tilde{u}_k \) is a minimizing sequence for \( \mathcal{I} \) on \( \mathcal{M}_M \) such that
\[
\int_{B_R} |\gamma(\tilde{u}_k)|^2 \geq M - \epsilon.
\]
Since \( \{\tilde{u}_k\} \) is a bounded sequence in \( H^1(\mathbb{R}^{N+1}_+) \), \( \tilde{u}_k \to u \) weakly in \( H^1(\mathbb{R}^{N+1}_+) \) and \( \tilde{w}_k = \gamma(\tilde{u}_k) \to w = \gamma(u) \) weakly in \( H^{1/2} \) and strongly in \( L^p_{\text{loc}}(\mathbb{R}^N) \) for \( p \in [2, 2N/(N-1)] \). As in the proof of Lemma 3.7 we deduce that \( \int_{\mathbb{R}^N} |\gamma(u)|^2 = M \).

Moreover we claim that, as \( k \to +\infty \),
\[
\int_{\mathbb{R}^N} (W * |\tilde{w}_k|^2) |\tilde{w}_k|^2 \to \int_{\mathbb{R}^N} (W * |w|^2) |w|^2.
\]
Indeed, by the weak Young inequality and the Hölder inequality we have
\[
\left| \int_{\mathbb{R}^N} (W * \tilde{w}_k^2) \tilde{w}_k^2 - \int_{\mathbb{R}^N} (W * w^2) w^2 \right| \leq \int_{\mathbb{R}^N} (W * (\tilde{w}_k^2 + w^2)) |\tilde{w}_k^2 - w^2| \leq C \|W\|_{q, w} \|\tilde{w}_k^2 + w^2\|_s |\tilde{w}_k^2 - w^2|_s \leq C |\tilde{w}_k - w|_{2s} \to 0
\]
since \( 2 < 2s = 4q/(2q - 1) < 2N/(N-1) \).
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Hence, finally, by the weakly lower semicontinuity of the $H^1$ and $L^p$ norms (the positive terms of the functional $I$), we conclude that

$$I(u) \leq \lim \inf_{k \to +\infty} I(\tilde{u}_k) = I(M),$$

which implies the $u$ is a minimizer for $I$ in $\mathcal{M}_M$.

Now we collect all the results obtained to conclude the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 3.5 there exists a function $u \in H^1(\mathbb{R}^{N+1}_+)$ which minimizes $I$ in $\mathcal{M}_M$. Therefore $u$ can always be assumed nonnegative and, by Lemma 2.1, to have the form

$$u(x, y) = F^{-1}(e^{-x}\sqrt{m^2 + |y|^2} F(w)),$$

where $w = \gamma(u) \in H^{1/2}(\mathbb{R}^N)$.

If $W$ is a nonincreasing radial function, then $w$ can be assumed to be a radial nonincreasing function. Indeed let $w^*$ be the spherically symmetric decreasing rearrangement of $w$ and define

$$u^*(x, y) = F^{-1}(e^{-x}\sqrt{m^2 + |y|^2} F(w^*)).$$

Then $I(u^*) = \mathcal{E}[w^*]$ (this also follows from Lemma 2.1). We can then use the properties of the spherically symmetric decreasing rearrangement, namely

(i) $w^*$ is a nonnegative, radial function;

(ii) $w \in L^p(\mathbb{R}^N)$ implies $w^* \in L^p(\mathbb{R}^N)$ and $|w^*|^p = |w|^p$;

(iii) symmetric decreasing rearrangement decreases kinetic energy (Lemma 7.17 in [10]), that is,

$$\int_{\mathbb{R}^N} w^*(\sqrt{-\Delta + m^2} - m) w^* dy \leq \int_{\mathbb{R}^N} w(\sqrt{-\Delta + m^2} - m) w dy;$$

(iv) Riesz’s rearrangement inequality (see Theorem 3.7 in [10]),

$$\int_{\mathbb{R}^N} (W \ast |w^*|^2)|w^*|^2 dy \geq \int_{\mathbb{R}^N} (W \ast |w|^2)|w|^2 dy$$

if $W(y) = W^*(|y|)$ (in particular if $W$ is radial and nonincreasing);

to deduce that

$$I(u^*) = \mathcal{E}[w^*] \leq \mathcal{E}[w] = I(u) = I(M).$$

Moreover, by the theory of Lagrange multipliers, any minimizer $u \in H^1(\mathbb{R}^{N+1}_+)$ of the functional $I$ on $\mathcal{M}_M$ is such that

$$\int_{\mathbb{R}^{N+1}_+} (\nabla u \nabla w + m^2 uw) dx dy - \int_{\mathbb{R}^N} m\gamma(u)\gamma(w) dy + \mu \int_{\mathbb{R}^N} \gamma(u)\gamma(w) dy$$

$$+ \eta \int_{\mathbb{R}^N} |\gamma(u)|^{p-2} \gamma(u)\gamma(w) dy - \sigma \int_{\mathbb{R}^N} (W \ast |\gamma(u)|^2) \gamma(u)\gamma(w) dy = 0$$

(3.4)
for all \( w \in H^1(\mathbb{R}^{N+1}_+) \), i.e., \( u \) is a weak solution of the nonlinear Neumann boundary condition problem

\[
\begin{aligned}
\left\{ \begin{array}{ll}
-\Delta u + m^2 u = 0 & \text{in } \mathbb{R}^{N+1}_+,
\frac{\partial u}{\partial x} + \mu u = mu - \eta |u|^{p-2} u + \sigma (W \ast |u|^2) u & \text{on } \mathbb{R}^N = \partial \mathbb{R}^{N+1}_+,
\end{array} \right.
\end{aligned}
\]

for some Lagrange multiplier \( \mu \in \mathbb{R} \). To prove that \( \mu > 0 \) we take \( w = u \) in (3.4) to get

\[
0 = \int_{\mathbb{R}^{N+1}_+} (|\nabla u|^2 + m^2 |u|^2) \, dx \, dy - \int_{\mathbb{R}^N} m |\gamma(u)|^2 \, dy + \mu \int_{\mathbb{R}^N} |\gamma(u)|^2 \, dy
\]

\[
+ \eta \int_{\mathbb{R}^N} |\gamma(u)|^p \, dy - \sigma \int_{\mathbb{R}^N} (W \ast |\gamma(u)|^2) |\gamma(u)|^2 \, dy
\]

\[
= 2\mathcal{I}(u) + \mu \int_{\mathbb{R}^N} |\gamma(u)|^2 \, dy + \eta \left(1 - \frac{2}{p}\right) \int_{\mathbb{R}^N} |\gamma(u)|^p \, dy
\]

\[
- \frac{\sigma}{2} \int_{\mathbb{R}^N} (W \ast |\gamma(u)|^2) |\gamma(u)|^2 \, dy.
\]

Since \( \mathcal{I}(u) < 0 \), we have in particular that

\[
\frac{\eta}{p} \int_{\mathbb{R}^N} |\gamma(u)|^p \, dy < \frac{\sigma}{4} \int_{\mathbb{R}^N} (W \ast |\gamma(u)|^2) |\gamma(u)|^2 \, dy
\]

and hence, since \( p \leq 2N/(N-1) \leq 4 \), for \( N \geq 2 \), we get

\[
\mu \int_{\mathbb{R}^N} |\gamma(u)|^2 \, dy = -2\mathcal{I}(u) - \eta \left(1 - \frac{2}{p}\right) \int_{\mathbb{R}^N} |\gamma(u)|^p \, dy + \sigma \frac{p}{2} \int_{\mathbb{R}^N} (W \ast |\gamma(u)|^2) |\gamma(u)|^2 \, dy
\]

\[
> \eta \left(\frac{4}{p} - 1\right) \int_{\mathbb{R}^N} |\gamma(u)|^p \, dy \geq 0.
\]

Finally the regularity, the strictly positivity and the exponential decay at infinity of the weak nonnegative solutions of (3.5) follow straightforwardly from Theorems 3.14 and 5.1 in [2].

\[\square\]

4. Appendix A

We prove that

\[
\int_{\mathbb{R}^N} |(W \ast w\gamma(v_k)) w\gamma(v_k)| \, dx + \int_{\mathbb{R}^N} |(W \ast (v_k)^2) w^2| \, dx + \int_{\mathbb{R}^N} |(W \ast w\gamma(v_k)) w^2| \, dx
\]

\[
+ \int_{\mathbb{R}^N} |(W \ast (v_k)^2) w\gamma(v_k)| \, dx \to 0 \quad \text{as } k \to +\infty,
\]

as claimed in the proof of Lemma 3.7. Indeed we have the following result.
Lemma 4.1. For any \( w \in H^{1/2}(\mathbb{R}^N) \) and for sequences \( \{f_n, g_n, h_n\} \) bounded in \( H^{1/2}(\mathbb{R}^N) \) and such that \( f_n \to 0 \) in \( L^2_{loc} \) we have
\[
\int_{\mathbb{R}^N} (W * |f_n g_n|) |w h_n| \to 0 \quad \text{as } n \to +\infty.
\]

Proof. It is convenient to introduce, for any given \( \delta > 0 \) and \( R > 0 \), \( W_\delta = W_1 W_{\geq \delta} \)
and
\[
W_\delta^R(y) = (W_\delta - R)^+ \|y| < R + W_\delta \|y| \geq R.
\]

Then for \( W \in L^p_{loc}(\mathbb{R}^N) \) we have \( W_\delta \in L^p(\mathbb{R}^N) \) for any \( p \in [1, q) \) and moreover
that \( |W_\delta^R|_p \to 0 \) as \( R \to +\infty \) for any \( \delta > 0 \). Define again also \( \Gamma_\delta^R = W_\delta - W_\delta^R \).
Note that \( \text{supp} \Gamma_\delta^R \subset B_R \) and \( 0 \leq \Gamma_\delta^R \leq R \).

From the Young inequality (with \( p = N/2, r = 2p/(2p - 1) = N/(N - 1) \)), the Hölder inequality and the Sobolev embedding theorem we have
\[
\int_{\mathbb{R}^N} (W * |f_n g_n|) |w h_n|
\leq \int_{\mathbb{R}^N} ((W - W_\delta) * |f_n g_n|) |w h_n| + \int_{\mathbb{R}^N} (W_\delta^R * |f_n g_n|) |w h_n|
+ \int_{\mathbb{R}^N} (\Gamma_\delta^R * |f_n g_n|) |w h_n|
\leq \delta |f_n g_n|_1 |w h_n|_1 + |W_\delta^R|_{N/2} |f_n g_n|_r |w h_n|_r + \int_{\mathbb{R}^N} (\Gamma_\delta^R * |f_n g_n|) |w h_n|
\leq C(\delta + |W_\delta^R|_{N/2}) + \int_{\mathbb{R}^N} (\Gamma_\delta^R * |f_n g_n|) |w h_n|.
\]

First we claim that
\[
\int_{\mathbb{R}^N} (\Gamma_\delta^R * |f_n g_n|) |w h_n| \to 0 \quad \text{as } n \to +\infty.
\]

Indeed, for any \( \epsilon > 0 \) we fix \( R_1 > 0 \) such that \( \|w_{B_1} \|_2 < \epsilon \), where \( B_1 = B_{R_1} \).
We define also \( R_2 = R_1 + R \) and \( B_2 = B_{R_2} \) so that for any \( y \in B_1 \) and \( z \in \mathbb{R}^N \setminus B_2 \),
we have \( |z - y| \geq R \) and hence \( \Gamma_\delta^R(z - y) = 0 \).

Now we estimate the term as follows:
\[
\int_{\mathbb{R}^N} (\Gamma_\delta^R * |f_n g_n|) |w h_n| = \int_{B_1} (\Gamma_\delta^R * (\|B_2|f_n g_n|)) |w h_n| + \int_{\mathbb{R}^N \setminus B_1} (\Gamma_\delta^R * |f_n g_n|) |w h_n|
\leq R \|B_2 f_n g_n\|_1 \|B_1 w h_n\|_1 + |\Gamma_\delta^R * (f_n g_n)|_\infty \|B_{N/2} h_n\|_2 \|B_2 w\|_2
\leq R \|g_n\|_2 \|h_n\|_1 \left( R \|B_2 f_n\|_2 \|w\|_2 + R \|f_n\|_2 \|B_2 w\|_2 \right)
\leq C R \left( \|B_2 f_n\|_2 + \|B_2 w\|_2 \right).
\]

Since \( f_n \to 0 \) as \( n \to +\infty \) in \( L^2(B_2) \), the claim is proved.

We conclude the proof of the lemma letting first \( n \to +\infty \), then \( R \to +\infty \) and finally \( \delta \to 0 \) in (4.1). \( \square \)
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