
Abstract. — In this note, we consider blow-up for solutions of the SU(3) Toda system on a compact surface $\Sigma$. In particular, we give a complete proof of the compactness result stated by Jost, Lin and Wang in [11] and we extend it to the case of singularities. This is a necessary tool to find solutions through variational methods.

Key words: Toda system, compactness of solutions, blow-up analysis, mass quantization

Mathematics Subject Classification: 35B44, 35J47, 35J60

1. Introduction

Let $(\Sigma, g)$ be a smooth, compact Riemannian surface. We consider the SU(3) Toda system on $\Sigma$:

\[ -\Delta u_i = \sum_{j=1}^{2} a_{ij} \rho_j \left( \frac{V_j e^{u_j}}{\int_{\Sigma} V_j e^{u_j} \, dv_g} - \frac{1}{|\Sigma|} \right) - 4\pi \sum_{j=1}^{l} \zeta_{ij} \left( \delta_{pj} - \frac{1}{|\Sigma|} \right) \quad i = 1, 2 \tag{1} \]

with $\rho_i > 0$, $0 < V_i \in C^\infty(\Sigma)$, $\zeta_{ij} > -1$, $p_j \in \Sigma$ given and

\[ A = (a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \]

is the SU(3) Cartan matrix.

The Toda system is widely studied in both geometry (description of holomorphic curves in $\mathbb{C}P^N$, see e.g. [4, 6, 8]) and mathematical physics (non-abelian Chern-Simons vortices theory, see [10, 18, 19]).

In the regular case, Jost, Lin and Wang [11] proved the following important mass-quantization result for sequences of solutions of (1).

Theorem 1.1. Suppose $\zeta_{ij} = 0$ for any $i, j$ and let $u_n = (u_{1,n}, u_{2,n})$ be a sequence of solutions of (1) with $\rho_i = \rho_{i,n}$. Define, for $x \in \Sigma$, $\sigma_1(x)$, $\sigma_2(x)$ as

\[ \sigma_i(x) := \lim_{r \to 0} \lim_{n \to +\infty} \rho_{i,n} \frac{\int_{B_r(x)} V_i e^{u_{i,n}} \, dv_g}{\int_{\Sigma} V_i e^{u_{i,n}} \, dv_g}. \tag{2} \]

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Then,

(3) \((\sigma_1(x), \sigma_2(x)) \in \{(0, 0), (0, 4\pi), (4\pi, 0), (4\pi, 8\pi), (8\pi, 4\pi), (8\pi, 8\pi)\}\).

In the same paper, the authors state that Theorem 1.1 immediately implies the following compactness result.

**Theorem 1.2.** Suppose \(x_{ij} = 0 \) for any \(i, j\) and let \(K_1, K_2\) be compact subsets of \(\mathbb{R}^+ \setminus 4\pi \mathbb{N}\). Then, the space of solutions of (1) with \(\rho_i \in K_i\) satisfying \(\int_\Sigma u_i \, dv_g = 0\) is compact in \(H^1(\Sigma)\).

Theorem 1.2 is a necessary step to find solutions of (1) by variational methods, as was done in [2, 16, 17].

Although Theorem 1.2 has been widely used, it was not explicitly proved how it follows from Theorem 1.1. Recently, in [13], a proof was given in the case \(\rho_1 < 8\pi\).

The purpose of this note is to give a complete proof of Theorem 1.2, extending it to the singular case as well. Actually, the proof follows quite directly from [7].

In the presence of singularities, that is when we allow the \(x_{ij}\) to be non-zero, it is convenient to write the system (1) in an equivalent form through the following change of variables:

\[
\begin{align*}
\tilde{u}_i &\rightarrow u_i + 4\pi \sum_{j=1}^{l} x_{ij} G_{p_j} \\
&\text{where } G_p \text{ solves } \begin{cases} -\Delta G_p = \delta_p - \frac{1}{|\Sigma|} \\
\int_{\Sigma} G_p \, dv_g = 0 \end{cases}.
\end{align*}
\]

The new \(u_i\)’s solve

(4) \(-\Delta u_i = \sum_{j=1}^{2} a_{ij} \rho_j \left( \frac{\tilde{V}_j e^{u_j}}{\int_{\Sigma} \tilde{V}_j e^{u_j} \, dv_g} - \frac{1}{|\Sigma|} \right) \quad i = 1, 2.\)

with

\[
\tilde{V}_i = \prod_{j=1}^{l} e^{-4\pi x_{ij} G_{p_j}} V_i \quad \Rightarrow \quad \tilde{V}_i \sim d(\cdot, p_j)^{2x_{ij}} \quad \text{near } p_j.
\]

In this case, we still have an analogue of Theorem 1.1 for the newly defined \(u_i\).

The finiteness of the local blow-up values has been proved in [14].

We will also show how this quantization result implies compactness of solutions outside a closed, zero-measure set of \(\mathbb{R}^+\).

**Theorem 1.3.** There exist two discrete subset \(\Lambda_1, \Lambda_2 \subset \mathbb{R}^+\), depending only on the \(x_{ij}\)’s, such that for any \(K_i \subset \mathbb{R}^+ \setminus \Lambda_i\), the space of solutions of (1) with \(\rho_i \in K_i\) satisfying \(\int_\Sigma u_i \, dv_g = 0\) is compact in \(H^1(\Sigma)\).

As in the regular case, Theorem 1.3 has an important application in the variational analysis of (1), see for instance [2, 1].
2. Proof of the main results

Let us consider a sequence $u_n$ of solutions of (1) with $\rho_i = \rho_{i,n} \to \bar{\rho}_i$ and let us define

\begin{equation}
    w_{i,n} := u_{i,n} - \log \int_{\Sigma} \tilde{V}_i e^{w_{i,n}} dv_g + \log \rho_{i,n},
\end{equation}

which solves

\begin{equation}
\begin{cases}
    -\Delta w_{i,n} = \sum_{j=1}^{2} a_{ij} (\tilde{V}_j e^{w_{j,n}} - \rho_{j,n}) ; \\
    \int_{\Sigma} \tilde{V}_i e^{w_{i,n}} dv_g = \rho_{i,n}
\end{cases}
\end{equation}

moreover,

\[
    \sigma_i(x) = \lim_{r \to 0} \lim_{n \to +\infty} \int_{B_r(x)} \tilde{V}_i e^{w_{i,n}} dv_g.
\]

Let us denote by $S_i$ the blow-up set of $w_{i,n}$:

\[
    S_i := \{ x \in \Sigma : \exists \{x_n\} \subseteq \Sigma, w_{i,n}(x_n) \to +\infty \}.
\]

For $w_{i,n}$ we have a concentration-compactness result from [15, 3]:

**Theorem 2.1.** Up to subsequences, one of the following alternatives holds:

- *(Compactness)* $w_{i,n}$ is bounded in $L^\infty(\Sigma)$ for $i = 1, 2$.
- *(Blow-up)* The blow-up set $S := S_1 \cup S_2$ is non-empty and finite and $\forall i \in \{1, 2\}$ either $w_{i,n}$ is bounded in $L^\infty_{loc}(\Sigma \setminus S)$ or $w_{i,n} \to -\infty$ locally uniformly in $\Sigma \setminus S$.

In addition, if $S_1 \setminus (S_1 \cap S_2) \neq \emptyset$, then $w_{i,n} \to -\infty$ locally uniformly in $\Sigma \setminus S$.

Moreover, denoting by $\mu_i$ the weak limit of the sequence of measures $\tilde{V}_i e^{w_{i,n}}$, one has

\[
    \mu_i = r_i + \sum_{x \in S_i} \sigma_i(x) \delta_x
\]

with $r_i \in L^1(\Sigma) \cap L^\infty_{loc}(\Sigma \setminus S_i)$ and $\sigma_i(x) \geq 2\pi \min\{1, 1 + \zeta_i(x)\}$ for $x \in S_i$, $i = 1, 2$, where

\[
    \zeta_i(x) = \begin{cases} 
        0 & \text{if } x \neq p_j, j = 1, \ldots, l \\
        2x_{ij} & \text{if } x = p_j.
    \end{cases}
\]

Here we want to show that one has $r_i \equiv 0$ for at least one $i \in \{1, 2\}$. It may actually occur that only one of the $r_i$'s is zero, as shown in [9]. Anyway, to prove Theorems 1.2 and 1.3 we only need one between $r_1$ and $r_2$ to be identically zero.
As a first thing, we can show that the profile near blow-up points resembles a combination of Green’s functions:

**Lemma 2.1.** \( w_{i,n} - \overline{w}_{i,n} \to \sum_{j=1}^{2} \sum_{x \in S_j} a_{ij} \sigma_j(x) G_x + s_i \) in \( L^\infty_{\text{loc}}(\Sigma \setminus S) \) and weakly in \( W^{1,q}(\Sigma) \) for any \( q \in (1, 2) \) with \( e^{s_i} \in L^p(\Sigma) \) \( \forall p \geq 1 \).

**Proof.** If \( q \in (1, 2) \)
\[
\int_{\Sigma} \nabla w_{i,n} \cdot \nabla \varphi \, dv_g \leq \|\Delta w_{i,n}\|_{L^1(\Sigma)} \|\varphi\|_{\infty} \leq C \|\varphi\|_{W^{1,q}(\Sigma)}
\]
\( \forall \varphi \in W^{1,q}(\Sigma) \) with \( \int_{\Sigma} \varphi = 0 \), hence one has \( \|\nabla w_{i,n}\|_{L^q(\Sigma)} \leq C \). In particular \( w_{i,n} - \overline{w}_{i,n} \) converges to a function \( w_i \in W^{1,q}(\Sigma) \) weakly in \( W^{1,q}(\Sigma) \) \( \forall q \in (1, 2) \) and, thanks to standard elliptic estimates, we get convergence in \( L^\infty_{\text{loc}}(\Sigma \setminus S) \).

The limit functions \( w_i \) are distributional solutions of
\[
-\Delta w_i = \sum_{j=1}^{2} a_{ij} \left( r_j + \sum_{x \in S_j} \sigma_j(x) \delta_x - \frac{\tilde{p}_j}{|\Sigma|} \right).
\]
In particular \( s_i := w_i - \sum_{j=1}^{2} \sum_{x \in S_j} a_{ij} \sigma_j(x) G_x \) solves
\[
-\Delta s_i = \sum_{j=1}^{2} a_{ij} \left( r_j + \frac{1}{|\Sigma|} \sum_{x \in S_j} \sigma_j(x) - \frac{\tilde{p}_j}{|\Sigma|} \right).
\]
Since \( -\Delta s_i \in L^1(\Sigma) \) we can exploit Remark 2 in [5] to prove that \( e^{s_i} \in L^p(\Sigma) \) \( \forall p \geq 1 \).

The following Lemma shows the main difference between the case of vanishing and non-vanishing residual.

**Lemma 2.2.**

- \( r_i \equiv 0 \Rightarrow \overline{w}_{i,n} \to -\infty \).
- \( r_i \not\equiv 0 \Rightarrow \overline{w}_{i,n} \) is bounded.

**Proof.** First of all, \( \overline{w}_{i,n} \) is bounded from above due to Jensen’s inequality.

Now, take any non-empty open set \( \Omega \subseteq \Sigma \setminus S \).
\[
\int_{\Omega} \tilde{V}_i e^{w_{i,n}} \, dv_g = e^{\overline{w}_{i,n}} \int_{\Omega} \tilde{V}_i e^{w_{i,n} - \overline{w}_{i,n}} \, dv_g
\]
and by Lemma 2.1
\[
\int_{\Omega} \tilde{V}_i e^{w_{i,n} - \overline{w}_{i,n}} \, dv_g \to_{n \to +\infty} \int_{\Omega} \tilde{V}_i e^{\sum_{j=1}^{2} \sum_{x \in S_j} a_{ij} \sigma_j(x) G_x + s_i} \, dv_g \in (0, +\infty).
\]
On the other hand,
\[
\int_{\Omega} \tilde{V}_i e^{u_{i,n}} \, dv_g \xrightarrow{n \to \infty} \mu_i(\Omega) = \int_{\Omega} r_i(x) \, dv_g(x).
\]

If \( r_i \equiv 0 \) one has \( \tilde{w}_{i,n} \to -\infty \). If instead \( r_i \not\equiv 0 \), choosing \( \Omega \) such that \( \int_{\Omega} r_i(x) \, dv_g > 0 \) we must have \( \tilde{w}_{i,n} \) necessarily bounded. \( \square \)

**Remark 2.1.** From the previous two lemmas, we can write \( r_i = \hat{V}_i e^{s_i} \), where
\[
\hat{V}_i := \tilde{V}_i e^{\sum_{j=1}^{p} \alpha_j(x) G_{x}}
\]
satisfies \( \hat{V}_i \sim d(\cdot, x) \frac{2a_i(x)}{x} \) around each \( x \in S_i \), provided \( r_i \not\equiv 0 \).

The key lemma is an extension of Chae-Ohtsuka-Suzuki [7] to the singular case. Basically, it gives necessary conditions on the \( \sigma_j \)'s to have non-vanishing residual.

**Lemma 2.3.** For both \( i = 1, 2 \) we have \( s_i \in W^{2,p}(\Sigma) \) for some \( p > 1 \). Moreover, if \( \sum_{j=1}^{2} a_{ij}\sigma_j(x_0) \geq 4\pi(1 + \alpha_i(x_0)) \) for some \( x_0 \in S_i \), then \( r_i \equiv 0 \).

**Proof.** If both \( r_1 \) and \( r_2 \) are identically zero, then also \( s_1 \) and \( s_2 \) are both identically zero, so there is nothing to prove.

Suppose now \( r_1 \not\equiv 0 \) and \( r_2 \equiv 0 \). In this case,
\[
\begin{cases}
-\Delta s_1 = 2(r_1 + \frac{1}{|\Sigma|} \sum_{x_0 \in S_1} \sigma_1(x_0) - \frac{p_i}{|\Sigma|}) \\
-\Delta s_2 = -(r_1 + \frac{1}{|\Sigma|} \sum_{x_0 \in S_1} \sigma_1(x_0) - \frac{p_i}{|\Sigma|})
\end{cases}
\]

Then, being \( G_s(y) \geq -C \) for all \( x, y \in \Sigma \) with \( x \neq y \), we get
\[
s_1(x) = \int_{\Sigma} G_s(y) 2r_1(y) \, dv_g(y) \geq -2C \int_{\Sigma} r_1 \, dv_g \geq -C'.
\]

Therefore, from the previous remark, around each \( x_0 \in S_1 \) we get
\[
r_1(y) \geq C d(x_0, y) \frac{2a_i(x_0)}{x} \frac{\sum_{j=1}^{2} a_{ij}(x_0)}{x},
\]
so being \( r_1 \in L^1(\Sigma) \), it must be \( \sum_{j=1}^{2} a_{ij}(x_0) < 4\pi(1 + \alpha_i(x_0)) \).

Moreover, being \( e^{q s_i} \in L^1(\Sigma) \) for any \( q \geq 1 \), from Holder’s inequality we get \( r_1 \in L^p(\Sigma) \) for some \( p > 1 \); therefore, standard estimates yield \( s_i \in W^{2,p}(\Sigma) \) for both \( i = 1, 2 \).

Consider now the case of both non-vanishing residuals, which means by Theorem 2.1 \( S_1 = S_2 = S \). In this case,
\[
-\Delta \left( \frac{2s_1 + s_2}{3} \right) = \left( r_1 + \frac{1}{|\Sigma|} \sum_{x_0 \in S_1} \sigma_1(x_0) - \bar{p}_1 \right)
\]
hence, arguing as before, \( \frac{2\sigma_1 + \sigma_2}{3} \geq -C \). Therefore, using the convexity of \( t \to e^t \) we get

\[
C \int_{\Sigma} \min \{ \hat{V}_1, \hat{V}_2 \} \, dv_g \leq \int_{\Sigma} \min \{ \hat{V}_1, \hat{V}_2 \} e^{\frac{2\sigma_1 + \sigma_2}{3}} \, dv_g \\
\leq \frac{2}{3} \int_{\Sigma} \hat{V}_1 e^{\sigma_1} \, dv_g + \frac{1}{3} \int_{\Sigma} \hat{V}_2 e^{\sigma_2} \, dv_g \\
= \frac{2}{3} \int_{\Sigma} r_1 \, dv_g + \frac{1}{3} \int_{\Sigma} r_2 \, dv_g < +\infty.
\]

Therefore, for any \( x_0 \in S \) there exists \( i \in \{1, 2\} \) such that \( \sum_{j=1}^{2} a_{ij} \sigma_j(x_0) < 4\pi(1 + \alpha_i(x_0)) \). Fix \( x_0 \) and suppose, without loss of generality, that this is true for \( i = 1 \). This implies that \( r_1 \in L^p(B_r(x_0)) \) for small \( r \), so for \( x \in B_{\frac{r}{2}}(x_0) \) we have

\[
s_2(x) = \int_{\Sigma} G_{x}(y) 2r_2(y) \, dv_g(y) - \int_{B_r(x_0)} G_{x}(y)r_1(y) \, dv_g(y) \\
- \int_{\Sigma \setminus B_r(x_0)} G_{x}(y)r_1(y) \, dv_g(y) \\
\geq -C - \sup_{z \in \Sigma} \| G_z \|_{L^{p'}(\Sigma)} \| r_1 \|_{L^p(B_r(x_0))} - \sup_{z \in B_{\frac{r}{2}}(x_0)} \| G_z \|_{L^\infty(\Sigma \setminus B_r(x_0))} \| r_1 \|_{L^1(\Sigma)} \\
\geq -C'.
\]

Therefore, arguing as before, we must have \( \sum_{j=1}^{2} a_{2j} \sigma_j(x_0) < 4\pi(1 + \alpha_2(x_0)) \) and \( r_2 \in L^p(B_{\frac{r}{2}}(x_0)) \). This implies \( -\Delta s_i \in L^p(B_{\frac{r}{2}}(x_0)) \) for both \( i \)'s. Hence, being \( x_0 \) arbitrary and \( -\Delta s_i \in L^p_{loc}(\Sigma \setminus S) \), by elliptic estimates the proof is complete. □

From Lemmas 2.1 and 2.3 we can deduce, through a Pohozaev identity, the following information about the local blow-up values. This was explicitly done in [12, 14].

**Lemma 2.4.** If \( x_0 \in S \) then

\[
\sigma_1^2(x_0) + \sigma_2^2(x_0) - \sigma_1(x_0)\sigma_2(x_0) = 4\pi(1 + \alpha_1(x_0))\sigma_1(x_0) + 4\pi(1 + \alpha_2(x_0))\sigma_2(x_0).
\]

**Lemma 2.5.** If \( x_0 \in S_1 \cap S_2 \) then there exists \( i \) such that \( \sum_{j=1}^{2} a_{ij} \sigma_j(x_0) \geq 4\pi(1 + \alpha_i(x_0)) \).

**Proof.** Suppose the statement is not true. Then, by Lemmas 2.3 and 2.4, we would have
which has no solution between positive $\sigma_1(x_0)$, $\sigma_2(x_0)$.

In fact, by multiplying the first equation by $\frac{\sigma_1(x_0)}{2}$ and the second by $\frac{\sigma_2(x_0)}{2}$ and summing, we get

$$\sigma_1^2(x_0) + \sigma_2^2(x_0) - \sigma_1(x_0)\sigma_2(x_0) < 2\pi(1 + \alpha_1(x_0))\sigma_1(x_0) + 2\pi(1 + \alpha_2(x_0))\sigma_2(x_0),$$

which contradicts the third equation.

The scenario is described by the picture.

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**Corollary 2.1.** Let $w_n$ be a sequence of solutions of (6). If $S \neq \emptyset$ then either $r_1 \equiv 0$ or $r_2 \equiv 0$. In particular there exists $i \in \{1, 2\}$ such that $\bar{p}_i = \sum_{x \in S_i} \sigma_i(x)$. 

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Figure 1. The algebraic conditions (7) satisfied by $\sigma_1(x_0)$, $\sigma_2(x_0)$.
Proof of Theorems 1.2 and 1.3. Let $u_n$ be a sequence of solutions of (1) with $\rho_i = \rho_{i,n} \to \bar{\rho}_i$ and $\int_{\Sigma} u_{1,n} \, dv_g = \int_{\Sigma} u_{2,n} \, dv_g = 0$ and let $w_{i,n}$ be defined by (5).

If both $w_{1,n}$ and $w_{2,n}$ are bounded from above, then by standard estimates $u_n$ is bounded in $W^{2,p}(\Sigma)$, hence is compact in $H^1(\Sigma)$.

Otherwise, from Corollary 2.1 we must have $\bar{\rho}_i = \sum_{x \in S_i} \sigma_i(x)$ for some $i \in \{1, 2\}$. In the regular case, from Theorem 1.1 follows that $\rho_i$ must be an integer multiple of $4\pi$, hence the proof of Theorem 1.2 is complete.

In the singular case, local blow-up values at regular points are still defined by (3), whereas for any $j = 1, \ldots, l$ there exists a finite $\Gamma_j$ such that $(\sigma_1(p_j), \sigma_2(p_j)) \in \Gamma_j$. Therefore, it must hold

$$\rho_i \in \Lambda_i := \left\{ 4\pi k + \sum_{j=1}^{l} n_j \sigma_j, \ k \in \mathbb{N}, \ n_j \in \{0, 1\}, \ \sigma_j \in \Pi_i(\Gamma_j) \right\},$$

where $\Pi_i$ is the projection on the $i^{th}$ component; being $\Lambda_i$ discrete we can also conclude the proof of Theorem 1.3.

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References


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