On the existence of dense ideals in LMC*-algebras

M. Fritzschke

In der Arbeit wird folgender Satz bewiesen: Besitzt eine LMC*-Algebra (mit Einselement) ein unbeschränktes Element, so gibt es in ihr ein dichtes Ideal.

In this paper we prove the following proposition: The existence of an unbounded element in an LMC*-algebra (with unity) implies the existence of a dense ideal in this algebra.

The concept of LMC*-algebras is a natural generalization of the concept of C*-algebras. LMC*-algebras were investigated in [2, 3, 5—7]. Many of the results on C*-algebras can be extended to the larger class of LMC*-algebras, nevertheless there are also essential differences between these classes of algebras. One of these is the existence of dense ideals in LMC*-algebras. In a C*-algebra with unity every maximal left (right, two-sided) ideal is automatically closed. This follows from the well known result that the closure of a proper regular ideal in a Banach algebra is again a proper ideal. Żelasko proved that in commutative lmc-algebras (locally multiplicatively-convex algebras) the existence of an unbounded element implies the existence of a dense ideal (of infinite codimension) [8].

In [2] we conjectured that the following theorem holds.

Theorem 1: The existence of an unbounded element in an LMC*-algebra (with unity) implies the existence of a dense ideal in this algebra.

For commutative LMC*-algebras this proposition is obviously a special case of the result of Żelasko; thanks to the isomorphy of such algebras to algebras C(X) of all continuous complex-valued functions on a topological space X (see Theorem 3) the structure of maximal (closed and dense) ideals is known [4]. In this paper we will prove the conjectured theorem. First of all we recall the definition and some basic properties of LMC*-algebras.

Definition 2 [6]: An LMC*-algebra is a complete locally convex *-algebra \( \mathcal{A}[\tau] \), whose topology \( \tau \) can be given by a system of seminorms \( p \) with the following properties:

(i) \( p(xy) \leq p(x) p(y) \) and

(ii) \( p(x^*x) = p(x)^2 \) \( \forall x, y \in \mathcal{A} \).

Such seminorms are called C*-seminorms.

We will always assume in this paper the existence of a unity \( e \) in an LMC*-algebra \( \mathcal{A}[\tau] \). For C*-seminorms \( p \) we have \( p(x^*) = p(x) \) and \( p(e) = 1 \). \( \Gamma^*_\text{max} \) denotes the set of all \( \tau \)-continuous C*-seminorms on \( \mathcal{A}[\tau] \), it is an upwards directed system under the order relation \( p \leq q \) iff \( p(x) \leq q(x) \) \( \forall x \in \mathcal{A} \). By \( \Gamma^* \) we denote a directed subsystem of \( \Gamma^*_\text{max} \), which is generating yet the topology \( \tau \).

6 Analysis Bd. 1, Heft 3 (1982)
The following result parallel to the Gelfand-Neumark-Theorem for C*-algebras is very useful for our considerations.

**Theorem 3 [6]:** For a commutative LMC*-algebra $\mathcal{A}[\tau]$ there exists a completely regular topological space $X$ such that

(i) $\mathcal{A}[\tau]$ is algebraically and topologically isomorphic to the Algebra $C(X)$ of all continuous complex-valued functions on $X$ equipped with a topology $\tau_0$ weaker than the compact-open topology.

(We will write $\mathcal{A}[\tau] \cong C(X)[\tau_0]$.)

(ii) Under this isomorphism $I$ the seminorms $p$ were converted into suprema on compact subsets of $X$, that means for $p \in \Gamma^\tau$ there is a compact subset $K_p$ of $X$ such that

$$p(x) = p_{K_p}(I(x)) = \sup_{t \in K_p} |x(t)| \quad \text{and} \quad \bigcup_{p \in \Gamma^\tau} K_p = X.$$ 

**Remark:** The image of an element of the algebra under $I$ we always denote by the same letter joining the argument.

The set $\mathcal{A}_b = \left\{ x \in \mathcal{A} \mid \sup_{p \in \Gamma^\tau} p(x) < \infty \right\}$ is called the bounded part of $\mathcal{A}$. Hence, unbounded elements are the elements of $\mathcal{A} \setminus \mathcal{A}_b$, $\mathcal{A}_b$ is $\tau$-dense in $\mathcal{A}$ and a C*-algebra under the norm $||x|| = \sup_{p \in \Gamma^\tau} p(x)$ [6].

The set $\mathcal{P}(\mathcal{A}) = \left\{ \sum_{\text{finite}} x_i \cdot x_i \mid x_i \in \mathcal{A} \right\}$ is called the positive cone of $\mathcal{A}$, it organizes the hermitian part $\mathcal{A}_h = \left\{ x \in \mathcal{A} \mid x = x^* \right\}$ of $\mathcal{A}$ to a partially ordered topological space. We have $\mathcal{P}(\mathcal{A}_b) = \mathcal{A}_b \cap \mathcal{P}(\mathcal{A})$ where $\mathcal{P}(\mathcal{A}_b) = \left\{ \sum_{\text{finite}} y_i \cdot y_i \mid y_i \in \mathcal{A}_b \right\}$ [6] and $\mathcal{P}(\mathcal{A}_b)$ is $\tau$-dense in $\mathcal{P}(\mathcal{A})$, even one can approximate elements of $\mathcal{P}(\mathcal{A})$ by increasing sequences of elements of $\mathcal{P}(\mathcal{A}_b)$. The simple proof of this fact is contained in the proof of our theorem.

A further essential result is that every LMC*-algebra is the projective limit of C*-algebras. For $p \in \Gamma^\tau$, the set $\mathcal{N}_p = \left\{ x \in \mathcal{A} \mid p(x) = 0 \right\}$ is a $\tau$-closed two-sided *-Ideal in $\mathcal{A}$. Let $\pi_p$ be the natural homomorphism of $\mathcal{A}$ on $\mathcal{A}_p = \mathcal{A}/\mathcal{N}_p$, $\mathcal{A}_p$ is a C*-algebra under the norm $||\pi_p(x)||_p = p(x)$ and $\mathcal{A}[\tau] = \lim_{p \in \Gamma^\tau} \text{proj} (\mathcal{A}_p, \parallel \cdot \parallel_p)$ [6].

The following facts about continuous linear functionals are immediately clear. For $f \in \mathcal{A}[\tau]'$ there exists $p \in \Gamma^\tau$ such that $||f(x)|| \leq cp(x) \forall x \in \mathcal{A}$ ($c$ is a positive constant). Then

$$f_p(\pi_p(x)) = f(x) \quad \text{(*)}$$

defines $f_p \in \mathcal{A}_p[[\parallel \cdot \parallel_p]]'$ and conversely, for $f_p \in \mathcal{A}_p[[\parallel \cdot \parallel_p]]'$ we get by (*) an element $f$ of $\mathcal{A}[\tau]'$, continuous with respect to $p$, $f$ is positive iff $f_p$ is positive. Further we have: $f$ is a continuous state iff $f_p$ is a state; $f$ is an extremal continuous state iff $f_p$ is an extremal state [5]. We denote by $S$ (resp. $S_p$) the set of all continuous states of $\mathcal{A}[\tau]$ (resp. $\mathcal{A}_p[[\parallel \cdot \parallel_p]]$), by $\text{ex} S$ (resp. $\text{ex} S_p$) the subsets of extremal states.

We will make use of the following result on the ideal structure of LMC*-algebras.

**Proposition 4 [2]:**

(i) Every maximal closed left ideal $\mathcal{I}$ in an LMC*-algebra $\mathcal{A}[\tau]$ is the left kernel of an extremal continuous state, i.e. $\exists \omega \in \text{ex} S$ such that $\mathcal{I} = \{ x \in \mathcal{A} \mid \omega(x^*x) = 0 \}$.

(ii) Every closed left ideal $\mathcal{I}$ in an LMC*-algebra is the intersection of all maximal closed left ideals containing $\mathcal{I}$. 

Now we prove a lemma on the possibility of extension of continuous states. This result is well known for C*-algebras (see for instance [1], 2.10.1.).
Lemma 5: Let $\mathcal{A}[\tau]$ be an LMC*-algebra, $\mathcal{B}$ a closed subalgebra of $\mathcal{A}$ and $a \in \mathcal{B}$. If $g$ is a continuous state of $\mathcal{B}$, then
(i) $g$ can be extended to a continuous state of $\mathcal{A}$ and
(ii) the extension can be chosen extremal for extremal $g$.

Proof: ad (i): There is a seminorm $p \in \mathcal{P}_N^a$ such that $|g(b)| \leq p(b) \forall b \in \mathcal{B}$. Regard the algebras $\mathcal{A}_p = \mathcal{A}/\mathcal{N}_p$ and $\mathcal{B}_p = \mathcal{B}/\mathcal{B} \cap \mathcal{N}_p$, $\pi_p: \mathcal{B} \to \mathcal{B}_p$ the natural homomorphism. $\mathcal{B}_p$ is a C*-algebra under the norm $\|\pi_p(b)\| = p(b)$ and $\pi_p'(b) \to \pi_p(b)$ is an imbedding of $\mathcal{B}_p$ in $\mathcal{A}_p$ preserving the norm, thus we can regard $\mathcal{B}_p$ as a C*-subalgebra of $\mathcal{A}_p$. $g_p$ is a state of $\mathcal{B}_p$ and so it can be extended to a state $f_p$ of $\mathcal{A}_p$. Define $f$ by (*)&. Then $f$ is a continuous state of $\mathcal{A}$ and for $b \in \mathcal{B}$ we have $f(b) = f_p(\pi_p(b)) = g_p(\pi_p(b)) = g(b)$.

ad (ii): For extremal $g$ $g_p$ is extremal too (Prop. 4). Then one can choose $f_p$ extremal [1] and so $f$ is extremal.

Remark: We cannot directly use extension theorems, because in general $e$ is not an inner point of the positive cone.

We are able now to prove our theorem:

Proof: Let $a \in \mathcal{A}$ be an unbounded element. Without loss of generality we can assume $a \in \mathcal{P}(\mathcal{A})$, since for unbounded $a$ $a^*a$ is unbounded too. Let us regard the commutative closed subalgebra $\mathcal{A}_0[\tau]$ of $\mathcal{A}[\tau]$ generated by $a$ and $e$. We have $\mathcal{A}_0[\tau] \cong \mathcal{C}(X)[\tau_0]$ (Th. 3). Then $a(t) \geq 0 \forall t \in X$ and $a(t)$ is an unbounded function. Set $a_n(t) = \min\{a(t), n\} \in \mathcal{C}(X) \forall n \in \mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers); $a_n = I^{-1}(a_n(t)) \in \mathcal{A}_0$. By Theorem 3 we get $\forall n \in \mathbb{N}$

$$0 \leq a_n < a, \quad a_n \in \mathcal{A}_0 \text{ with } \|a_n\| = n, \quad a_n \leq a_{n+1}$$

and $a = \tau\text{-lim } a_n$. Therefore $0 < a_n \leq ne$ and there is a number $n_0 \in \mathbb{N}$ such that $a_n < ne \forall n \geq n_0$.

In the following we consider only indices $n \geq n_0$. Put $b_n = ne - a_n$. Regarding the functions $b_n(t)$ one finds: $0 < b_n \leq b_{n+1}$. For $F_n = \{t \in X | b_n(t) = 0\}$ we obtain

$$F_n = \emptyset, \quad F_n = X \quad \text{and} \quad F_n \supseteq F_{n+1}. \quad (**)$$

Further, the extremal continuous states of $\mathcal{A}_0$ are the "point functionals" of $\mathcal{C}(X)$, i.e. the states $\omega_n(a) = a(t_0)$ ($t_0 \in X$). These states can be extended to elements of $\text{ex } \mathcal{S}$ by Lemma 5. From this considerations and (**) it follows for the sets

$$R_n = \{\omega \in \text{ex } \mathcal{S} | \omega(b_n) = 0\};$$

$$R_n = \emptyset \forall n \in \mathbb{N} \quad \text{and} \quad R_n = \text{ex } \mathcal{S}; \quad R_{n+1} \subseteq R_n \quad \text{since} \ b_n \leq b_{n+1}. \quad (**)$$

Consider now the sets

$$\mathcal{I}_n = \cap \mathcal{I}_w \quad \text{where} \quad \mathcal{I}_w = \{x \in \mathcal{A} | \omega(x^*x) = 0\}. \quad \text{with } \omega \in \mathcal{A}_0$$

Then, $\mathcal{I}_n$ is a closed left ideal in $\mathcal{A}[\tau]$, $b_n \in \mathcal{I}_n$ (and hence $\mathcal{I}_n = \{0\}$) and $\mathcal{I}_n \subseteq \mathcal{I}_{n+1}$.

Now, let us regard $\mathcal{I} = \bigcup \mathcal{I}_n$. Obviously, $\mathcal{I}$ is a proper left ideal in $\mathcal{A}$. Now we show that $\mathcal{I}$ is dense. Assuming the converse then by Prop. 4 there is an element $\sigma \in \text{ex } \mathcal{S}$ such that $\mathcal{I} \subseteq \mathcal{I}_\sigma = \{x \in \mathcal{A} | \sigma(x^*x) = 0\}$. But $\sigma(b_n) = \sigma(ne - a_n) = n - \sigma(a_n) \geq n - \sigma(a) > 0$ for sufficiently large $n$, hence $b_n \notin \mathcal{I}_\sigma$ for such $n$, and so we have a contradiction. Thus, our proof is complete.

Remarks: 1. The converse of our theorem is not true, i.e. there are LMC*-algebras without unbounded elements containing dense ideals. Such algebras one
can find already in the commutative case. To see this take a pseudocompact and locally compact, but not compact, completely regular space $X$. (An example of such a space one finds in [4], it is a space of ordinals with suitable chosen topology.) We take the algebra $\mathcal{A} = C(X)$ with the topology $\tau$ given by the seminorms $p_K(x) = \sup_{t \in K} |x(t)|$ where $K$ runs over all compact subsets of $X$. $\mathcal{A}[\tau]$ is a LMC*-algebra, the completeness is given by the locally compactness of $X$. Since $X$ is pseudocompact, every continuous function on $X$ is bounded, hence $\mathcal{A}_b = \mathcal{A}$. But there is at least one dense ideal in $\mathcal{A}[\tau]$. To see that take the one-point-compactification $X'$ of $X$ and the ideal of all functions vanishing in a neighbourhood of the adjoint point.

2. In the commutative case, the dense maximal ideals are in one-to-one correspondence to the extremal states of $\mathcal{A}_b(\| \cdot \|)$, not extendable to continuous states of $\mathcal{A}[\tau]$. The question, whether it is so in the general (noncommutative) case, is yet open. The structure of dense maximal ideals was described only in the case that the LMC*-algebra is a direct product of C*-algebras [2].

REFERENCES: