A Non-Differentiability Result
for the Inversion Operator between
Sobolev Spaces

G. Farkas and B. M. Garay

Abstract. The order of differentiability of the inversion operator $J$ between certain spaces or manifolds of distributionally differentiable functions is shown to be sharp in the following sense. Up to a certain order $k$ guaranteed by inverse function arguments, the operator $J$ is everywhere differentiable and $J^{(k)}$ is continuous. On the other hand, $J$ is nowhere $k + 1$ times differentiable.

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1. Introduction

Let $\mathcal{M} = \mathcal{M}_1$ and $\mathcal{M}_2$ be compact $C^\infty$ manifolds without boundary and let $n_i$ be the dimension of $\mathcal{M}_i$ $(i = 1, 2)$; $n = n_1$. For $\frac{n}{2} < s \in \mathbb{N}$ define

$$H^s(\mathcal{M}_1, \mathcal{M}_2) = \left\{ \mathcal{F} : \mathcal{M}_1 \rightarrow \mathcal{M}_2 \mid \text{for any } x \in \mathcal{M}_1, \psi \circ \mathcal{F} \circ \varphi^{-1} \in H^s(\varphi(U), \mathbb{R}^{n_2}) \right\}$$

and note that $H^s(\mathcal{M}_1, \mathcal{M}_2) \subset C(\mathcal{M}_1, \mathcal{M}_2)$. Following Marsden [7] we briefly recall the manifold structure of $H^s(\mathcal{M}_1, \mathcal{M}_2)$. Denote the tangent bundle of $\mathcal{M}_1$ and $\mathcal{M}_2$ by $T\mathcal{M}_1$ and $T\mathcal{M}_2$, respectively. For each $\mathcal{F} \in H^s(\mathcal{M}_1, \mathcal{M}_2)$ define

$$T_{\mathcal{F}}H^s(\mathcal{M}_1, \mathcal{M}_2) = \left\{ \chi \in H^s(\mathcal{M}_1, T\mathcal{M}_2) : \pi_{\mathcal{M}_2} \circ \chi = \mathcal{F} \right\}$$

and

$$\exp_{\mathcal{F}} : T_{\mathcal{F}}H^s(\mathcal{M}_1, \mathcal{M}_2) \rightarrow H^s(\mathcal{M}_1, \mathcal{M}_2), \quad \exp_{\mathcal{F}}(\chi) = \exp \circ \chi$$

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where \( \pi_{\mathcal{M}} : T\mathcal{M}_2 \to \mathcal{M}_2 \) is the canonical projection and \( \exp : T\mathcal{M}_2 \to \mathcal{M}_2 \) is an exponential map. The properties of the exponential map we need in the sequel are that, for any \( y \in \mathcal{M}_2 \), \( \exp_y = \exp |_{T_y\mathcal{M}_2} \) is a \( C^\infty \) diffeomorphism of \( T_y\mathcal{M}_2 \) onto a neighborhood of \( y \) in \( \mathcal{M}_2 \), and \( \exp_y(0) = y \), \( (\exp_y)'(0) = \text{id}_{T_y\mathcal{M}_2} \). It can be shown that \( \{\exp_{\mathcal{F}}\}_{\mathcal{F} \in H^s(\mathcal{M}_1, \mathcal{M}_2)} \) gives rise to a \( C^\infty \) atlas on \( H^s(\mathcal{M}_1, \mathcal{M}_2) \). For any \( \mathcal{F} \in H^s(\mathcal{M}_1, \mathcal{M}_2) \) fixed, \( \exp_{\mathcal{F}} \) is a \( C^\infty \) diffeomorphism of (the Banachable space) \( T\mathcal{F}H^s(\mathcal{M}_1, \mathcal{M}_2) \) onto a neighborhood of \( \mathcal{F} \) in \( H^s(\mathcal{M}_1, \mathcal{M}_2) \), and \( \exp_{\mathcal{F}}(0) = \mathcal{F} \), \( (\exp_{\mathcal{F}})'(0) = \text{id}_{T\mathcal{F}H^s(\mathcal{M}_1, \mathcal{M}_2)} \).

Define

\[
\mathcal{D}^s(\mathcal{M}) = \left\{ \mathcal{F} \in H^s(\mathcal{M}, \mathcal{M}) \mid \mathcal{F} \text{ is one-to-one, orientation preserving and } \mathcal{F}^{-1} \in H^s(\mathcal{M}, \mathcal{M}) \right\}.
\]

From now on we assume that \( s > \frac{n}{2} + 1 \), which guarantees that \( \mathcal{D}^s(\mathcal{M}) \) is open in \( H^s(\mathcal{M}, \mathcal{M}) \) and henceforth \( \mathcal{D}^s(\mathcal{M}) \) is a manifold. Given \( \mathcal{F} \in \mathcal{D}^s(\mathcal{M}) \) arbitrarily, consider the mapping

\[
\mathcal{A}_{\mathcal{F}} : T_{\text{id}}H^s(\mathcal{M}, \mathcal{M}) \to T\mathcal{F}H^s(\mathcal{M}, \mathcal{M}), \quad \mathcal{A}_{\mathcal{F}}(v) = v \circ \mathcal{F}.
\]

For brevity, we write \( \mathcal{X}^s(\mathcal{M}) = T_{\text{id}}H^s(\mathcal{M}, \mathcal{M}) \), the (Banachable) space of \( H^s \) vector fields on \( \mathcal{M} \). The norm on \( \mathcal{X}^s(\mathcal{M}) \) will be denoted by \( | \cdot |_s \). This norm comes from the standard Sobolev norm \( \| \cdot \|_{H^s} = | \cdot |_s \) on coordinate charts. Anticipating Lemma 1/(ii) we see that \( \mathcal{A}_{\mathcal{F}} \) is a \( C^\infty \) diffeomorphism of \( \mathcal{X}^s(\mathcal{M}) \) onto \( T\mathcal{F}H^s(\mathcal{M}, \mathcal{M}) \). Hence \( \mathcal{D}^s(\mathcal{M}) \) is a \( C^\infty \) manifold over \( \mathcal{X}^s(\mathcal{M}) \) and \( \{\exp_{\mathcal{F}} \circ \mathcal{A}_{\mathcal{F}}\}_{\mathcal{F} \in \mathcal{D}^s(\mathcal{M})} \) gives rise to a \( C^\infty \) atlas on \( \mathcal{D}^s(\mathcal{M}) \).

In the following lemma we collect some basic results on \( \mathcal{D}^s(\mathcal{M}) \) (for similar results in the category of continuously differentiable functions we refer to Franks [3] and Irwin [4: Appendix]).

**Lemma 1.**

(i) \( \mathcal{D}^s \) is a group under composition.

(ii) If \( \mathcal{F} \in \mathcal{D}^s(\mathcal{M}) \), then the map

\[
\mathcal{N}(\cdot, \mathcal{F}) : \mathcal{D}^s(\mathcal{M}) \to \mathcal{D}^s(\mathcal{M}), \quad \mathcal{N}(\mathcal{G}, \mathcal{F}) = \mathcal{G} \circ \mathcal{F}
\]

is of class \( C^\infty \) (\( \alpha \)-lemma).

(iii) If \( \mathcal{F} \in \mathcal{D}^s(\mathcal{M}) \), then the map

\[
\mathcal{N}(\mathcal{F}, \cdot) : \mathcal{D}^s(\mathcal{M}) \to \mathcal{D}^s(\mathcal{M}), \quad \mathcal{N}(\mathcal{F}, \mathcal{G}) = \mathcal{F} \circ \mathcal{G}
\]

is continuous (\( \omega \)-lemma global).

(iv) More generally, the composition or Nemytskii operator

\[
\mathcal{N}(\cdot, \cdot) : \mathcal{D}^{s+k}(\mathcal{M}) \times \mathcal{D}^s(\mathcal{M}) \to \mathcal{D}^{s-q}(\mathcal{M}), \quad \mathcal{N}(\mathcal{F}, \mathcal{G}) = \mathcal{F} \circ \mathcal{G}
\]

is of class \( C^{k+q} \), for any \( k \in \mathbb{N} \) and \( q \in [0, s - \frac{n}{2} - 1] \cap \mathbb{N} \).
(v) $D^s(M)$ is a topological group. In particular, the inversion operator

$$J : D^s(M) \to H^s(M, M), \quad J(F) = F^{-1}$$

is continuous.

(vi) More generally, the map

$$J : D^{s+k}(M) \to H^s(M, M), \quad J(F) = F^{-1}$$

is of class $C^k$ for any $k \in \mathbb{N}$.

**Proof.** Proofs for statements (i) - (iii), the case $q = 0$ of statement (iv), and statement (v) can be found in Marsden [7]. The case $q \neq 0$ of statement (iv) can be proved similarly. A simple proof (in the case $k > 0$) of statement (vi) is presented here for convenience. Set

$$R : \mathcal{X}^{s+k}(M) \times \mathcal{X}^s(M) \to \mathcal{X}^s(M)$$

$$R(v, w) = (\exp_{id} \circ A_{id})^{-1} N\left(\exp_F \circ A_F(v), \exp_{F^{-1}} \circ A_{F^{-1}}(w)\right).$$

By a direct calculation, $R(0, 0) = 0$ and

$$([R'_w(0, 0)]w)(x) = \left(\exp^{-1}_x \left(\exp_{F(x)} \circ A_{F^{-1}}(w(F^{-1}(x)))\right)\right)'(x)$$

$$= \text{id}_{T_xM} \cdot F'(F^{-1}(x)) \cdot \text{id}_{T_{F^{-1}(x)}M} \cdot w(F^{-1}(x))$$

$$= F'(F^{-1}(x)) \cdot w(F^{-1}(x))$$

which shows that $R'_w(0, 0)$ is an invertible element of $L(\mathcal{X}^s(M), \mathcal{X}^s(M))$. Differentiability properties of $N$ (given in the $q = 0$ case of statement (iv)) and the implicit function theorem yield a $C^k$ coordinate representation of $J$ at $F$.\[\square\]

The aim of this paper is to show that statement (vi) of Lemma 1 is optimal in the sense that

$$J : D^{s+k}(M) \to H^s(M, M), \quad J(F) = F^{-1}$$

is nowhere $k + 1$ times differentiable. The corresponding result on manifolds of continuously differentiable functions was proved in our earlier paper [2]. Though the main line of argumentation remains the same, several technical modifications are needed throughout and, reflecting the difference between $L^2$ and maximum norms, the construction in the later Lemma 8 for proving the result is new.

**Theorem 1.** Let $s, k \in \mathbb{N}$ and assume that $s > \frac{n}{2} + 2$. Then the operator

$$J : D^{s+k}(M) \to H^s(M, M), \quad J(F) = F^{-1}$$

is nowhere $k + 1$ times differentiable.

The proof is postponed to Section 3 below. Its core is a reductio ad absurdum argument. Assuming $J$ is $k + 1$ times differentiable at some $F_0 \in D^{s+k}(M)$, a formula
for (coordinate representations of) $J^{(k+1)}(F_0)$ is derived. We do this by exploiting the assumption $s > \frac{n}{2} + 2$ to point out that the formula for $J^{(j)}(F_0)$, $j = k$ (obtained in Section 2 by standard methods of combinatorial enumeration) remains then valid for $j = k + 1$, too. The last step is to demonstrate that the formula for $J^{(k+1)}(F_0)$ leads to a contradiction.

We do not know whether the assumption $s > \frac{n}{2} + 2$ can be weakened to $s > \frac{n}{2} + 1$. Topological properties alone do not seem to make inequality $s > \frac{n}{2} + 2$ necessary.

Our basic references for composition, inversion and differentiation are [1, 7, 8, 10]. It is a challenging question to characterize those pairs /scales of Banach/Fréchet spaces/manifolds in which Theorem 1 holds true. It is worth mentioning here that Lemmata 1 - 7 remain valid for Sobolev spaces of fractional order. Thus the extension of Theorem 1 for $s \notin \mathbb{N}$ requires only a construction in proving Lemma 8 that works for any real $s$ satisfying $s > \frac{n}{2} + 2$. (On the other hand, extensions for fractional differentiation seem to be much harder. The $k \notin \mathbb{N}$ version of Lemma 1 seems to be unknown, too.)

Of course, differentiation and differentiability are understood in the sense of Fréchet throughout.

2. Local formulae for the derivatives of $J$

Let $U \neq \emptyset$ be a bounded open subset of $\mathbb{R}^n$. Define

$$D^s_U = \left\{ f \in H^s(U, \mathbb{R}^n) : f \text{ is invertible with a } H^s \text{ inverse} \right\}$$

and consider an $f_0 \in D^s_U$ arbitrarily choosen. Assuming $\partial U$ smooth enough, our standing assumption $s > \frac{n}{2} + 1$ implies that $f_0 \in C^1(U, \mathbb{R}^n)$ with $f_0(U)$ open and $\text{cl}(f_0(U))$ compact in $\mathbb{R}^n$. Finally, let $V$ and $W$ be open subsets of $\mathbb{R}^n$ satisfying

$$\emptyset \neq W \subset \text{cl}(W) \subset f_0(U) \subset \text{cl}(f_0(U)) \subset V.$$

Then there is an $\varepsilon > 0$ with the properties as follows. For any $f \in D^s_U$ with $\|f - f_0\|_{H^s} < \varepsilon$ and $g \in D^s_V$, the composition function $g \circ f$ is defined and belongs to $D^s_U$. We write $g \circ f = N(g, f)$ (the local version of Lemma 1/(iv)). Moreover, for any $f \in D^s_U$ with $\|f - f_0\|_{H^s} < \varepsilon$, there exists a unique $h \in H^s(W, \mathbb{R}^n)$ such that $f \circ h = \text{id}_W$. We write $h = J(f)$ (the local version of Lemma 1/(vi)).

This section is devoted to local properties where domains of the underlying $H^s$ functions play no rule and are omitted. For brevity, we write $D^s$ and $H^s$ instead of $D^s_U$ and $H^s(U, \mathbb{R}^n)$.

Formulae for higher order derivatives of $J$ contain an exponentially growing number of summation terms. In order to write them in a compact form we follow Rybakowski's version [9] of the method of equating coefficients in Taylor expansions for implicitly defined maps and use graphs as summation indices. This approach has been worked out in [2] for the operator $J$ between spaces of continuously differentiable functions. Formulae obtained in [2: Section 2] remain valid in the Sobolev space setting as well. Lemmata 2 - 5 below contain formulae for the higher order derivatives of the operator $J$ between
If a subtree $(C-DA3)$

If the vertex $(C-DA2)$

$v \in (C-DA1)$

$j$ stands for a maximal collection of pairwise non-equivalent Cayley trees on $B\tau$ trees at vertex $v$ as follows. By definition, $\lambda$ is injective. For a fixed $r \in V$ it is required that, given a vertex $v \in V \setminus \{r\}$ arbitrarily, $r$ is the starting point of a directed path that terminates at $v$. It is easily seen that vertex $r$, the root of $\tau$, is uniquely determined. Two Cayley trees $\tau = (V, \lambda, E)$ and $\tilde{\tau} = (\tilde{V}, \tilde{\lambda}, \tilde{E})$ of type $C_j$ are isomorphic if there are bijections $B : V \rightarrow \tilde{V}$ and $b : \{\lambda(v_i)\}_{i=1}^{j} \rightarrow \{\tilde{\lambda}(\tilde{v_i})\}_{i=1}^{j}$ such that $\tilde{\lambda}(B(v)) = b(\lambda(v))$ for each $v \in V$ as well as $(v, w) \in E$ if and only if $(B(v), B(w)) \in \tilde{E}$. From now on, letter $C_j$ stands for a maximal collection of pairwise non-equivalent Cayley trees on $j$ vertices $(j \in \mathbb{N})$. The labelling set is chosen for $\{1, 2, \ldots, j\}$. Consider also

$$R_{1+j}^{-1} = \left\{ \tau \in C_{j+1} : \text{root } r \text{ labelled by } j+1 \right\},$$

the set of Cayley trees on $1 + j$ vertices with a fixed (labelling of the) root.

Consider a Cayley tree $\tau = (V, \lambda, E) \in C_j$ with $j \leq k$ and let $h_1, \ldots, h_j \in (D^{s+k})^j$. Labelling $\lambda$ gives rise to a differential assignment according to the rules as follow:

(C-DA1) If the root $r$ is of degree $d$, then the differential monomial $h^{(d)}_{\lambda(r)}$ is assigned to $r$.

(C-DA2) If the vertex $v \in V \setminus \{v\}$ is of degree $d$, then the differential monomial $h^{(d-1)}_{\lambda(v)}$ is assigned to $v$.

In particular, vertices of degree one are associated with $h_l$ for some $l \in \{1, 2, \ldots, j\}$ as above. This gives the possibility of assigning differential expressions to subtrees via the inductive bracket rules.

(C-DA3) If a subtree $\tau'$ with root $v \neq r$ is chosen in such a way that the components $\tau_1, \ldots, \tau_{d-1}$ of the forest $\tau' \setminus \{v\} \cup \{\text{adjoint edges}\}$ are already associated with the differential expressions $E_1, \ldots, E_{d-1}$, then the differential expression

$$[h^{(d-1)}_{\lambda(v)}](E_1, \ldots, E_{d-1})$$
is assigned to $\tau'$.

(\textbf{C-DA4}) If components $\tau_1, \ldots, \tau_d$ of a forest $\tau \setminus \{r \text{ plus adjacent edges}\}$ are already associated with the differential expressions $E_1, \ldots, E_d$, then the differential expression

$$[d_{\tau}^C(J; \text{id})](h_1, \ldots, h_j) = [h_{\lambda(r)}^{(d)}](E_1, \ldots, E_d)$$

is assigned to the tree $\tau$ itself.

Now we have

\textbf{Lemma 2.} Consider the operator $J : D^{s+k} \rightarrow H^s$, $J(f) = f^{-1}$. Then

$$[J^{(j)}(\text{id})](h_1, \ldots, h_j) = (-1)^j \sum_{\tau \in \mathcal{R}_j} [d_{\tau}^C(J; \text{id})](h_1, \ldots, h_j)$$

whenever $(h_1, \ldots, h_j) \in (D^{s+k})^j$ ($j = 0, 1, \ldots, k$).

Similarly, consider a rooted tree $\tau = (V, \lambda, E) \in \mathcal{R}_{1+j}$ with $j \leq k$ and let $f \in D^{s+k}$ and $(h_1, \ldots, h_j) \in (D^{s+k})^j$. For brevity, we write $H_i = h_i \circ f^{-1}$ ($i = 1, \ldots, j$). Labelling $\lambda$ gives rise to a differential assignment according to the rules as follow:

(\textbf{R-DA1}) If a root $r$ is of degree $d$, then the differential monomial $(f^{-1})^{(d)}$ is assigned to $r$.

(\textbf{R-DA2}) If a vertex $v \in V \setminus \{r\}$ is of degree $d$, then the differential monomial $H_{\lambda(v)}^{(d-1)}$ is assigned to $v$.

(\textbf{R-DA3}) If a subtree $\tau'$ with root $v \neq r$ is chosen in such a way that the components $\tau_1, \ldots, \tau_{d-1}$ of forest $\tau' \setminus \{v \text{ plus adjacent edges}\}$ are already associated with the differential expressions $E_1, \ldots, E_{d-1}$, then the differential expression

$$[H_{\lambda(v)}^{(d-1)}](E_1, \ldots, E_{d-1})$$

is assigned to $\tau'$.

(\textbf{R-DA4}) If components $\tau_1, \ldots, \tau_d$ of a forest $\tau \setminus \{r \text{ plus adjacent edges}\}$ are already associated with the differential expressions $E_1, \ldots, E_d$, then the differential expression

$$[d_{\tau}^R(J; f)](h_1, \ldots, h_j) = [(f^{-1})^{(d)}](E_1, \ldots, E_d)$$

is assigned to the tree $\tau$ itself.

\textbf{Lemma 3.} Let $f \in D^{s+k}$. Then

$$[J^{(j)}(f)](h_1, \ldots, h_j) = (-1)^j \sum_{\tau \in \mathcal{R}_{1+j}} [d_{\tau}^R(J; f)](h_1, \ldots, h_j)$$

whenever $(h_1, \ldots, h_j) \in (D^{s+k})^j$ ($j = 0, 1, \ldots, k$).

The case of real functions has the peculiarity that both $J^{(j)}(\text{id})$ and $J^{(j)}(f)$ can be written in a more compact form.
Lemma 4. Let \( n = 1 \). Then

\[
[J^{(j)}(\text{id})](h_1, \ldots, h_j) = (-1)^j (h_1 \cdots h_j)^{(j-1)}
\]

for \( j = 1, 2, \ldots, k \).

Lemma 5. Let \( n = 1 \). Then

\[
[J^{(j)}(f)](h_1, \ldots, h_j) = (-1)^j ((f^{-1})' \cdot h_1 \circ f^{-1} \cdots h_j \circ f^{-1})^{(j-1)},
\]

for \( j = 1, 2, \ldots, k \).

3. Proof of Theorem 1

We prove Theorem 1 by means of a series of lemmas.

Lemma 6. Assume that \( \mathcal{J} \) is \( k + 1 \) times differentiable at some \( \mathcal{F}_0 \in \mathcal{D}^{s+k}(\mathcal{M}) \). Then \( \mathcal{F}_0 \in \mathcal{D}^{s+k+1}(\mathcal{M}) \).

Proof. Let \( x_0 \in \mathcal{M} \) be chosen arbitrarily. We will show that \( \mathcal{F}_0^{-1} \) is of class \( H^{s+k+1} \) near \( x_0 \).

By the Whitney embedding theorem we may assume that \( \mathcal{M} \subset \mathbb{R}^{2n+1} \). Moreover, by choosing a suitable embedding we may assume that there is an open neighborhood \( V \) of \( x_0 \) in \( \mathcal{M} \) such that \( V \subset \mathbb{R}^n \subset \mathbb{R}^{2n+1} \) and that our atlas on \( \mathcal{M} \) contains the special charts \((\text{id}_U, U)\) and \((\text{id}_{\mathcal{F}_0^{-1}(U)}, \mathcal{F}_0^{-1}(U))\) where \( U \) is an open subset of \( V \) with \( x_0 \in U \subset \text{cl}(U) \subset V \). The exponential map is chosen in such a way that

\[
\exp_y w = w + y \quad \text{whenever } y, w + y \in U
\]

\[
\exp_y^{-1} w = w - y \quad \text{whenever } y, w \in \mathcal{F}_0^{-1}(U).
\]

Consider the coordinate representation of \( \mathcal{J} \) at \( \mathcal{F}_0 \)

\[
\tilde{\mathcal{J}} : \mathcal{X}^{s+k}(\mathcal{M}) \to \mathcal{X}^s(\mathcal{M}), \quad \tilde{\mathcal{J}}(v) = (\exp_{\mathcal{F}_0^{-1}} \circ \mathcal{A}_{\mathcal{F}_0^{-1}})^{-1} \mathcal{J} \exp_{\mathcal{F}_0} \circ \mathcal{A}_{\mathcal{F}_0}(v).
\]

By the indirect hypothesis \( \tilde{\mathcal{J}} \) is \( k + 1 \) times differentiable at 0.

Next consider

\[
\mathcal{X}^{s+k}_{B, \delta}(\mathcal{M}) := \left\{ v \in \mathcal{X}^{s+k}(\mathcal{M}) : v(x) = 0 \text{ if } x \in \mathcal{M} \setminus B, |v|_{s+k} < \delta \right\}
\]

and the natural chart representation of \( \tilde{\mathcal{J}}|_{\mathcal{X}^{s+k}_{B, \delta}(\mathcal{M})} \), where \( B \) is a fixed compact ball in \( U \) centered at \( x_0 \) and \( \delta \) is a small positive number we specify below. Writing out the details, set

\[
\mathcal{G}(v) = \exp_{\mathcal{F}_0(v)} v(\mathcal{F}_0(\cdot)) \quad \text{for each } v \in \mathcal{X}^{s+k}(\mathcal{M}).
\]
Note that $G(0) = F_0$ and $G(v) \in D^{s+k}(M)$ for $|v|_{s+k}$ small enough. Using continuity we see there is a positive $\delta$ for which $v \in \mathcal{X}^{s+k}_{B,\delta}(M)$ implies

$$
(\tilde{J}(v))(x) = \exp^{-1}(G(v))^{-1}(F_0(x)) \quad \text{whenever } x \in M \\
(G(v))(x) = F_0(x) \\
(\tilde{J}(v))(x) = 0 \\
\text{whenever } x \in M \setminus F_0^{-1}(B) \\
F_0(x) + v(F_0(x)) \in U \\
(G(v))^{-1}(F_0(x)) \in F_0^{-1}(U) \\
\text{whenever } x \in F_0^{-1}(B).
$$

Since

$$
v(F_0(x)) = 0 \\
(G(v))^{-1}(F_0(x)) = x
$$

we conclude that

$$
(G(v))(x) = F_0(x) + v(F_0(x)) \\
(\tilde{J}(v))(x) = (G(v))^{-1}(F_0(x)) - x
$$

or, equivalently,

$$
(\tilde{J}(v))(x) = F_0^{-1}(id + v)^{-1}F_0(x) - x \quad \text{for all } x \in F_0^{-1}(U).
$$

Now we pass from $\mathcal{X}^{s+k}_{B,\delta}(M)$ to (the open $\delta$-ball of $C(B, \mathbb{R}^n) \supset H^{s+k}_0(B, \mathbb{R}^n)$, a Sobolev space with vanishing trace on the boundary. By letting

$$
(\mathcal{K}(v))(x) = F_0^{-1}(id + v)^{-1}F_0(x) - x \quad \text{if } x \in F_0^{-1}(B),
$$

a $C^k$ mapping

$$
\mathcal{K} : H^{s+k}_0(B, \mathbb{R}^n) \hookrightarrow H^s_0(F_0^{-1}(B), \mathbb{R}^n)
$$

is defined (for $|v|_{s+k}$ small enough). The mapping $\mathcal{K}$ decomposes as

$$
\mathcal{K} = L_s \circ K \circ L_{s+k}
$$

where

$$
K : H^{s+k}_0(F_0^{-1}(B), \mathbb{R}^n) \hookrightarrow H^s_0(B, \mathbb{R}^n) \\
(Kw)(x) = (F_0 + w)^{-1}(x) - F_0^{-1}(x) \quad \text{if } x \in B
$$

and

$$
(L_j(v))(x) = v(F_0(x)) \quad \text{whenever } x \in F_0^{-1}(B), \ v \in H^j_0(B, \mathbb{R}^n) \text{ and } j = s, s+k.
$$

Note that $L_s$ and $L_{s+k}$ are linear isomorphisms. In view of $K = L_s^{-1} \circ \mathcal{K} \circ L_{s+k}^{-1}$, the mapping $K$ is of class $C^k$ and the indirect hypothesis implies that $K$ is $k+1$ times differentiable at 0.
Recall that \( s > \frac{n}{2} + 2 \). As a simple corollary of the case \( q = 1 \) of Lemma 1/(iv), the operator

\[
\tilde{K} = (H_0^s(B, \mathbb{R}^n) \xrightarrow{\text{inclusion}} H_0^{s-1}(B, \mathbb{R}^n)) \circ K
\]
is \( k + 1 \) times differentiable. In particular,

\[
\tilde{K}^{(k+1)}(0) = (H_0^s(B, \mathbb{R}^n) \xrightarrow{\text{inclusion}} H_0^{s-1}(B, \mathbb{R}^n)) \circ K^{(k+1)}(0).
\]

Consequently, Lemma 3 applies to \( K^{(k+1)}(0) \) and

\[
[K^{(k+1)}(0)](w_1, \ldots, w_{k+1}) \in H_0^s(B, \mathbb{R}^n)
\]
or, equivalently, by passing to the leading term in (1),

\[
[(\mathcal{F}_0^{-1})^{(k+1)}](w_1 \circ \mathcal{F}_0^{-1}, \ldots, w_{k+1} \circ \mathcal{F}_0^{-1}) \in H_0^s(B, \mathbb{R}^n)
\]

whenever \( w_1, \ldots, w_{k+1} \in H_0^{s+k}(\mathcal{F}_0^{-1}(B), \mathbb{R}^n) \). In fact, all other summation terms in (1) correspond to Cayley graphs for which \( d(r) \), the degree of the root, is less than \( k + 1 \). Correspondingly, the order of each differentiation in those remaining summation terms is not greater than \( k \). Since \( H_0^s(B, \mathbb{R}) \) is closed under pointwise multiplication, the smoothness properties \( \mathcal{F}_0 \in D^{s+k}(\mathcal{M}) \) and \( w_1 \circ \mathcal{F}_0^{-1}, \ldots, w_{k+1} \circ \mathcal{F}_0^{-1} \in H_0^{s+k}(B, \mathbb{R}^n) \) imply coordinatewise that all the remaining summation terms belong to \( H_0^s(B, \mathbb{R}^n) \).

Next we apply a simplified version of the inverse method of Lanza [5, 6] and conclude that \( (\mathcal{F}_0^{-1})^{(k+1)} \) is of class \( H^s \) at interior points of \( B \). The \((k+1)\)-linear symmetric operator \([[(\mathcal{F}_0^{-1})^{(k+1)}]]\) can be reconstructed from a carefully chosen finite collection of points on its graph. The \((k+1)\)-th order mixed partial derivatives of the coordinate functions of \( \mathcal{F}_0^{-1} \) satisfy a system of linear algebraic equations with coefficients in \( H_0^{s+k}(\mathcal{F}_0^{-1}(B), \mathbb{R}) \) and inhomogenities in \( H_0^s(\mathcal{F}_0^{-1}(B), \mathbb{R}) \). Locally, at each interior point of \( B \), a density argument implies that all entries of the coefficient matrix can be made \( C^\infty \) smooth and the determinant can be made non-zero. Cramer’s rule implies that \( (\mathcal{F}_0^{-1})^{(k+1)} \) is of class \( H^s \) and, a fortiori, \( \mathcal{F}_0^{-1} \) is of class \( H^{s+k+1} \) near \( x_0 \).

This holds true for any \( x_0 \in \mathcal{M} \) implying \( \mathcal{F}_0 \in D^{s+k+1}(\mathcal{M}) \).

**Lemma 7.** We may assume that \( \mathcal{F}_0 = \text{id}_\mathcal{M} \).

**Proof.** This is an easy combination of Lemmata 1 and 6. In fact, consider the identity

\[
\mathcal{J}(\mathcal{F}) = \mathcal{N}(\mathcal{F}_0, \mathcal{J}(\mathcal{N}(\mathcal{F}, \mathcal{F}_0))) \quad \text{for each } \mathcal{F} \in D^{s+k}(\mathcal{M}).
\]

The inner composition operator \( \mathcal{N} \) is understood as a mapping of \( D^{s+k}(\mathcal{M}) \times \{\mathcal{F}_0\} \) to \( D^{s+k}(\mathcal{M}) \) and is of class \( C^\infty \) in \( \mathcal{F} \). On both sides, the operator \( \mathcal{J} \) is understood as a mapping of \( D^{s+k}(\mathcal{M}) \) to \( D^s(\mathcal{M}) \) and is of class \( C^k \). However, the outer composition operator \( \mathcal{N} \) is understood as a mapping of \( \{\mathcal{F}_0\} \times D^{s+k}(\mathcal{M}) \) to \( D^s(\mathcal{M}) \) and is of class \( C^{k+1} \). Since \( \mathcal{J} \) is \( k + 1 \) times differentiable at \( \mathcal{F}_0 \), it follows that each side of (3) is \( k + 1 \) times differentiable at \( \mathcal{F} = \text{id}_\mathcal{M} \).
Lemma 8. The operator
\[ J : D^{s+k}(M) \to H^s(M, M) \]
is not \( k + 1 \) times differentiable at \( \text{id}_M \).

Proof. It is enough to show that \( K \), with the special choice \( F_0 = \text{id}_M \), is not \( k + 1 \) times differentiable at 0. By (2), formula (1) applies to \( K^{(k+1)}(0) \) just as to all preceding derivatives \( K^{(j)}(w) \). We distinguish two cases according \( k = 0 \) or \( k \neq 0 \). There is no loss of generality in assuming that \( x_0 = 0 \) and \( B = \{ x \in \mathbb{R}^n : |x| \leq 4n \} \).

The Case \( k = 0 \): Thus \( K'(0)w = -w \) for \( w \in H^s_0(B, \mathbb{R}^n) \) and, by definition of the derivative as that of a multilinear mapping,
\[
| (\text{id} + w)^{-1} - \text{id} + w |_s = o(|w|_s)
\]
where (equivalently to the standard norm calculated on the basis of mixed partial derivatives)
\[
|w|_s = \left( \sum_{i=0}^{s} \int_B \| w^{(i)}(x) \|_{L((\mathbb{R}^n)^i, \mathbb{R}^n)} dx \right)^{\frac{1}{2}}.
\]
Using (2) again, we have
\[
| (\text{id} + w)^{-1} - \text{id} + w |_{s-1} = o(|w|_s).
\]
We arrive thus at a contradiction to (4) if we construct a sequence \( \{W_l\} \subset H^s_0(B, \mathbb{R}^n) \) for which \( |W_l|_s = O(a_l) \) and
\[
\left\| ( (\text{id} + W_l)^{-1} - \text{id} + W_l )^{(s)} \right\|_{L_2(B, L((\mathbb{R}^n)^s, \mathbb{R}^n))} \geq c_1 \cdot a_l
\]
where \( \{a_l\} \) is some positive zero sequence and \( c_1 \) is a positive constant. The construction of \( W_l \) can be reduced to the case \( n = 1 \). In fact, it is enough to construct a sequence \( \{w_l\} \subset H^s_0([-4, 4], \mathbb{R}) \) satisfying
\[
|w_l|_s = O(a_l)
\]
and, with some positive constant \( c_2 \),
\[
\left\| ( (\text{id} + w_l)^{-1} - \text{id} + w_l )^{(s)} \right\|_{L_2([-1, 1], L(\mathbb{R}^n, \mathbb{R}))} \geq c_2 \cdot a_l.
\]
Having done this, we can simply take \( w_l = 0 \) outside \([-4, 4]\) and set
\[
W_l(x) = \mu(x) \cdot (w_l(x_1), w_l(x_2), \ldots, w_l(x_n))
\]
for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) where \( \mu : \mathbb{R}^n \to [0, 1] \) is a \( C^\infty \) function with
\[
\mu(x) = \begin{cases} 
1 & \text{if } x \in \frac{1}{2}B \\
0 & \text{if } x \notin \frac{2}{3}B.
\end{cases}
\]
For \( l \in \mathbb{N} \) and \( x \in [-4, 4] \), we define

\[
\alpha_l(x) = \begin{cases} 1 & \text{if } x \in \left(\frac{i}{l}, \frac{i+1}{l}\right) \quad (i = -l + 2t, t = 0, 1, \ldots, l - 1) \\ 0 & \text{otherwise} \end{cases}
\]

\[
\beta_l(x) = \int_{-2}^{x} \int_{-21}^{t_1} \cdots \int_{-2^{s-1}}^{t_s} \alpha_l(t_s) \, dt_s \cdots dt_2 \, dt_1
\]

\[
\gamma_l(x) = \nu(x) \cdot \beta_l(x) \quad (\nu : [-4, 4] \to [0, 1] \text{ of } C^\infty \text{-- type}, \nu(x) = \begin{cases} 1 & \text{if } |x| \leq 3 \\ 0 & \text{if } |x| \geq \frac{7}{2} \end{cases})
\]

By construction, \( \gamma_l \in H^{s}_{0}([-4, 4], \mathbb{R}) \). The \( s \)-th derivative of \( \gamma_l \) cannot be defined in the classical but only in the distributional sense. Nevertheless, the \( L_2 \)-representation of \( \gamma^{(s)}_l \) can be computed via the pointwise Leibniz rule of differentiation (valid for products of \( C^\infty_0 \) and \( H^s \) functions) and we are justified in writing

\[
\gamma^{(s)}_l(x) = \sum_{j=0}^{s-1} \binom{s}{j} \nu^{(s-j)}(x) \beta^{(j)}_l(x) + \nu(x)\alpha_l(x) \quad (x \in [-4, 4])
\]

and in particular, with a positive constant \( c_3 \),

\[
|\gamma^{(j)}_l(x)| \leq c_3 \quad (x \in [-4, 4]; \, j = 0, 1, \ldots, s). \tag{8}
\]

In what follows \( c_4, \ldots, c_{11} \) stay for positive constants that are independent of \( l \) and \( x \) (but may depend on \( s \)). Observe that \( |\gamma_l| \leq c_4 \) and \( c_5 \leq \gamma_l(x) \leq c_6 \) whenever \( |x| \leq \frac{3}{2} \).

Now we are in a position to define \( w_l \). For \( 1 \leq l \in \mathbb{N} \) and \( x \in [-4, 4] \), set

\[
w_l(x) = \frac{\gamma_l(x)}{c_6 l}.
\]

For \( l \) sufficiently large, say \( l > l_0 \), the case \( j = 1 \) of (8) implies \( |w'_l(x)| \leq \frac{1}{2} \) for each \( x \in [-4, 4] \) and thus \( \text{id} + w_l \) is an increasing \( H^s \) self-diffeomorphism of \([-4, 4]\). Furthermore,

\[
x + \frac{c_5}{c_6 l} \leq (\text{id} + w_l)(x) \leq x + \frac{1}{l} \quad \text{whenever } |x| \leq \frac{3}{2}.
\]

By taking the inverse functions, we obtain the crucial inequality

\[
x - \frac{1}{l} \leq (\text{id} + w_l)^{-1}(x) \leq x - \frac{c_5}{c_6 l} \quad \text{whenever } |x| \leq 1. \tag{9}
\]

Choose \( a_l = \frac{1}{l} \) and observe that property (6) is satisfied.

For \( l > l_0 \), define

\[
S_l = \bigcup \left\{ \left[ \frac{i}{l}, \frac{i+1}{l} + \frac{c_5}{c_6 l} \right] \mid i = -l + 2t \text{ with } t = 0, 1, \ldots, l - 1 \right\}.
\]
Since $c_5 < c_6$, $S_l$ is a collection of disjoint intervals in $[-1, 1]$ and thus its measure is equal to $\frac{c_5}{c_6}$. Consequently, the proof of inequality (7) reduces to checking

$$\left| ((\text{id} + w_l)^{-1} - \text{id} + w_l)^{(s)}(x) \right| \geq \frac{c_7}{l}$$
whenever $x \in S_l$. \hfill (10)

We begin by observing that

$$\left| ((\text{id} + w_l)^{-1} - \text{id} + w_l)^{(s)}(x) \right|$$

$$= \left| \left[ 1 + w_l'(\text{id} + w_l)^{-1} \right]^{-1} - 1 + w_l' \right)^{(s-1)}(x) \right|$$

$$= \left| \left[ 1 - w_l'(\text{id} + w_l)^{-1} + (w_l'(\text{id} + w_l)^{-1})^2 - \ldots \right] - 1 + w_l' \right)^{(s-1)}(x) \right|$$

$$\geq \left| ( - w_l'(\text{id} + w_l)^{-1} + w_l' \right)^{(s-1)}(x) \right| - \sum_{k=2}^{\infty} \left| (w_l'(\text{id} + w_l)^{-1})^k \right)^{(s-1)}(x) \right|$$

for each $x \in [-4, 4]$ and $l > l_0$. Since

$$((w_l'(\text{id} + w_l)^{-1})^k)^{(s-1)} = \frac{1}{(c_6 l)^{2k}} \left( (\gamma_l'(\text{id} + w_l)^{-1})^2 \cdot (w_l'(\text{id} + w_l)^{-1} - 1)^{k-2} \right)^{(s-1)}$$

for $2 \leq k \in \mathbb{N}$, the polynomial version of the Leibniz rule plus a repeated use of inequalities (8) and $|w_l'(x)| \leq \frac{1}{2}$ show that

$$\sum_{k=2}^{\infty} \left| ((w_l'(\text{id} + w_l)^{-1})^k)^{(s-1)}(x) \right| \leq \frac{c_8}{l^2} \sum_{k=2}^{\infty} \frac{k^{s-1}}{2^{(k-2)-(s-1)}}.$$ 

Furthermore, by standard manipulations with geometric series,

$$\left| ((\text{id} + w_l)^{-1} - \text{id} + w_l)^{(s)}(x) \right|$$

$$\geq \left| ( - w_l'(\text{id} + w_l)^{-1} + w_l' \right)^{(s-1)}(x) \right| - \frac{c_9}{l^2}$$

$$= \left| ( - w_l''(\text{id} + w_l)^{-1} \cdot [1 + w_l'(\text{id} + w_l)^{-1}]^{-1} + w_l'' \right)^{(s-2)}(x) \right| - \frac{c_9}{l^2}$$

$$\geq \left| ( - w_l''(\text{id} + w_l)^{-1} + w_l'' \right)^{(s-2)}(x) \right| - \frac{c_{10}}{l^2} - \frac{c_9}{l^2}$$

$$\vdots$$

(inductively)

$$\geq \left| - w_l^{(s)}((\text{id} + w_l)^{-1}(x)) + w_l^{(s)}(x) \right| - \frac{c_{11}}{l^2}$$

for each $x \in [-4, 4]$ and $l > l_0$. In particular, property (9) and the identification $\beta_l^{(s)} = \alpha_l$ lead to the inequality

$$\left| ((\text{id} + w_l)^{-1} - \text{id} + w_l)^{(s)}(x) \right|$$

$$\geq \left| - w_l^{(s)}((\text{id} + w_l)^{-1}(x)) + w_l^{(s)}(x) \right| - \frac{c_{11}}{l^2}$$

$$= \frac{1}{c_6 l} \cdot \left| - \alpha_l((\text{id} + w_l)^{-1}(x)) + \alpha_l(x) \right| - \frac{c_{11}}{l^2}$$

$$= \frac{1}{c_6 l} - \frac{c_{11}}{l^2}.$$
whenever $x \in S_l$. For $l$ large enough, inequality (10) follows.

The case $k > 0$: As in the proof of the case $k = 0$ we can assume that $n = 1$. By definition,

$$\sup_{|w_1|_s \cdots |w_k|_s} \left| K^{(k)}(w) - K^{(k)}(0) - K^{(k+1)}(0)w(w_1, \ldots, w_k) \right|_{s-k} = o(|w|_{s+k}) \tag{11}$$

where the supremum is taken over all $0 \neq w_i \in H^{s+k}_0([-4,4], \mathbb{R})$. As in the case $k = 0$ above, (11) reduces to

$$\sup_{|w_1|_s \cdots |w_k|_s} \left\| \left( K^{(k)}(w) - K^{(k)}(0) - K^{(k+1)}(0)w \right)(w_1, \ldots, w_k) \right\|_{L_2[-4,4]} = o(|w|_{s+k}).$$

To arrive at a contradiction, it is enough to construct a sequence $\{z_l\} \subset H^{s+k}_0([-4,4], \mathbb{R})$ and to find non-zero elements $v_1, \ldots, v_{k-1}, q_l \in H^{s+k}_0([-4,4], \mathbb{R})$ such that

$$|z_l|_{s+k} \to 0 \tag{12}$$

but

$$\left\| \left( K^{(k)}(z_l) - K^{(k)}(0) - K^{(k+1)}(0)z_l \right)(v_1, \ldots, v_{k-1}, q_l) \right\|_{L_2[-4,4]} \nrightarrow 0 \tag{13}$$

as $l \to \infty$.

In what follows $d_1, \ldots, d_5$ stay for positive constants that are independent of $l$ and $x$ (but may depend on $k$ and $s$).

For $1 \leq l \in \mathbb{N}$ and $|x| \leq 4$, we set $z_l(x) = \nu(x) \cdot \frac{1+x}{l}$ and observe that, for $l$ sufficiently large, say $l > l_0$, $(\text{id} + z_l)^{-1}$ is a $C^\infty$ self-diffeomorphism of $[-4,4]$,

$$(\text{id} + z_l)^{-1}(x) = \frac{x - \frac{1}{l}}{1 + \frac{1}{l}} \quad \text{if} \ |x| < 1 \quad \text{and} \quad \frac{d_1}{l} \leq |z_l|_{s+k} \leq \frac{d_2}{l}. \tag{14}$$

Thus (12) is satisfied. Similarly, set

$$v_i(x) = \nu(x) \quad \text{whenever} \ |x| \leq 4 \text{ and } i = 1, \ldots, k-1.$$

The definition of $q_l$ requires a little more care. Consider a 3-periodic $C^\infty$ function $Q : \mathbb{R} \to \mathbb{R}$ with the property that $Q(y) = y^{k+s+1}$ for each $y \in [-1,1]$ and set

$$q_l(x) = \nu(x) \cdot l^{-k-s}Q(lx) \quad \text{whenever} \ |x| \leq 4 \text{ and } l > l_0.$$

It is readily checked that $\|q_l^{(j)}\|_{L_2[-4,4]} \to 0$ as $l \to \infty$ for $j = 0, \ldots, k+s-1$ and, using the 3-periodicity of $Q$,

$$\|q_l^{(j)}\|^2_{L_2[-4,4]} \geq \int_{-3}^3 |Q^{(j)}(lx)|^2 dx = \int_{-3}^3 |Q^{(j)}(y)|^2 dy.$$
for \( j = k + s \). Together with an easy upper estimate, we conclude that
\[
d_3 \leq |q_l|_{s+k} \leq d_4.
\] (15)

In view of (14) and (15) we see that (13) is implied by the property
\[
\| (K^{(k)}(z_l) - K^{(k)}(0) - K^{(k+1)}(0)z_l) \|_{L_2[-1,1]} (s) \rightarrow 0
\] (16)
as \( l \rightarrow \infty \).

For each \( x \in [-1,1] \), the particularly simple form of \( z_l \) and of \( v_1, \ldots, v_{k-1} \) implies via Lemmata 4 and 5 that
\[
\left( [K^{(k)}(z_l)](v_1, \ldots, v_{k-1}, q_l) \right)^{(s)} (x) = (-1)^k (\text{id} + z_l)^{-1} \cdot q_l (\text{id} + z_l)^{-1} (x)
\] (k−1+s)
\[
= \frac{(-1)^k}{(1 + \frac{1}{t})^{k+s}} q_l^{(k+s-1)} \left( \frac{x - 1}{1 + \frac{1}{t}} \right)
\] \( = \frac{(-1)^k}{(1 + \frac{1}{t})^{k+s}} \cdot \frac{1}{l} Q^{(k+s-1)} \left( \frac{x - 1}{1 + \frac{1}{t}} \right) \)
and
\[
\left( [K^{(k)}(0)](v_1, \ldots, v_{k-1}, q_l) \right)^{(s)} (x) = (-1)^k q_l^{(k-1+s)} (x) = (-1)^k \frac{1}{l} Q^{(k-1+s)} (lx)
\]
and
\[
\left( [K^{(k+1)}(0)z_l](v_1, \ldots, v_{k-1}, q_l) \right)^{(s)} (x) = (-1)^{k+1} q_l (k+s) (x)
\]
\[
= (-1)^{k+1} \left( \frac{1 + x}{l} q_l^{(k+s)} (x) + \frac{k + s}{l} q_l^{(k+s-1)} (x) \right)
\] \( = (-1)^{k+1} \left( \frac{1 + x}{l} Q^{(k+s)} (lx) + \frac{k + s}{l^2} Q^{(k+s-1)} (lx) \right) \).

It follows immediately that (16) is a direct consequence of the slightly stronger property
\[
\int_{-1}^1 \left| \frac{1}{(1 + \frac{1}{t})^p} \cdot Q^{(p-1)} \left( \frac{lx - 1}{1 + \frac{1}{t}} \right) - Q^{(p-1)} (lx) + (1 + x)Q^{(p)} (lx) \right|^2 \; dx \geq d_5
\] (17)
where \( p = k + s > 3 \) and \( l > l_p = 10^p \).

We fix parameters \( \gamma = \gamma(p) \in \left[ \frac{9}{10}, 1 \right) \) and \( \Gamma = \Gamma(p) \in \left( 0, \frac{1}{10} \right] \) in such a way that the inequality
\[
\frac{\gamma^2}{2^p} - \left( 1 - \gamma + 3 \Gamma + \frac{3}{l_p} \right)^2 - 4 \left( 1 - \gamma + 3 \Gamma + \frac{3}{l_p} \right) \geq \frac{1}{2^p}
\] (18)
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holds true. With $[\Gamma l]$ denoting the integer part of $\Gamma l$, we define

$$T_l = \bigcup \left\{ \left( \frac{3i}{l} + \frac{3i-1}{l^2}, \frac{3i}{l} + \frac{1-\gamma}{l} + \frac{3i-\gamma}{l^2} \right) \mid i = 1, 2, \ldots, [\Gamma l] + 1 \right\} \quad (l > l_p).$$

Observe that $T_l$ is a collection of disjoint intervals in $[0,1]$ and that the measure of $T_l$ is at least $(\lfloor \Gamma l \rfloor + 1)\gamma > \Gamma(1-\gamma)$. It is crucial that $x \in T_l$ is equivalent to

$$3i - 1 < \frac{lx - 1}{1 + \frac{1}{l}} < 3i - \gamma \quad \text{for some} \ i \in \{1, 2, \ldots, [\Gamma l] + 1\}.$$

Consequently, for each $x \in T_l \ (l > l_p)$ we have

$$\left| \frac{1}{(1 + \frac{1}{l})^p} \cdot Q^{(p-1)}\left( \frac{lx - 1}{1 + \frac{1}{l}} \right) - Q^{(p-1)}(lx) + (1 + x)Q^{(p)}(lx) \right|$$

$$\geq \frac{1}{2^p} \left| Q^{(p-1)}\left( \frac{lx - 1}{1 + \frac{1}{l}} \right) \right| - |Q^{(p-1)}(lx)| - 2|Q^{(p)}(lx)|$$

$$\geq \frac{1}{2^p} \frac{(p + 1)!}{2!} \gamma^2 - \frac{(p + 1)!}{2!} \left( \left| 1 - \gamma + \frac{3(\lfloor \Gamma l \rfloor + 1)}{l} \right|^2 - 4 \left| 1 - \gamma + \frac{3(\lfloor \Gamma l \rfloor + 1)}{l} \right| \right)$$

$$\geq \frac{(p + 1)!}{2!} \left( \frac{\gamma^2}{2^p} - \left( 1 - \gamma + 3\Gamma + \frac{3}{l_p} \right)^2 - 4 \left( 1 - \gamma + 3\Gamma + \frac{3}{l_p} \right) \right).$$

As a direct consequence of (18), inequality (17) and, a fortiori, (16) and (13) follow.\[\blacksquare\]

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**References**


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