Generalized Euler-Frobenius Polynomials

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Abstract. An initial value problem for the two-dimensional difference equation $a_{n+1,\nu+1} = a_{n+1,\nu} + (1 - z)a_{n\nu}$ is solved by means of the generating function and their functional equation. Special values of the solution are the well known Euler-Frobenius polynomials.

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Let $k$ be an arbitrary but fixed non-negative integer. We start from the two-dimensional difference equation

$$a_{n+1,\nu+1} = a_{n+1,\nu} + (1 - z)a_{n\nu} \quad (1)$$

for $0 \leq k \leq n$ and $0 \leq \nu \leq n$ with $n, k, \nu \in \mathbb{Z}$, a real parameter $z$, and the initial conditions

$$a_{k0} = a_{k1} = \ldots = a_{kk} = 1 \quad (2)$$

as well as

$$a_{n+1,0} = z \sum_{\nu=0}^{n} a_{n\nu} \quad (3)$$

for $n \geq k$. Obviously, the solutions of this initial value problem are uniquely determined polynomials $a_{n\nu} = a_{n\nu}(z)$ of order $n - k$, in particular

$$a_{k+1,\nu}(z) = \nu + (k + 1 - \nu)z \quad (4)$$

for $\nu = 0, \ldots, k + 1$. The general solution is given by

**Theorem 1:** For $0 \leq k \leq n$ and $0 \leq \nu \leq n$, the difference equation (1) with the initial conditions (2) and (3) has the solution

$$a_{n\nu}(z) = (1 - z)^{n-k+1} \sum_{m=0}^{\infty} \sum_{\mu=0}^{\nu} \binom{n-\nu}{k-\mu} \binom{\nu}{\mu} (m+1)^{\nu-\mu} m^{n+\mu-k-\nu} \quad (5)$$

Note that the binomial coefficients $\binom{n-\nu}{k-\mu}$ vanish for $k < \mu$. For the proof of this theorem we need some preliminaries.

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**Lemma 1:** For \( n \geq k + 1 \), we have

\[
a_{n0}(z) = za_{nn}(z). \tag{6}
\]

**Proof:** By summation, (1) implies

\[
a_{n+1,n+1} - a_{n+1,0} = (1-z) \sum_{\nu=0}^{n} a_{n\nu}
\]

for \( n \geq k \), and by (3) we obtain

\[
za_{n+1,n+1} - za_{n+1,0} = (1-z)a_{n+1,0},
\]

i.e. (6) \( \blacksquare \)

Next, we introduce the generating function of \( a_{n\nu} \) by the formal power series

\[
F(x, y, z) = \sum_{n=k}^{\infty} \sum_{\nu=0}^{n} a_{n\nu}(z)y^\nu x^n. \tag{7}
\]

From (1), we obtain

\[
\sum_{n=k+1}^{\infty} \sum_{\nu=1}^{n} a_{n\nu}(z)y^{\nu-1}x^{n-1} = \sum_{n=k+1}^{\infty} \sum_{\nu=0}^{n-1} a_{n\nu}(z)y^{\nu}x^{n-1} + (1-z)F(x, y, z).
\]

In view of (2), the left-hand side is equal to

\[
\frac{1}{xy} \left( F(x, y, z) - F(x, 0, z) - \sum_{\nu=1}^{k} y^\nu x^k \right),
\]

and the sum on the right-hand side equals to

\[
\frac{1}{x} \left( F(x, y, z) - \sum_{n=k+1}^{\infty} a_{nn}(z)y^n x^n - \sum_{\nu=0}^{k} y^\nu x^k \right),
\]

where (6) implies

\[
\sum_{n=k+1}^{\infty} a_{nn}(z)y^n x^n = \frac{1}{z} \left( F(xy, 0, z) - y^k x^k \right).
\]

Using the abbreviations \( F = F(x, y, z) \) and \( F(z) = F(x, 0, z) \), we obtain the equation

\[
F - F(x) - \sum_{\nu=1}^{k} y^\nu x^k = y \left( F - \frac{1}{z} (F(xy) - y^k x^k) - \sum_{\nu=0}^{k} y^\nu x^k \right) + xy(1-z)F,
\]
and finally, the functional equation

\[(1 - y - xy(1 - z))F = F(x) - \frac{y}{z}F(xy) + y^{k+1}x^k \left( \frac{1}{z} - 1 \right). \tag{8}\]

**Theorem 2:** Equation (8) has the solution

\[F(x, y, z) = (1 - z)x^k \sum_{m=0}^{\infty} z^m x^{k+1} \left( \frac{1}{z} - 1 \right) \tag{9}\]

\[\times \sum_{\mu=0}^{k} \left( 1 - (m + 1)xy(1 - z) \right)^{-\mu-1} \left( 1 - mx(1 - z) \right)^{\mu-k} y^\mu\]

for \(|z| < 1\) and \(x(1 - z) \neq \frac{1}{n}, \ xy(1 - z) \neq \frac{1}{n} \ (n \in \mathbb{N}).\)

**Proof:** In order to solve (8), it is necessary to determine \(F(x)\). The easiest way would be to put \(y = 0\) in (8), however, then we get an identity. Hence, we introduce a new variable \(u\) with \(u(1 - z) \neq \frac{1}{n} \ (u \in \mathbb{N})\), and choose

\[x = \frac{u}{1 - u(1 - z)} \quad \text{and} \quad y = 1 - u(1 - z).\]

Then \(1 - y - xy(1 - z) = 0\), and (8) turns into

\[F(u) = (1 - z)u^k + \frac{z}{1 - u(1 - z)} F\left( \frac{u}{1 - u(1 - z)} \right). \tag{10}\]

By iteration, we find the series

\[F(u) = (1 - z)u^k \sum_{m=0}^{\infty} \frac{z^m}{(1 - m(1 - z))^{k+1}}, \tag{11}\]

which converges for \(|z| < 1\). Since \(F\left( \frac{u}{1 - u(1 - z)} \right) = (1 - z)u^k \sum_{m=0}^{\infty} \frac{z^m (1 - u(1 - z))}{(1 - (m + 1)u(1 - z))^{k+1}}\), we see that (11) is indeed a solution of (10).

Now, the right-hand side of (8) can be written as

\[(1 - z)x^k \left( \sum_{m=0}^{\infty} \frac{z^m}{(1 - mx(1 - z))^{k+1}} - \sum_{m=1}^{\infty} \frac{z^{m-1} y^{k+1}}{(1 - mxy(1 - z))^{k+1}} \right)\]

\[= (1 - z)x^k \sum_{m=0}^{\infty} \frac{(1 - (m + 1)xy(1 - z))^{k+1} - (y - mxy(1 - z))^{k+1}}{(1 - mx(1 - z))(1 - (m + 1)xy(1 - z))^{k+1}} z^m,\]

and after division by \(1 - y - xy(1 - z)\), we easily find (9).
Corollary: For \( k = 0 \), (9) simplifies to

\[
F(x, y, z) = (1 - z) \sum_{m=0}^{\infty} \frac{z^m}{(1 - mx)(1 - (m + 1)x(1 - z))}.
\] (12)

Proof of Theorem 1: By means of binomial series and

\[
(-1)^i \binom{-\mu - 1}{i} = \binom{\mu + i}{i} \quad \text{and} \quad (-1)^j \binom{\mu - k - 1}{j} = \binom{k + j - \mu}{j},
\]

we obtain from (9) the formal power series

\[
F(x, y, z) = \sum_{m=0}^{\infty} z^m \sum_{i=0}^{k} \sum_{j=0}^{\infty} \binom{\mu + i}{i} \times (m + 1)^i \binom{k + j - \mu}{j} m^j (1 - z)^{i+j+1} y^{\mu+i} x^{i+j+k}.
\]

Choosing \( n = i + j + k \) and \( \nu = \mu + i \), we find by comparison with (7) that equation (5) holds.

In particular, for \( \nu = 0 \) we have

\[
a_{n0}(z) = (1 - z)^{n-k+1} \binom{n}{k} \sum_{m=0}^{\infty} m^{n-k} z^m.
\] (13)

For \( n > k \), these functions can be expressed by the Euler-Frobenius polynomials

\[
E_n(z) = (1 - z)^{n+1} \sum_{m=1}^{\infty} m^n z^{m-1} = \sum_{m=0}^{n-1} \sum_{\nu=0}^{m} \binom{n+1}{\nu} (-1)^\nu (m + 1 - \nu)^n z^m
\]

of degree \( n - 1 \) for \( n \geq 1 \) (cf. Chui [2] and Schoenberg [3]), namely

\[
a_{n0}(z) = \binom{n}{k} z E_{n-k}(z).
\]

The polynomial character of (13) implies that the functions (5) are also polynomials, which we call generalized Euler-Frobenius polynomials.

By means of the notation \( D = z \frac{d}{dz} \), the polynomials \( E_n \) have the representation

\[
E_n(z) = \frac{1}{z} (1 - z)^{n+1} D^n (1 - z)^{-1},
\]

which can be generalized in the following way.
Lemma 2: The polynomials (5) have the representation

\[ a_{n\nu}(z) = (1 - z)^{n-k+1} \sum_{j=0}^{n} \binom{n - \nu}{j} \binom{\nu}{n - k - j} D^j(1 + D)^{n-k-j}(1 - z)^{-1}. \]  

Proof: If we use the equations

\[ (1 + D)^{n-k-j} = \sum_{i=0}^{n-k-j} \binom{n - k - j}{i} D^i \quad \text{and} \quad D^{i+j}(1 - z)^{-1} = \sum_{m=0}^{\infty} m^{i+j} z^m \]

and the substitution \( \mu = k + \nu + j - n \), we obtain

\[ \sum_{i=0}^{n-k-j} \binom{n - k - j}{i} m^i = (m + 1)^{\nu - \mu}, \]

and (14) turns into (5) \( \square \)

Application: In [1], there appear (in different notations) the \((\nu + 1) \times (\nu + 1)\)-matrices

\[ Y_\nu(z) = \begin{pmatrix} 1 & \cdots & 1 \\ z & 1 & \cdots \\ \vdots & \ddots & \vdots \\ z & \cdots & z & 1 \end{pmatrix}, \]

the direct sums \( Y_\nu^n(z) = I_{n-\nu} \oplus Y_\nu(z) \) with

\[ I_\mu \oplus Y_\nu = \begin{pmatrix} I_\mu & O^T \\ O & Y_\nu \end{pmatrix}, \]

where \( I_\mu \) is the \( \mu \)-dimensional unit matrix with dummy \( I_0 \), and the products

\[ P_n(z) = Y_1^n(z) \cdots Y_n^n(z) \]  

with \( n \geq 1 \). For clearness, we denote the functions (5) more precisely by \( a_{n\nu}^k(z) \). Then for the \(((n + 1) \times (n + 1))\)-dimensional matrices \( P_n(z) \), we obtain

Lemma 3: The entries of \( P_n(z) \) are the polynomials \( a_{nj}^{n-i}(z) \) \((i, j = 0, \ldots, n)\), where \( i \) is the row index and \( j \) the column index.

Proof: The matrices (16) satisfy the recursion

\[ P_{n+1}(z) = (I_1 \oplus P_n(z)) Y_{n+1}(z). \]

Denoting the entries of \( P_n(z) \) by \( a_{nj}^{n-1} \), this equation implies \( a_{nj+1,j}^{n+1} = 1 \) for \( j = 0, \ldots, n + 1 \), i.e. (2), and

\[ a_{nj+1,j}^{n+1-i} = \sum_{\ell=0}^{j-1} a_{nj}^{n-(i-1)} + z \sum_{\ell=j}^{n} a_{n\ell}^{n-(i-1)} \]
for \( i = 1, \ldots, n + 1 \). For \( j = 0 \), the last equation equals to (3). For \( j = 0, \ldots, n \), we subtract it from

\[
a_{n+1,j+1}^{n+1-i} = \sum_{\ell=0}^{j} a_{n\ell}^{n+1-i} + z \sum_{\ell=j+1}^{n} a_{n\ell}^{n+1-i}
\]

and obtain

\[
a_{n+1,j+1}^{n+1-i} = a_{n+1,j+1}^{n+1-i} + (1 - z) a_{n,j}^{n+1-i},
\]

i.e. (1). In view of

\[
Y_1(z) = \begin{pmatrix} 1 & 1 \\ z & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ z & 1 \end{pmatrix}
\]

or already \( P_0(z) = 1 \), the lemma is proved by induction.

Further examples are

\[
P_2(z) = \begin{pmatrix} 1 & 1 & 1 \\ 2z & z+1 & 2 \\ z^2 + z & 2z & z+1 \end{pmatrix},
\]

\[
P_3(z) = \begin{pmatrix} 1 & 1 & 1 \\ 3z & 2z + 1 & z+2 \\ 3z^2 + 3z & z^2 + 5z & 5z + 1 \\ z^3 + 4z^2 + z & 4z^2 + 2z & 2z^2 + 4z \end{pmatrix}.
\]

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References


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