Analyticity of some Kernels

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Abstract. The object here is to prove that the series \( \sum_{n \in \mathbb{N}} \lambda_n \mathcal{H}_n(x) \mathcal{H}_n(y) \) is analytic if the sequence \( (\lambda_n)_{n \in \mathbb{N}} \) is such that there exist constants \( \rho > 2 \) and \( C > 0 \) with \( |\lambda_n| \leq \frac{C}{\rho^n} \) for all \( n \in \mathbb{N} \). This is important in the study of eigenvalues of integral operators in relation to their kernels and is a partial affirmative response to a question raised by A.L. Brown.

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1. Introduction

The object of this paper is to prove a theorem, which is a partial affirmative answer to the question asked A.L. Brown whether \( \sum_{n \in \mathbb{N}} \lambda_n \mathcal{H}_n(x) \mathcal{H}_n(y) \) is analytic for all sequences \( (\lambda_n)_{n \in \mathbb{N}} \in \Gamma^- \) where \( \Gamma^- \) is some set of sequences defined below. This question is important in the study of eigenvalues of integral operators in relation to their kernels.

We prove below that the series \( \sum_{n \in \mathbb{N}} \lambda_n \mathcal{H}_n(x) \mathcal{H}_n(y) \) is analytic for some sequences \( (\lambda_n)_{n \in \mathbb{N}} \) forming a subclass of the sequence space \( \Gamma^- \). Namely, the following theorem will be true.

**Theorem.** Let \( (\lambda_n)_{n \in \mathbb{N}} \) be a sequence such that, for some constants \( \rho > 2 \) and \( C > 0 \), \( |\lambda_n| \leq \frac{C}{\rho^n} \) for all \( n \in \mathbb{N} \). Then the series \( \sum_{n \in \mathbb{N}} \lambda_n \mathcal{H}_n(x) \mathcal{H}_n(y) \) is an analytic function in \( \mathbb{R}^2 \) where \( \mathcal{H}_n \) are the Hermite functions as defined in [2: p. 261].

Before we proceed to prove this theorem we give some preliminaries.

2. Preliminaries

The following definitions of Hermite functions and Hermite polynomials are taken from [2: p. 261]. The \( m \)th Hermite polynomial (\( m \) a non-negative integer) is defined as

\[
H_m(x) = \frac{1}{c_m} e^{2\pi x^2} \frac{d^m}{dx^m} e^{-2\pi x^2} \quad \text{where} \quad c_m = (-1)^m \sqrt{m!} 2^{m-1/4} \pi^{m/2}.
\]
The $m^{\text{th}}$ Hermite function ($m$ a non-negative integer) is defined as

$$
H_m(x) = H_m(x)e^{-x^2}.
$$

The following recurrence formula is well known and can be easily proved using the definitions of Hermite functions:

$$
\sqrt{m}H_m = 2\sqrt{\pi}xH_{m-1} - \sqrt{(m-1)}H_{m-2} \quad \text{for all } m \geq 2.
$$

Consequently, we have for Hermite polynomials the recurrence formula

$$
\sqrt{m}H_m = 2\sqrt{\pi}xH_{m-1} - \sqrt{(m-1)}H_{m-2}, \quad (1)
$$

**Definition 1.** A sequence $(\mu_n)_{n \in \mathbb{N}} \subset C$ is said to be rapidly decreasing if for all $k \in \mathbb{N}$ the inequality

$$
\sup_{n \in \mathbb{N}} n^k |\mu_n| < +\infty
$$

is fulfilled.

It is proved in [2: pp. 261 - 262] that, for a sequence $(a_n)_{n \in \mathbb{N}} \in l^2$, the series

$$
\sum_{n=1}^{m} a_n H_n(x)
$$

converges in $L^2$ as $m \to \infty$ to a function of $S(\mathbb{R})$, the Schwartz class of functions, if and only if $(a_n)_{n \in \mathbb{N}}$ is rapidly decreasing. Moreover, using Sobolev's lemma it can be proved that if the sequence $(a_n)_{n \in \mathbb{N}}$ is rapidly decreasing, then the series

$$
\sum_{n=1}^{m} a_n H_n(x)
$$

converges in $S(\mathbb{R})$ as $m \to \infty$.

In the same way, it can be easily seen that if the sequence $(a_n)_{n \in \mathbb{N}}$ is rapidly decreasing, then the series

$$
\sum_{n=1}^{m} a_n H_n(x)H_n(y)
$$

converges in $S(\mathbb{R}^2)$ as $m \to \infty$.

If $|\lambda_n| \leq \frac{C}{\rho^n}$ for some $\rho > 1$, then it is clear that the sequence $(\lambda_n)_{n \in \mathbb{N}}$ is rapidly decreasing. Hence, the series

$$
\sum_{n=1}^{m} \lambda_n H_n(x)H_n(y)
$$

converges to a function $g \in S(\mathbb{R}^2)$ in the topology of $S(\mathbb{R}^2)$. Hence,

$$
D^\alpha g(x, y) = \sum_{n=1}^{\infty} \lambda_n D^\alpha_1 H_n(x)D^\alpha_2 H_n(y) \quad (2)
$$

for all $\alpha = (\alpha_1, \alpha_2)$ and $(x, y) \in \mathbb{R}^2$. To prove that $g$ is analytic, we make use of the following well-known criterion.

**Criterion.** A function $f \in C^\infty(\mathbb{R}^2)$ is real analytic if and only if for any $R > 0$ there exists a real number $C_R > 0$ such that the inequality

$$
\sup_{(x, y) : x^2 + y^2 \leq R^2} |D^\alpha f(x, y)| \leq C_R^{||\alpha||+1} \alpha!
$$

for all $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2$ non-negative integers is true.
For Proof see, for example, [1: Proposition 1.1.14]

To apply this criterion, we are therefore forced to estimate \( D^a g(x,y) \) in a ball \( B(0,R) \). Because of (2) we in turn are forced to estimate \( D^k \mathcal{H}_n(x) \) for any \( k, n \in \mathbb{N} \) and all \( x \) with \( |x| \leq R \). Since \( \mathcal{H}_n(x) = e^{-\pi x^2} H_n(x) \),

\[
D^k \mathcal{H}_n(x) = \sum_{r=0}^{k} \binom{k}{r} D^r H_n(x) D^{k-r} e^{-\pi x^2}.
\]

Thus, to estimate \( D^k \mathcal{H}_n(x) \), we need estimates for \( D^r H_n(x) \) and \( D^l e^{-\pi x^2} \) for any integers \( r \) and \( l \).

3. Necessary estimates

**Estimating \( D^r H_n \).** Here we shall obtain the necessary estimates for \( D^r H_n \). Remember that \( H_k(x) = e^{\pi x^2} \mathcal{H}_k(x) \). Therefore,

\[
H'_k(x) = 2\pi x e^{\pi x^2} \mathcal{H}_k(x) + e^{\pi x^2} \mathcal{H}'_k(x)
= e^{\pi x^2} (\mathcal{H}'_k(x) + 2\pi x \mathcal{H}_k(x))
= e^{\pi x^2} \tau_+ \mathcal{H}_k(x)
\]

where \( \tau_+ \varphi = \varphi'(x) + 2\pi x \varphi \) for all \( \varphi \in C^1 \). It can be easily proved that \( \tau_+ \mathcal{H}_k = 2\sqrt{\pi k} \mathcal{H}_{k-1} \). Hence,

\[
H'_k(x) = 2\sqrt{\pi k} H_{k-1}(x) \quad \text{for all } k \geq 1.
\]

Thus, by induction, we have for all \( p \leq k \)

\[
D^p H_k(x) = (\sqrt{\pi})^p \sqrt{k(k-1)\cdots(k-p+1)} H_{k-p}(x).
\]

Since \( \mathcal{H}_k \) is a \( k \)th degree polynomial, \( D^p \mathcal{H}_k(x) = 0 \) for all \( p > k \).

**Estimating \( D^r e^{-\pi x^2} \).** By the definition of the \( r \)th Hermite polynomial, \( D^r e^{-2\pi x^2} = c_r H_r(x) e^{-2\pi x^2} \). Let \( y = x/\sqrt{2} \) and \( g(x) = e^{-2\pi x^2} \). Then, \( g(y) = e^{-2\pi y^2} = e^{-\pi y^2} \). We have therefore the equality

\[
\frac{d}{dx} e^{-\pi x^2} = \frac{d}{dx} g(y) m \quad \text{where} \quad \frac{d}{dx} g(y) = \frac{d g(y)}{dy} \frac{dy}{dx} = \frac{1}{\sqrt{2}} \frac{d}{dy} g(y).
\]

Therefore, by induction,

\[
\frac{d^r}{dx^r} g(y) = \frac{d^r}{dy^r} g(y) \cdot \left( \frac{1}{\sqrt{2}} \right)^r
= c_r H_r(y) e^{-2\pi y^2} \left( \frac{1}{\sqrt{2}} \right)^r
= c_r \left( \frac{1}{\sqrt{2}} \right)^r e^{-\pi y^2} H_r \left( \frac{x}{\sqrt{2}} \right).
\]
Thus
\[ D^r e^{-\pi z^2} = c_r \left( \frac{1}{\sqrt{2}} \right)^r e^{-\pi z^2} H_r \left( \frac{x}{\sqrt{2}} \right). \] (5)

From the expressions in (2) and (3) we see that to estimate \( D^k H_n(x) \) and \( D^l e^{-\pi x^2} \) for \( |x| \leq R \), we need to estimate \( H_{n-k}(x) \) and \( H_l(x) \) for \( |x| \leq R \). This is the object of the next proposition. Before we go to that, we give the following definition.

Definition 2. By \( \Gamma^- \) we denote the space of all sequences \( (a_n)_{n \in \mathbb{N}} \) such that, for all \( R > 0 \) and all \( \varepsilon, 0 < \varepsilon < 1 \), the inequality
\[ \sup_{n \in \mathbb{N}} R^{1-\varepsilon} |a_n| < +\infty \]
is true.

As can be easily seen the inclusion \( (a_n)_{n \in \mathbb{N}} \in \Gamma^- \) implies that, for all \( R > 0 \) and all \( \varepsilon > 0 \),
\[ \sum_{n=1}^{\infty} R^{1-\varepsilon} |a_n| < +\infty. \]
Since \( |\lambda_n| \leq \frac{C}{\rho^n} \), the inclusion \( (\lambda_n)_{n \in \mathbb{N}} \in \Gamma^- \) is true.

Proposition. Let \( 0 < \varepsilon < \frac{1}{2} \). Then, for all \( R > 1 \), there exists a constant \( C(R, \varepsilon) \) such that
\[ |H_n(x)| \leq C(R, \varepsilon) R^{1-\varepsilon} \]
for all \( n \in \mathbb{N} \) and all \( x \) such that \( |x| \leq R \).

Proof. Let us give the proof by induction on \( n \), using the recurrence formula (1). Let us assume that, for some constant \( C(R, \varepsilon) \) and some \( n \geq 2 \),
\[ |H_{n-1}(x)| \leq C(R, \varepsilon) R^{(n-1)1-\varepsilon} \quad \text{and} \quad |H_{n-2}(x)| \leq C(R, \varepsilon) R^{(n-2)1-\varepsilon} \]
for all \( x \) with \( |x| \leq R \). Then we have to prove this for \( H_n(x), |x| \leq R \). By (1),
\[ H_n(x) = \frac{2\sqrt{\pi x}}{\sqrt{n}} H_{n-1}(x) - \frac{\sqrt{n-1}}{\sqrt{n}} H_{n-2}(x). \]
It follows that
\[ |H_n(x)| \leq \frac{2\sqrt{\pi R}}{\sqrt{n}} C(R, \varepsilon) R^{(n-1)1-\varepsilon} + \sqrt{1 - \frac{1}{n}} C(R, \varepsilon) R^{(n-2)1-\varepsilon} \]
for all \( x \) with \( |x| \leq R \), i.e., since \( R > 1 \),
\[ |H_n(x)| \leq C(R, \varepsilon) R^{(n-1)1-\varepsilon} \left( \frac{2\sqrt{\pi R}}{\sqrt{n}} + \sqrt{1 - \frac{1}{n}} \right). \]
Therefore, if the right-hand side can be estimated from above by $C(R, \varepsilon)R^{n-1+\varepsilon}$, we are through. Such estimation is true if

$$R^{(n-1)^{1-\varepsilon}} \left( \frac{2\sqrt{\pi} R}{\sqrt{n}} + \sqrt{1 - \frac{1}{n}} \right) \leq R^{n^{1-\varepsilon}},$$

i.e. if

$$\frac{2\sqrt{\pi} R}{\sqrt{n}} + 1 - \frac{1}{n} \leq R^{n^{1-\varepsilon} - (n-1)^{1-\varepsilon}}.$$

Observe that

$$n^{\varepsilon} - (n-1)^{1-\varepsilon} = n^{1-\varepsilon} - n^{1-\varepsilon} \left( 1 - \frac{1}{n} \right)^{1-\varepsilon} \quad \text{and} \quad \left( 1 - \frac{1}{n} \right)^{1-\varepsilon} \leq 1 - \frac{1 - \varepsilon}{n}.$$

This implies the inequality

$$n^{1-\varepsilon} - n^{1-\varepsilon} \left( 1 - \frac{1}{n} \right)^{1-\varepsilon} \geq n^{1-\varepsilon} - n^{1-\varepsilon} + \frac{1 - \varepsilon}{n} n^{1-\varepsilon} = \frac{1 - \varepsilon}{n^{\varepsilon}}.$$

Thus $R^{n^{1-\varepsilon} - (n-1)^{1-\varepsilon}} \geq R^{(1-\varepsilon)/n^{\varepsilon}}$. Therefore, if

$$\frac{2\sqrt{\pi} R}{\sqrt{n}} + \sqrt{1 - \frac{1}{n}} \leq R^{(1-\varepsilon)/n^{\varepsilon}},$$

then automatically

$$\frac{2\sqrt{\pi} R}{\sqrt{n}} + \sqrt{1 - \frac{1}{n}} \leq R^{n^{1-\varepsilon} - (n-1)^{1-\varepsilon}}.$$

Now

$$R^{(1-\varepsilon)/n^{\varepsilon}} = e^{\log R^{(1-\varepsilon)/n^{\varepsilon}}} = e^{(1-\varepsilon)/n^{\varepsilon} \log R} \geq 1 + \frac{1 - \varepsilon}{n^{\varepsilon}} \log R.$$

Therefore, if

$$\frac{2\sqrt{\pi} R}{\sqrt{n}} + \sqrt{1 - \frac{1}{n}} \leq 1 + \frac{1 - \varepsilon}{n^{\varepsilon}} \log R,$$

then automatically

$$\frac{2\sqrt{\pi} R}{\sqrt{n}} + \sqrt{1 - \frac{1}{n}} \leq R^{(1-\varepsilon)/n^{\varepsilon}}.$$

As $\sqrt{1 - \frac{1}{n}} < 1$, we will have

$$\frac{2\sqrt{\pi} R}{\sqrt{n}} + \sqrt{1 - \frac{1}{n}} \leq 1 + \frac{1 - \varepsilon}{n^{\varepsilon}} \log R \quad \text{if} \quad \frac{2\sqrt{\pi} R}{\sqrt{n}} \leq \frac{1 - \varepsilon}{n^{\varepsilon}} \log R,$$

i.e. if

$$\frac{2\sqrt{\pi} R}{(1 - \varepsilon) \log R} \leq n^{1/2-\varepsilon}, \quad \text{i.e. if} \quad n \geq \left( \frac{2\sqrt{\pi} R}{(1 - \varepsilon) \log R} \right)^{1/(1/2-\varepsilon)}.$$
This makes sense as $\varepsilon < 1/2$. Let

$$N(R, \varepsilon) = \left( \frac{2\sqrt{\pi} R}{(1 - \varepsilon) \log R} \right)^{1/(1/2 - \varepsilon)} + 1.$$ 

Since $H_n$ is an $n$th degree polynomial, there exists a constant $C_1(R, \varepsilon) > 0$ such that $|H_n(x)| \leq C_1(R, \varepsilon) R^n$ for all $x$ with $|x| \leq R$ and all $n \leq N(R, \varepsilon)$. This implies that

$$|H_n(x)| \leq C_1(R, \varepsilon) R^{n-1} \frac{R^n}{R^{n-1}}.$$ 

As $\frac{R^n}{R^{n-1}}$ is bounded for all $n \leq N(R, \varepsilon)$, the proposition is proved.

4. Proof of the theorem

We have to show that

$$\sup_{x \in \mathbb{R}^2, \|x\| < R} |D^\alpha g(x, y)| \leq C_R^{\alpha+1} \alpha!$$

for some constant $C_R$ and all $\alpha$, where $D^\alpha g(x, y)$ is given by (2). By (3), for any $k \in \mathbb{N}$ we have

$$D^k H_n(x) = D^k \left( e^{-\pi x^2} H_n(x) \right) = \sum_{r=0}^{k} \binom{k}{r} D^r H_n(x) D^{k-r} e^{-\pi x^2}.$$ 

By (4), we have

$$D^r H_n(x) = \begin{cases} (2\sqrt{\pi})^r \sqrt{n(n-1) \cdots (n-r+1)} H_{n-r}(x) & \text{if } r \leq n, \\ 0 & \text{otherwise} \end{cases}$$

Therefore, by the Proposition, we have for all $r, n \in \mathbb{N}$ and $|x| \leq R$,

$$|D^r H_n(x)| \leq C(R, \varepsilon)(2\sqrt{\pi})^r \sqrt{n(n-1) \cdots (n-r+1)} \frac{R^{n-r}}{R^{n-1}}$$

as $r \leq k$ and $\binom{n}{r} \leq 2^n$. By (5), we have

$$D^{k-r} e^{-\pi x^2} = c_{k-r} \left( \frac{1}{\sqrt{2}} \right)^{k-r} H_{k-r} \left( \frac{x}{\sqrt{2}} \right) e^{-\pi x^2}.$$
Therefore, if $|x| \leq R$, then by Proposition

$$|D^{k-r}e^{-\pi x^2}| \leq C(R, \varepsilon)c_{k-r} \left| H_{k-r} \left( \frac{x}{\sqrt{2}} \right) \right|$$

$$\leq C(R, \varepsilon)c_{k-r}R^{(k-r)\varepsilon}$$

$$\leq C(R, \varepsilon)^{(k-r)!}2^{k-r}R^{1/4}(k-r)^2/k!$$

$$\leq C(R, \varepsilon)^{k!}2^{k}\pi^{k/2}R^k.$$

Therefore, for $|x| \leq R$, we have by (3)

$$|D^k \mathcal{H}_n(x)| \leq C^2(R, \varepsilon)\sum_{r=0}^{k} \left( \frac{k}{r} \right) (2\sqrt{\pi})^k k! \sqrt{2^n R^{n-1}} \sqrt{k!} 2^{k} \pi^{k/2} R^k$$

$$\leq C^2(R, \varepsilon)2^{2k}(\sqrt{\pi})^k k! \pi^{k/2} R^k \sqrt{2^n R^{n-1}} \sum_{r=0}^{k} \left( \frac{k}{r} \right)$$

$$\leq C^2(R, \varepsilon)2^{2k}(\sqrt{\pi})^k k! \pi^{k/2} R^k \sqrt{2^n R^{n-1}} 2^k$$

$$= C^2(R, \varepsilon)(8\pi R)^k k! 2^{n/2} R^{n-1}.$$
Since $p > 2$, the number $p > 1$ can be chosen such that $\frac{2p}{p} < 1$. Since $(\lambda_n)_{n \in N} \in \Gamma^-$, we have $\sum_{n \in N} |\lambda_n| R^{2n^{1-\varepsilon}} < +\infty$. Hence $\sum_{n \in N} |\lambda_n| s^n R^{2n^{1-\varepsilon}}$ is a real number depending only on $R$. Thus, defining

\[ S = S(R, \varepsilon) = C^2(R, \varepsilon) \sum_{n=1}^{\infty} |\lambda_n| 2^n R^{2n^{1-\varepsilon}} \]

we have

\[ |D^\alpha g(x, y)| \leq S(8\pi R)^{\alpha_1 + \alpha_2} \alpha_1! \alpha_2! \]
\[ \leq (S + 1)^{\alpha_1 + \alpha_2 + 1} (8\pi R)^{\alpha_1 + \alpha_2 + 1} \alpha_1! \alpha_2! \]
\[ = (8\pi R(S + 1))^{\alpha_1 + \alpha_2 + 1} \alpha_1! \alpha_2! \]

Hence the proposition is proved \( \blacksquare \)

**Remark.** The above method of proof does not tell anything about the analyticity or otherwise of the series

\[ \sum_{n=1}^{\infty} \lambda_n H_n(x) H_n(y) \]

when the sequence $(\lambda_n)_{n \in N}$ is such that $|\lambda_n| \leq \frac{C}{2^n}$ $(n \in N)$. When $\lambda_n = \frac{1}{2^n}$ $(n \in N)$, the analyticity of that series follows from Mehler's formula (see [3: Lemma 1.1.1]).

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Added in proof: Subsequently the author has settled the question of A. L. Brown in the affirmative by proving a more general result. This possibly will be the object of a forthcoming paper in this journal.

**References**


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