Parametric Weighted Integral Inequalities
for
A-Harmonic Tensors

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Abstract. We prove the $A_r(\Omega)$-weighted Hardy-Littlewood inequality, the $A_r(\Omega)$-weighted weak reverse Hölder inequality and the $A_r(\Omega)$-weighted Caccioppoli-type estimate for $A$-harmonic tensors all being generalizations of classical results.

Keywords: $A_r$-weights, inequalities, $A$-harmonic equation, differential forms

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1. Introduction

The purpose of this paper is to develop parametric versions of the $A_r(\Omega)$-weighted integral inequalities for $A$-harmonic tensors. These results are of interest in nonlinear potential theory, degenerate elliptic equations, continuum mechanics, and the $L^p$ theory. They can be used to study the integrability of $A$-harmonic tensors and to estimate the integrals for $A$-harmonic tensors. $A$-harmonic tensors are differential forms which satisfy the $A$-harmonic equation. They are interesting and important extensions of $p$-harmonic tensors. In the meantime, $p$-harmonic tensors are extensions of harmonic functions and $p$-harmonic functions, $p > 1$. Many interesting results of $A$-harmonic tensors and their applications in different fields, such as quasiregular mappings and the theory of elasticity, have been found recently (see [1 - 4, 8 - 12, 14]).

We always assume that $\Omega$ is a connected open subset of $\mathbb{R}^n$. We write $\mathbb{R} = \mathbb{R}^1$. Balls are denoted by $B$, and $\sigma B$ is the ball with the same center as $B$ and with $\text{diam}(\sigma B) = \sigma \text{diam}(B)$. We do not distinguish the balls from cubes throughout this paper. The $n$-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$. We call $w$ a weight if $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e. Also, in general $d\mu = w \, dx$ where $w$ is a weight. For $0 < p < \infty$ we denote the weighted $L^p$-norm of a measurable function $f$ over $E$ by

$$
\|f\|_{p,E,w} = \left( \int_E |f(x)|^p w(x) \, dx \right)^{1/p}.
$$

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard unit basis of $\mathbb{R}^n$. Assume that $\wedge^l = \wedge^l(\mathbb{R}^n)$ is the linear space of $l$-vectors spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$,
corresponding to all ordered \( l \)-tuples \( I = (i_1, i_2, \ldots, i_l) \) \((1 \leq i_1 < i_2 < \ldots < i_l \leq n; \ l = 0, 1, \ldots, n)\). The Grassman algebra \( \wedge = \oplus \wedge^l \) is a graded algebra with respect to the exterior products. For \( \alpha = \sum \alpha^I e_I \in \wedge \) and \( \beta = \sum \beta^I e_I \in \wedge \) the inner product in \( \wedge \) is given by

\[
\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I
\]

with summation over all \( l \)-tuples \( I = (i_1, i_2, \ldots, i_l) \) and all integers \( l = 0, 1, \ldots, n \). We define the Hodge star operator

\[
\star : \wedge \rightarrow \wedge
\]

by the rule

\[
\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n \quad \text{and} \quad \alpha \wedge \star \beta = \beta \wedge \star \alpha = \langle \alpha, \beta \rangle (\star 1)
\]

for all \( \alpha, \beta \in \wedge \). The norm of \( \alpha \in \wedge \) is given by the formula

\[
|\alpha|^2 = \langle \alpha, \alpha \rangle = \star (\alpha \wedge \star \alpha) \in \wedge^0 = \mathbb{R}.
\]

The Hodge star is an isometric isomorphism on \( \wedge \) with \( \star : \wedge^l \rightarrow \wedge^{n-l} \) and \( \star (-1)^{(l-n)} \star : \wedge^l \rightarrow \wedge^l \).

A differential \( l \)-form \( \omega \) on \( \Omega \) is a de Rham current (see [13: Chapter III]) on \( \Omega \) with values in \( \wedge^l (\mathbb{R}^n) \). We use \( D^l(\Omega, \wedge^l) \) to denote the space of all differential \( l \)-forms and \( L^p(\Omega, \wedge^l) \) to denote the \( l \)-forms

\[
\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1i_2\cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}
\]

with \( \omega_I \in L^p(\Omega, \mathbb{R}) \) for all ordered \( l \)-tuples \( I \). Thus \( L^p(\Omega, \wedge^l) \) is a Banach space with norm

\[
\|\omega\|_{p, \Omega} = \left( \int_{\Omega} |\omega(x)|^p dx \right)^{\frac{1}{p}} = \left( \int_{\Omega} \left( \sum_I |\omega_I(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.
\]

Similarly, \( W^1_p(\Omega, \wedge^l) \) are the differential \( l \)-forms on \( \Omega \) whose coefficients are in \( W^1_p(\Omega, \mathbb{R}) \). The notations \( W^1_{p,loc}(\Omega, \mathbb{R}) \) and \( W^1_{p,loc}(\Omega, \wedge^l) \) are self-explanatory. We denote the exterior derivative by

\[
d : D^l(\Omega, \wedge^l) \rightarrow D^{l+1}(\Omega, \wedge^{l+1})
\]

for \( l = 0, 1, \ldots, n \). Its formal adjoint operator

\[
d^* : D^{l+1}(\Omega, \wedge^l) \rightarrow D^l(\Omega, \wedge^l)
\]

is given by

\[
d^* = (-1)^{nl+1} \star d \star \quad \text{on} \quad D^l(\Omega, \wedge^{l+1}) \quad (l = 0, 1, \ldots, n).
\]

Many interesting results have been established in the study of the \( A \)-harmonic equation

\[
d^* A(x, d) = 0 \quad (1.1)
\]
for differential forms, where \( A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n) \) satisfies the conditions
\[
|A(x, \xi)| \leq a|\xi|^{p-1} \quad \text{and} \quad \langle A(x, \xi), \xi \rangle \geq |\xi|^p
\]
for almost every \( x \in \Omega \) and all \( \xi \in \wedge^l(\mathbb{R}^n) \). Here \( a > 0 \) is a constant and \( 1 < p < \infty \) is a fixed exponent associated with equation (1.1). A solution to equation (1.1) is an element of the Sobolev space \( W^{1,p}_{\text{loc}}(\Omega, \wedge^{l-1}) \) such that
\[
\int_{\Omega} \langle A(x, d\omega), d\varphi \rangle = 0
\]
for all \( \varphi \in W^{1,p}_p(\Omega, \wedge^{l-1}) \) with compact support.

**Definition 1.1.** We call \( u \) an \( A \)-harmonic tensor in \( \Omega \) if \( u \) satisfies the \( A \)-harmonic equation (1.1) in \( \Omega \).

A differential \( l \)-form \( u \in D'(\Omega, \wedge^l) \) is called a closed form if \( du = 0 \) in \( \Omega \). Similarly, a differential \((l + 1)\)-form \( v \in D'(\Omega, \wedge^{l+1}) \) is called a co-closed form if \( d^*v = 0 \). The equation
\[
A(x, du) = d^*v
\]
is called the conjugate \( A \)-harmonic equation. For example, \( du = d^*v \) is an analogue of a Cauchy-Riemann system in \( \mathbb{R}^n \). Clearly, the \( A \)-harmonic equation is not affected by adding a closed form to \( u \) and co-closed form to \( v \). Therefore, any type of estimates between \( u \) and \( v \) must be modulo such forms. Suppose that \( u \) is a solution to equation (1.1) in \( \Omega \). Then, at least locally in a ball \( B \), there exists a form \( v \in W^1_q(B, \wedge^{l+1}) \) \((\frac{1}{p} + \frac{1}{q} = 1)\) such that (1.3) holds. Throughout this paper, we always assume that \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Definition 1.2.** When \( u \) and \( v \) satisfy (1.3) in \( \Omega \) and \( A^{-1} \) exists in \( \Omega \), we call \( u \) and \( v \) conjugate \( A \)-harmonic tensors in \( \Omega \).

Iwaniec and Lutoborski prove the following result in [9]:

Let \( Q \subset \mathbb{R}^n \) be a cube or a ball. To each \( y \in Q \) there corresponds a linear operator
\[
K_y : C^\infty(Q, \wedge^l) \to C^\infty(Q, \wedge^{l-1})
\]
defined by
\[
(K_y \omega)(x; \xi_1, \ldots, \xi_l) = \int_0^1 t^{l-1} \omega(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1}) \, dt
\]
and the decomposition \( \omega = d(K_y \omega) + K_y (d\omega) \).

We define another linear operator
\[
T_Q : C^\infty(Q, \wedge^l) \to C^\infty(Q, \wedge^{l-1})
\]
by averaging \( K_y \) over all points \( y \) in \( Q \):
\[
T_Q \omega = \int_Q \varphi(y) K_y \omega \, dy
\]
where \( \varphi \in C_0^\infty(Q) \) is normalized by \( \int_Q \varphi(y) \, dy = 1 \). We define the \( l \)-form \( \omega_Q \in D'(Q, \wedge^l) \) by
\[
\omega_Q = \begin{cases} |Q|^{-1} \int_Q \omega(y) \, dy & \text{if } l = 0 \\ d(T_Q \omega) & \text{if } l = 1, 2, \ldots, n \end{cases}
\]
for all \( \omega \in L^p(Q, \wedge^l) \) \((1 \leq p < \infty)\).
2. The $A_r(\Omega)$-weighted Hardy-Littlewood inequality

In this section, we prove different versions of the $A_r(\Omega)$-weighted Hardy-Littlewood inequality.

**Definition 2.1.** A weight $w = w(x)$ is called an $A_r$-weight for some $r > 1$ in a domain $\Omega$, write $w \in A_r(\Omega)$, if $w > 0$ a.e. and

$$
\sup_B \left( \frac{1}{|B|} \int_B w \, dx \right)^{r-1} \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{\frac{1}{r}} \, dx \right)^{r} < \infty \tag{2.1}
$$

for any ball $B \subset \Omega$.

See [5, 7] for properties of $A_r(\Omega)$-weights. We will need the following generalized Hölder inequality.

**Lemma 2.2.** Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $\frac{1}{s} = \frac{1}{\alpha} + \frac{1}{\beta}$. If $f$ and $g$ are measurable functions on $\mathbb{R}^n$, then

$$
\|fg\|_{s,\Omega} \leq \|f\|_{\alpha,\Omega}\|g\|_{\beta,\Omega}
$$

for any $\Omega \subset \mathbb{R}^n$.

We also need the following lemma [5].

**Lemma 2.3.** If $w \in A_r(\Omega)$, then there exist constants $\beta > 1$ and $C > 0$, independent of $w$, such that

$$
\|w\|_{\beta,B} \leq C|B|^\frac{1-\beta}{p}\|w\|_{1,B}
$$

for all balls $B \subset \mathbb{R}^n$.

Hardy and Littlewood prove the following inequality for conjugate harmonic functions in the unit disk $D$ in [6]:

**Theorem A.** For each $p > 0$, there is a constant $C > 0$ such that

$$
\int_D |u - u(0)|^p \, dx \leq C \int_D |v - v(0)|^p \, dx
$$

for all analytic functions $f = u + iv$ in the unit disk $D$.

The above Hardy-Littlewood inequality has been generalized into different versions. In [12] Nolder proves the following version of it.

**Theorem B.** Let $u$ and $v$ be conjugate $A$-harmonic tensors in $\Omega \subset \mathbb{R}^n$, $\sigma > 1$, and $0 < s, t < \infty$. Then there exists a constant $C > 0$, independent of $u$ and $v$, such that

$$
\|u - u_B\|_{s,B} \leq C|B|^{\beta}\|v - c\|_{t,\sigma B}^\frac{n}{p}
$$

for all balls $B$ with $\sigma B \subset \Omega$. Here $c$ is any form in $W^{1,\text{loc}}(\Omega, \Lambda)$ with $d^* c = 0$ and

$$
\beta = \frac{1}{s} + \frac{1}{n} - \left(\frac{1+\frac{1}{s}}{p}\right).
$$

Now we prove the following parametric weighted Hardy-Littlewood inequality.
**Theorem 2.4.** Let $u$ and $v$ be conjugate $A$-harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$ and $w \in A_r(\Omega)$ for some $r > 1$. Let $0 < s, t < \infty$. Then there exists a constant $C > 0$, independent of $u$ and $v$, such that
\[
\left( \int_B |u - u_B|^s w^{\alpha} dx \right)^{\frac{1}{s}} \leq C|B|^\gamma \left( \int_{\sigma B} |v - c|^t w^{\frac{pt}{qt}} dx \right)^{\frac{q}{pt}} \tag{2.2}
\]
for all balls $B$ with $\sigma B \subset \Omega \subset \mathbb{R}^n$, $\sigma > 1$ and $0 < \alpha \leq 1$. Here $c$ is any form in $W^{1}_{q,loc}(\Omega, \Lambda)$ with $d^* c = 0$ and $\gamma = \frac{1}{s} + \frac{1}{n} - \left( \frac{1}{s} + \frac{1}{n} \right) q$.

As mentioned in Section 1, the $A$-harmonic equation is not affected by adding a closed form to $u$ and co-closed form to $v$. Therefore, any type of estimates between $u$ and $v$ must be modulo such forms. Thus, (2.2) is equivalent to
\[
\left( \int_B |u|^s w^{\alpha} dx \right)^{\frac{1}{s}} \leq C|B|^\gamma \left( \int_{\sigma B} |v - c|^t w^{\frac{pt}{qt}} dx \right)^{\frac{q}{pt}} \tag{2.2}'
\]
Note that (2.2) can also be written as the symmetric form
\[
\left( \frac{1}{|B|} \int_B |u - u_B|^s w^{\alpha} dx \right)^{\frac{1}{q}} \leq C|B|^{\frac{1}{q} - \frac{1}{n}} \left( \frac{1}{|B|} \int_{\sigma B} |v - c|^t w^{\frac{pt}{qt}} dx \right)^{\frac{1}{pt}} \tag{2.2}''
\]

**Proof of Theorem 2.4.** We first show that (2.2) holds for $0 < \alpha < 1$. Let $k = \frac{s}{1 - \alpha}$. Using Lemma 2.2 we have
\[
\left( \int_B |u - u_B|^s w^{\alpha} dx \right)^{\frac{1}{s}} = \left( \int_B (|u - u_B| w^{\frac{s}{n}})^k dx \right)^{\frac{1}{k} - \frac{s}{k}} \leq \|u - u_B\|_{k,B} \left( \int_B w^{\frac{ks}{n} - k} dx \right)^{\frac{k-s}{k}} \tag{2.3}
\]
\[
= \|u - u_B\|_{k,B} \left( \int_B w dx \right)^{\frac{k-s}{k}}.
\]
Choose $m = \frac{qst}{qs + \alpha pt(r-1)}$. Then $m < t$. By Theorem B we have
\[
\|u - u_B\|_{k,B} \leq C_1|B|^\beta \|v - c\|^{\frac{q}{m,\sigma B}}_{m,\sigma B} \tag{2.4}
\]
where $\beta = \frac{1}{k} + \frac{1}{n} - \left( \frac{1}{s} + \frac{1}{n} \right) q$. Substituting (2.4) into (2.3) yields
\[
\left( \int_B |u - u_B|^s w^{\alpha} dx \right)^{\frac{1}{s}} \leq C_1|B|^\beta \|v - c\|^{\frac{q}{m,\sigma B}}_{m,\sigma B} \left( \int_B w dx \right)^{\frac{k-s}{k}}. \tag{2.5}
\]
Since $\frac{1}{m} = \frac{1}{t} + \frac{t-m}{mt}$, by Lemma 2.2 again we find that
\[
\|v - c\|_{m,\sigma B} = \left( \int_{\sigma B} \left( |v - c| w^{\frac{p}{qt}} w^{-\frac{p}{q}} \right)^m dx \right)^{\frac{1}{m}} \leq \left( \int_{\sigma B} \left( |v - c| w^{\frac{p}{qt}} \right)^{\frac{pt}{qt}} dx \right)^{\frac{1}{t}} \left( \int_{\sigma B} \left( \frac{1}{w} \right)^{\frac{p}{pt} - \frac{1}{mt}} dx \right)^{\frac{t-m}{mt}} \leq \left( \int_{\sigma B} \left( |v - c| w^{\frac{p}{qt}} \right)^{\frac{pt}{qt}} dx \right)^{\frac{1}{t}} \left( \int_{\sigma B} \left( \frac{1}{w} \right)^{\frac{p}{pt} - \frac{1}{mt}} dx \right)^{\frac{t-m}{mt}} \frac{p\alpha(r-1)}{q^a}.\]
Hence
\[ \|v - c\|_{m, \sigma B}^{\frac{q}{p}} \leq \left( \int_{\sigma B} \left( \frac{1}{w} \right)^{\frac{1}{r-1}} \, dx \right)^{\frac{\alpha (r-1)}{s}} \left( \int_{\sigma B} |v - c|^t w^{\frac{pt\alpha}{q}} \, dx \right)^{\frac{q}{pt}}. \]  

Combining (2.5) and (2.6) we obtain
\[ \left( \int_B |u - u_B|^s w^\alpha \, dx \right)^{\frac{1}{s}} \leq C_1 |B|^\beta \left( \int_B w \, dx \right)^{\frac{\alpha}{s}} \left( \int_{\sigma B} \left( \frac{1}{w} \right)^{\frac{1}{r-1}} \, dx \right)^{\frac{\alpha (r-1)}{s}} \left( \int_{\sigma B} |v - c|^t w^{\frac{pt\alpha}{q}} \, dx \right)^{\frac{q}{pt}}. \]  

Using the condition that \( w \in A_r(\Omega) \) yields
\[ \left( \int_B w \, dx \right)^{\frac{\alpha}{s}} \left( \int_{\sigma B} \left( \frac{1}{w} \right)^{\frac{1}{r-1}} \, dx \right)^{\frac{\alpha (r-1)}{s}} \leq |\sigma B|^\frac{\alpha}{s} \left( \frac{1}{|\sigma B|} \int_B w \, dx \right)^{\frac{\alpha}{s}} \left( \frac{1}{|\sigma B|} \int_{\sigma B} \left( \frac{1}{w} \right)^{\frac{1}{r-1}} \, dx \right)^{r-1} \leq C_2 |\sigma B|^{\frac{\alpha r}{s}} = C_3 |B|^{\frac{\alpha r}{s}}. \]  

Substituting (2.8) into (2.7) and noting that \( \beta + \frac{\alpha r}{s} = \frac{1}{k} + \frac{1}{n} - \frac{(\frac{1}{r} + \frac{1}{t})q}{p} + \frac{r}{\alpha s} = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{r} + \frac{1}{t})q}{p} \), we have
\[ \left( \int_B |u - u_B|^s w^\alpha \, dx \right)^{\frac{1}{s}} \leq C_4 |B|^{\gamma} \left( \int_{\sigma B} |v - c|^t w^{\frac{pt\alpha}{q}} \, dx \right)^{\frac{q}{pt}} \]  

where \( \gamma = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{r} + \frac{1}{t})q}{p} \).

Next, we prove that Theorem 2.4 holds if \( \alpha = 1 \). By Lemma 2.3 there exist constants \( \beta_1 > 1 \) and \( C_5 > 0 \), independent of \( w \), such that
\[ \|w\|_{1, \sigma B} \leq C_5 |B|^{\frac{1}{\beta_1 s} - \frac{\beta_1}{\beta_1 s}} \|w\|_{1, \sigma B}. \]  

Since \( \frac{1}{\beta_1 s} + \frac{\beta_1 - 1}{\beta_1 s} = \frac{1}{s} \), then by Lemma 2.2 we have
\[ \|u - u_B\|_{s, B, w} \leq \|w\|_{\beta_1, B}^{\frac{1}{\beta_1 s}} \|u - u_B\|_{\frac{\beta_1 s}{\beta_1 - 1}, B}. \]  

By Theorem B, there is a constant \( C_6 > 0 \), independent of \( u \) and \( v \), such that for any \( t' > 0 \) we have
\[ \|u - u_B\|_{\frac{\beta_1 s}{\beta_1 - 1}, B} \leq C_6 |B|^{\beta'} \|v - c\|_{t', \sigma B}^{\frac{q}{r}}. \]
where $\beta' = \frac{\beta_1 - 1}{\beta_1 s} + \frac{1}{n} - \frac{(\frac{1}{p} + \frac{1}{s})q}{p}$. Combining (2.10) and (2.11) we obtain
\[
\|u - u_B\|_{s, B, w} \leq C_6|B|^\beta' \|w\|_{1, s, B}^\frac{1}{\beta_1 s} \|v - c\|_{t', \sigma B}. \tag{2.12}
\]

Now, choose $t' = \frac{t}{k_1}$ where $k_1$ is to be determined later. Since $|v - c| = w^{-\frac{q}{ps}}|v - c|^w$, by Lemma 2.2 we obtain
\[
\|v - c\|_{t', \sigma B} \leq \left(\frac{1}{w}\right)^{\frac{pt}{q}} \left(\frac{1}{w}\right)^{\frac{pt}{q}} \|v - c\|_{t, w} \left(\int_{\sigma B} |v - c|^t w^\frac{pt}{q} dx\right)^{\frac{q}{pt}}. \tag{2.13}
\]

From (2.9), (2.12) and (2.13) we have
\[
\|u - u_B\|_{s, B, w} \leq C_7|B|^\beta' \left(\frac{1}{w}\right)^{\frac{pt}{q}} \left(\frac{1}{w}\right)^{\frac{pt}{q}} \|v - c\|_{t, w} \left(\int_{\sigma B} |v - c|^t w^\frac{pt}{q} dx\right)^{\frac{q}{pt}}. \tag{2.14}
\]

Set $k_1 = 1 + \frac{pt(r - 1)q}{qs}$, then $\frac{(k_1 - 1)q}{pt} = r - 1$. By $w \in A_r(\Omega)$ we know that
\[
\|w\|_{1, s, B} \geq \left(\frac{1}{w}\right)^{\frac{pt}{q}} \left(\frac{1}{w}\right)^{\frac{pt}{q}} \|v - c\|_{t, w} \left(\int_{\sigma B} |v - c|^t w^\frac{pt}{q} dx\right)^{\frac{q}{pt}} \leq C_8|B|^\gamma \left(\int_{\sigma B} |v - c|^t w^\frac{pt}{q} dx\right)^{\frac{q}{pt}}, \tag{2.15}
\]

Combining (2.14) and (2.15) we have
\[
\|u - u_B\|_{s, B, w} \leq C_9|B|^\gamma \left(\int_{\sigma B} |v - c|^t w^\frac{pt}{q} dx\right)^{\frac{q}{pt}} \leq C_{10}|B|^\gamma \left(\int_{\sigma B} |v - c|^t w^\frac{pt}{q} dx\right)^{\frac{q}{pt}}.
\]

where
\[
\gamma = \beta' + \frac{1 - \alpha}{\alpha s} + \frac{1}{s} + \frac{q(k - 1)}{pt} = -\frac{nq + t(q - p)}{npt} + \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{t} + \frac{1}{n})q}{p}.
\]

Therefore, (2.2) holds if $\alpha = 1$. We have completed the proof of Theorem 2.4.$\blacksquare$

We need the following properties of the Whitney covers appearing [12].

**Lemma 2.5.** Each $\Omega$ has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that
\[
\cup_i Q_i = \Omega, \quad \sum_{Q \in \mathcal{V}} \chi_{\sqrt{\frac{1}{2}}Q} \leq N \chi_\Omega
\]

for all $x \in \mathbb{R}^n$ and some $N > 1$, and if $Q_i \cap Q_j \neq \phi$, then there exists a cube $R$ (this cube does not need be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if $\Omega$ is $\delta$-John, then there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q$ from $\mathcal{V}$ and such that $Q \subset \rho Q_i$ for some $\rho = \rho(n, \delta)$.

As applications of Theorem 2.4 we prove the following global $A_r(\Omega)$-weighted Hardy-Littlewood inequality.
**Theorem 2.6.** Let \( u \in D'(\Omega, \Lambda^{l-1}) \) and \( v \in D'(\Omega, \Lambda^{l+1}) \) be conjugate \( A \)-harmonic tensors. Let \( q \leq p \), \( v - c \in L^t(\Omega, \Lambda^{l+1}) \) \( (l = 1, 2, \ldots, n - 1) \) and \( w \in A_r(\Omega) \). If \( s \) is defined by

\[
s = \frac{npt}{nq + t(q - p)} \quad (0 < t < \infty),
\]

then there exists a constant \( C > 0 \), independent of \( u \) and \( v \), such that

\[
\left( \int_{\Omega} |u|^s w^\alpha dx \right)^{\frac{1}{s}} \leq C \left( \int_{\Omega} |v - c|^t w^{\frac{pt}{nq}} dx \right)^{\frac{q}{pt}}
\]

for any domain \( \Omega \subset \mathbb{R}^n \) with \( |\Omega| < \infty \). Here \( c \) is any form in \( W^1_{q, loc}(\Omega, \Lambda) \) with \( d^* c = 0 \).

**Proof.** From (2.2)' we have

\[
\left( \int_{Q} |u|^s w^\alpha dx \right)^{\frac{1}{s}} \leq C |Q|^\gamma \left( \int_{\sigma Q} |v - c|^t w^{\frac{pt}{nq}} dx \right)^{\frac{q}{pt}}
\]

(2.17)

where \( \gamma = \frac{1}{s} + \frac{1}{n} - \left( \frac{q + \frac{1}{p}}{p} \right) \). Substituting (2.16) into the expression of \( \gamma \) we get

\[
\gamma = \frac{1}{s} + \frac{1}{n} - \left( \frac{q}{pt} + \frac{q}{np} \right) = \frac{nq + t(q - p)}{npt} + \frac{1}{n} - \left( \frac{q}{pt} + \frac{q}{np} \right) = 0.
\]

(2.18)

Thus we find that (2.17) reduces to

\[
\left( \int_{Q} |u|^s w^\alpha dx \right)^{\frac{1}{s}} \leq C \left( \int_{\sigma Q} |v - c|^t w^{\frac{pt}{nq}} dx \right)^{\frac{q}{pt}}.
\]

(2.19)

Combining (2.19) and Lemma 2.5, we get

\[
\left( \int_{\Omega} |u|^s w^\alpha dx \right)^{\frac{1}{s}} \leq \sum_{Q \in V} \left( \int_{Q} |u|^s w^\alpha dx \right)^{\frac{1}{s}}
\]

\[
\leq \sum_{Q \in V} \left( \int_{Q} |u|^s w^\alpha \chi_{\sqrt{\frac{q}{4}}Q} dx \right)^{\frac{1}{s}}
\]

\[
\leq \sum_{Q \in V} \left( \int_{Q} |u|^s w^\alpha dx \right)^{\frac{1}{s}} \chi_{\sqrt{\frac{q}{4}}Q}
\]

\[
\leq \sum_{Q \in V} C_1 \left( \int_{\sigma Q} |v - c|^t w^{\frac{pt}{nq}} dx \right)^{\frac{q}{pt}} \chi_{\sqrt{\frac{q}{4}}Q}
\]

\[
\leq \sum_{Q \in V} C_1 \left( \int_{\Omega} |v - c|^t w^{\frac{pt}{nq}} dx \right)^{\frac{q}{pt}} \sum_{Q \in V} \chi_{\sqrt{\frac{q}{4}}Q}
\]

\[
\leq C_1 \left( \int_{\Omega} |v - c|^t w^{\frac{pt}{nq}} dx \right)^{\frac{q}{pt}} \sum_{Q \in V} \chi_{\sqrt{\frac{q}{4}}Q}
\]

\[
\leq C_2 \left( \int_{\Omega} |v - c|^t w^{\frac{pt}{nq}} dx \right)^{\frac{q}{pt}}.
\]

The proof of Theorem 2.6 has been completed.
Note that $\alpha \in (0, 1]$ is arbitrary in Theorem 2.4. Hence, if we choose $\alpha$ to be some special values, we will have some different versions of the Hardy-Littlewood inequality. For example, if we let $\alpha = qs$, $qs \leq 1$. By Theorem 2.4, we have the following symmetric version of the Hardy-Littlewood inequality.

**Corollary 2.7.** Let $u$ and $v$ be conjugate A-harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$ and $w \in A_r(\Omega)$ for some $r > 1$. Let $0 < t < \infty$ and $qs \leq 1$. Then there exists a constant $C > 0$, independent of $u$ and $v$, such that

$$\left( \int_B |u - u_B|^s w^{qs} dx \right)^{\frac{1}{qs}} \leq C |B|^\gamma \left( \int_{\sigma B} |v - c|^t w^{pt} dx \right)^{\frac{1}{pt}}$$

for all balls $B$ with $\sigma B \subset \Omega \subset \mathbb{R}^n$ and $\sigma > 1$. Here $c$ is any form in $W_{q,loc}^1(\Omega, \Lambda)$ with $d^* c = 0$ and $\gamma = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{s} + \frac{1}{t})q}{p}$.

If we choose $\alpha = \frac{1}{pt}$ and $pt \geq 1$ in Theorem 2.4, we obtain the following symmetric version.

**Corollary 2.8.** Let $u$ and $v$ be conjugate A-harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$ and $w \in A_r(\Omega)$ for some $r > 1$. Let $0 < t < \infty$ and $pt \geq 1$. Then there exists a constant $C > 0$, independent of $u$ and $v$, such that

$$\left( \int_B |u - u_B|^s w^{\frac{1}{pt}} dx \right)^{\frac{1}{qs}} \leq C |B|^\gamma \left( \int_{\sigma B} |v - c|^t w^{\frac{1}{pt}} dx \right)^{\frac{1}{pt}}$$

for all balls $B$ with $\sigma B \subset \Omega \subset \mathbb{R}^n$ and $\sigma > 1$. Here $c$ is any form in $W_{q,loc}^1(\Omega, \Lambda)$ with $d^* c = 0$ and $\gamma = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{s} + \frac{1}{t})q}{p}$.

If we choose $\alpha = \frac{1}{p}$ in Theorem 2.4, we obtain the following result.

**Corollary 2.9.** Let $u$ and $v$ be conjugate A-harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$ and $w \in A_r(\Omega)$ for some $r > 1$. Let $0 < s, t < \infty$. Then there exists a constant $C > 0$, independent of $u$ and $v$, such that

$$\left( \int_B |u - u_B|^s w^{\frac{1}{t}} dx \right)^{\frac{1}{qs}} \leq C |B|^\gamma \left( \int_{\sigma B} |v - c|^t w^{\frac{1}{t}} dx \right)^{\frac{1}{pt}}$$

for all balls $B$ with $\sigma B \subset \Omega \subset \mathbb{R}^n$ and $\sigma > 1$. Here $c$ is any form in $W_{q,loc}^1(\Omega, \Lambda)$ with $d^* c = 0$ and $\gamma = \frac{1}{s} + \frac{1}{n} - \frac{(\frac{1}{s} + \frac{1}{t})q}{p}$.

If we choose $\alpha = 1$ in Theorem 2.4, we have the following corollary.

**Corollary 2.10.** Let $u$ and $v$ be conjugate A-harmonic tensors in a domain $\Omega \subset \mathbb{R}^n$ and $w \in A_r(\Omega)$ for some $r > 1$. Let $0 < s, t < \infty$. Then there exists a constant $C > 0$, independent of $u$ and $v$, such that

$$\left( \int_B |u - u_B|^s w dx \right)^{\frac{1}{qs}} \leq C |B|^\gamma \left( \int_{\sigma B} |v - c|^t w^{\frac{pt}{pt}} dx \right)^{\frac{1}{pt}}$$
for all balls $B$ with $\sigma B \subset \Omega \subset \mathbb{R}^n$ and $\sigma > 1$. Here $c$ is any form in $W_{q,\text{loc}}^1(\Omega, \Lambda)$ with $d^*c = 0$ and $\gamma = \frac{1}{s} + \frac{1}{n} - \frac{(1 + \frac{1}{q})q}{p}$.

**Remark.** By making different choices for $\alpha$ in Theorem 2.6, we shall have different versions of the global Hardy-Littlewood inequality. Considering the length of the paper, we do not list them here.

3. The $A_r(\Omega)$-weighted weak reverse Hölder inequality

In [12], Nolder obtains the following Caccioppoli-type inequality.

**Theorem C.** Let $u$ be an $A$-harmonic tensor in $\Omega$ and let $\sigma > 1$. Then there exists a constant $C > 0$, independent of $u$, such that

$$\|du\|_{s,B} \leq C \text{diam}(B)^{-1}\|u - c\|_{s,\sigma B}$$

for all balls or cubes $B$ with $\sigma B \subset \Omega$ and all closed forms $c$. Here $1 < s < \infty$.

The following weak reverse Hölder inequality appears in [12].

**Theorem D.** Let $u$ be an $A$-harmonic tensor in $\Omega$, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant $C > 0$, independent of $u$, such that

$$\|u\|_{s,B} \leq C |B|^{\frac{s}{1-s}} \|u\|_{t,\sigma B}$$

for all balls or cubes $B$ with $\sigma B \subset \Omega$.

Using the same method as those used in Section 2, we prove the following $A_r(\Omega)$-weighted weak reverse Hölder inequality with parameter $\alpha$ for $A$-harmonic tensors.

**Theorem 3.1.** Let $u \in D'(\Omega, \wedge^l)$ ($l = 0, 1, \ldots, n$) be an $A$-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$, $\sigma > 1$. Assume that $0 < s, t < \infty$ and $w \in A_r(\Omega)$ for some $r > 1$. Then there exists a constant $C > 0$, independent of $u$, such that

$$\left(\frac{1}{|B|} \int_B |u|^s w^\alpha dx\right)^{\frac{1}{2}} \leq C \left(\frac{1}{|B|} \int_{\sigma B} |u|^t w^{\alpha_1} dx\right)^{\frac{1}{2}}$$

for all balls $B$ with $\sigma B \subset \Omega$ and any real number $\alpha$ with $0 < \alpha \leq 1$.

**Proof.** First, we suppose that $0 < \alpha < 1$. Let $k = \frac{s}{1-\alpha}$. From Lemma 2.2 we find that

$$\left(\int_B |u|^s w^\alpha dx\right)^{\frac{1}{2}} = \left(\int_B (|u| w^{\frac{\alpha}{2}})^s dx\right)^{\frac{1}{2}} \leq \left(\int_B |u|^k dx\right)^{\frac{1}{2}} \left(\int_B (w^{\frac{\alpha}{2}})^{\frac{k-\alpha}{k}} dx\right)^{\frac{k-\alpha}{k}} (3.2)$$

$$\leq \|u\|_{k,B} \left(\int_B w dx\right)^{\frac{\alpha}{2}}$$
for all balls $B$ with $\sigma B \subset \Omega$. Let $m = \frac{st}{s+\alpha t(r-1)}$. By Theorem D we obtain
\begin{equation}
\|u\|_{k,B} \leq C_1 |B|^{\frac{m-k}{km}} \|u\|_{m,\sigma B} .
\end{equation}

Using the Hölder inequality with $\frac{1}{m} = \frac{1}{t} + \frac{t-m}{mt}$ yields
\begin{align}
\|u\|_{m,\sigma B} &= \left( \int_{\sigma B} (|u| w^\frac{\alpha}{s} w^{-\frac{\alpha}{t}})^m \, dx \right)^{\frac{1}{m}} \\
&\leq \left( \int_{\sigma B} |u|^t w^{\frac{\alpha t}{s}} \, dx \right)^{\frac{1}{t}} \left( \int_{\sigma B} \left( \frac{1}{w} \right)^{\frac{\alpha(m-t)}{s(t-m)}} \, dx \right)^{\frac{t-m}{mt}} \\
&= \left( \int_{\sigma B} |u|^t w^{\frac{\alpha t}{s}} \, dx \right)^{\frac{1}{t}} \left( \int_{\sigma B} \left( \frac{1}{w} \right)^{\frac{1}{r-1}} \, dx \right)^{\frac{\alpha(r-1)}{s}} .
\end{align}

Combining (3.2) - (3.4) we find that
\begin{equation}
\left( \int_{B} |u|^s w^{\alpha} \, dx \right)^{\frac{1}{s}} \leq C_1 |B|^{\frac{m-k}{km}} \left( \int_{B} w \, dx \right)^{\frac{\alpha}{s}} \left( \int_{\sigma B} \left( \frac{1}{w} \right)^{\frac{1}{r-1}} \, dx \right)^{\frac{\alpha(r-1)}{s}} \left( \int_{\sigma B} |u|^t w^{\frac{\alpha t}{s}} \, dx \right)^{\frac{1}{t}} .
\end{equation}

Since $w \in A_r(\Omega)$, then we have
\begin{align}
\left( \int_{B} w \, dx \right)^{\frac{\alpha}{s}} \left( \int_{\sigma B} \left( \frac{1}{w} \right)^{\frac{1}{r-1}} \, dx \right)^{\frac{\alpha(r-1)}{s}} &= \left( \left( \int_{B} w \, dx \right) \left( \int_{\sigma B} \left( \frac{1}{w} \right)^{\frac{1}{r-1}} \, dx \right)^{r-1} \right)^{\frac{\alpha}{s}} \\
&\leq |\sigma B|^{\frac{\alpha r}{s}} \left( \left( \frac{1}{|\sigma B|} \int_{B} w \, dx \right) \left( \frac{1}{|\sigma B|} \int_{\sigma B} \left( \frac{1}{w} \right)^{\frac{1}{r-1}} \, dx \right)^{r-1} \right)^{\frac{\alpha}{s}} \\
&\leq C_2 |\sigma B|^{\frac{\alpha r}{s}} \\
&= C_3 |B|^{\frac{\alpha r}{s}} .
\end{align}

Substituting (3.6) into (3.5) we obtain
\begin{equation}
\left( \int_{B} |u|^s w^{\alpha} \, dx \right)^{\frac{1}{s}} \leq C_4 |B|^{\frac{\alpha r}{s}} \left( \int_{\sigma B} |u|^t w^{\frac{\alpha t}{s}} \, dx \right)^{\frac{1}{t}} .
\end{equation}

Then (3.1) holds if $0 < \alpha < 1$.

For the case $\alpha = 1$, by Lemma 2.3, there exist constants $\beta > 1$ and $C_5 > 0$ such that
\begin{equation}
\|w\|_{\beta,B} \leq C_5 |B|^{\frac{1-\beta}{\beta}} \|w\|_{1,B} .
\end{equation}
for any cube or any ball \( B \subset \mathbb{R}^n \). Choose \( k = \frac{s\beta}{\beta-1} \). Then \( s < k \) and \( \beta = \frac{k}{k-s} \). By (3.7) and Lemma 2.2 we have
\[
\left( \int_B |u|^s w \, dx \right)^{\frac{1}{s}} \leq \left( \int_B |u|^k \, dx \right)^{\frac{1}{k}} \left( \int_B (w^{\frac{1}{s}})^{\frac{s}{k-s}} \, dx \right)^{\frac{k-s}{s}} \\
= \|u\|_{k,B} \|w\|_{\beta,B}^{\frac{1}{2}} \\
\leq C_6 |B|^{\frac{1-\beta}{n}} \|w\|_{1,B} \|u\|_{k,B} \\
= C_6 |B|^{-\frac{t}{k}} \|w\|^{\frac{1}{1}}_{1,B} \|u\|_{k,B}. \tag{3.8}
\]
Selecting \( m = \frac{st}{s+t(r-1)} \) and repeating the same procedure as the case \( 0 < \alpha < 1 \), we see that (3.1) is also true for \( \alpha = 1 \). This ends the proof of Theorem 3.1.

As application of Theorem 3.1, we choose the parameter \( \alpha = 1 \) in Theorem 3.1. Then, we have the following version of the reverse Hölder inequality.

**Corollary 3.2.** Let \( u \in D'(\Omega, \Lambda^l) \) \( (l = 0, 1, \ldots, n) \) be an \( A \)-harmonic tensor in a domain \( \Omega \subset \mathbb{R}^n \), \( \sigma > 1 \). Assume that \( 0 < s, t < \infty \) and \( w \in A_r(\Omega) \) for some \( r > 1 \). Then there exists a constant \( C > 0 \), independent of \( u \), such that
\[
\left( \frac{1}{|B|} \int_B |u|^s w \, dx \right)^{\frac{1}{s}} \leq C \left( \frac{1}{|B|} \int_{\sigma B} |u|^t w^{\frac{t}{s}} \, dx \right)^{\frac{1}{t}}
\]
for all balls \( B \) with \( \sigma B \subset \Omega \).

Let \( \alpha = s \) with \( 0 < s \leq 1 \) in Theorem 3.1. We obtain the following symmetric version.

**Corollary 3.3.** Let \( u \in D'(\Omega, \Lambda^l) \) \( (l = 0, 1, \ldots, n) \) be an \( A \)-harmonic tensor in a domain \( \Omega \subset \mathbb{R}^n \), \( \sigma > 1 \). Assume that \( 0 < t < \infty \), \( 0 < s \leq 1 \) and \( w \in A_r(\Omega) \) for some \( r > 1 \). Then there exists a constant \( C > 0 \), independent of \( u \), such that
\[
\left( \frac{1}{|B|} \int_B |u|^s w^s \, dx \right)^{\frac{1}{s}} \leq C \left( \frac{1}{|B|} \int_{\sigma B} |u|^t w^{t} \, dx \right)^{\frac{1}{t}}
\]
for all balls \( B \) with \( \sigma B \subset \Omega \).

Let \( \alpha = \frac{1}{t} \) with \( t \geq 1 \) in Theorem 3.1. Then we have the following.

**Corollary 3.4.** Let \( u \in D'(\Omega, \Lambda^l) \) \( (l = 0, 1, \ldots, n) \) be an \( A \)-harmonic tensor in a domain \( \Omega \subset \mathbb{R}^n \), \( \sigma > 1 \). Assume that \( t \geq 1 \), \( 0 < s < \infty \) and \( w \in A_r(\Omega) \) for some \( r > 1 \). Then there exists a constant \( C > 0 \), independent of \( u \), such that
\[
\left( \frac{1}{|B|} \int_B |u|^s w^t \, dx \right)^{\frac{1}{s}} \leq C \left( \frac{1}{|B|} \int_{\sigma B} |u|^t w^{t} \, dx \right)^{\frac{1}{t}}
\]
for all balls \( B \) with \( \sigma B \subset \Omega \).

We prove the following global result.
Theorem 3.5. Let \( u \in D'(\Omega, \wedge^l) \) \((l = 0, 1, \ldots, n)\) be an \( A \)-harmonic tensor in a domain \( \Omega \subset \mathbb{R}^n \) with \( |\Omega| < \infty \). Assume that \( 0 < s \leq t < \infty \) and \( w \in A_r(\Omega) \) for some \( r > 1 \). Then

\[
\left( \frac{1}{|\Omega|} \int_\Omega |u|^s w^\alpha dx \right)^{\frac{1}{s}} \leq \left( \frac{1}{|\Omega|} \int_\Omega |u|^t w^{\frac{\alpha t}{s}} dx \right)^{\frac{1}{t}} \tag{3.9}
\]

for any real number \( \alpha \) with \( 0 < \alpha \leq 1 \).

**Proof.** It is clear that (3.9) is true if \( s = t \). Now we assume that \( s < t \). Using Lemma 2.2 with \( \frac{1}{s} = \frac{1}{t} + \frac{t-s}{st} \), we have

\[
\left( \int_\Omega |u|^s w^\alpha dx \right)^{\frac{1}{s}} = \left( \int_\Omega \left( |u|^{\alpha/s} \right)^s dx \right)^{\frac{1}{s}} \\
\leq \left( \int_\Omega 1 dx \right)^{\frac{t-s}{st}} \left( \int_\Omega (|u|^{s})^t dx \right)^{\frac{1}{t}} \\
= |\Omega|^{\frac{t-s}{st}} \left( \int_\Omega |u|^t w^{\frac{\alpha t}{s}} dx \right)^{\frac{1}{t}}
\]

which is equivalent to (3.9). The proof of Theorem 3.5 is completed \( \square \)

**Remark.** Theorem 3.5 can be proved by using Theorem 3.1 directly (see [11: Proof of Theorem 2.3]). Here we have the stronger condition \( 0 < s \leq t < \infty \). But the result is also stronger: the constant \( C \) in Theorem 3.1 now reduces to \( C = 1 \). By choosing \( \alpha \) to be some special values in (3.9), we have some global results as we did for the local case.

4. The \( A_r(\Omega) \)-weighted Caccioppoli-type estimate

We prove the following \( A_r(\Omega) \)-weighted Caccioppoli-type estimate with parameter \( \alpha \) for \( A \)-harmonic tensors.

**Theorem 4.1.** Let \( u \in D'(\Omega, \wedge^l) \) \((l = 0, 1, \ldots, n)\) be an \( A \)-harmonic tensor in a domain \( \Omega \subset \mathbb{R}^n \) and \( \rho > 1 \). Assume that \( 1 < s < \infty \) is a fixed exponent associated with the \( A \)-harmonic equation and \( w \in A_r(\Omega) \) for some \( r > 1 \). Then there exists a constant \( C > 0 \), independent of \( u \), such that

\[
\left( \int_B |du|^s w^\alpha dx \right)^{\frac{1}{s}} \leq \frac{C}{\text{diam}(B)} \left( \int_{\rho B} |u-c|^s w^\alpha dx \right)^{\frac{1}{s}} \tag{4.1}
\]

for all balls \( B \) with \( \rho B \subset \Omega \) and all closed forms \( c \). Here \( \alpha \) is any constant with \( 0 < \alpha \leq 1 \).

**Proof.** First, we assume that \( 0 < \alpha < 1 \). Choose \( t = \frac{s}{1-\alpha} \). Since \( \frac{1}{s} = \frac{1}{t} + \frac{t-s}{st} \),
using Lemma 2.2 and Theorem C, we obtain
\[
\left( \int_B |du|^s w^\alpha dx \right)^{\frac{1}{s}} = \left( \int_B (|du|w^{\frac{s}{2}})^s dx \right)^{\frac{1}{s}} \\
\leq \left( \int_B |du|^t dx \right)^{\frac{1}{t}} \left( \int_B (w^{\frac{s}{2}})^{\frac{t}{s}} dx \right)^{\frac{t-s}{t}} \\
\leq \|du\|_{t,B} \left( \int_B w dx \right)^{\frac{\alpha}{s}} \\
= C_1 \text{diam}(B)^{-1} \|u - c\|_{t,\sigma B} \left( \int_B w dx \right)^{\frac{\alpha}{s}}
\]
for all balls $B$ with $\sigma B \subset \Omega$ and all closed forms $c$. Since $c$ is a closed form and $u$ is an $A$-harmonic tensor, then $u - c$ is still an $A$-harmonic tensor. Taking $m = \frac{s}{1+\alpha(r-1)}$, then $m < s < t$. By Theorem D we have
\[
\|u - c\|_{t,\sigma B} \leq C_2 |B| \left( \frac{m-1}{mt} \right) \|u - c\|_{m,\sigma^2 B} = C_2 |B| \left( \frac{m-1}{mt} \right) \|u - c\|_{m,\rho B}
\]
where $\rho = \sigma^2$. Substituting (4.3) into (4.2) we get
\[
\left( \int_B |du|^s w^\alpha dx \right)^{\frac{1}{s}} \leq C_3 \text{diam}(B)^{-1} |B| \left( \frac{m-1}{mt} \right) \|u - c\|_{m,\rho B} \left( \int_B w dx \right)^{\frac{\alpha}{s}}.
\]
Using Lemma 2.2 with $\frac{1}{m} = \frac{1}{s} + \frac{s-m}{sm}$ we obtain
\[
\|u - c\|_{m,\rho B} = \left( \int_{\rho B} |u - c|^m dx \right)^{\frac{1}{m}} \\
= \left( \int_{\rho B} (|u - c| w^{\frac{\alpha}{s}} w^{-\frac{\alpha}{s}})^m dx \right)^{\frac{1}{m}} \\
\leq \left( \int_{\rho B} |u - c|^s w^\alpha dx \right)^{\frac{1}{s}} \left( \int_{\rho B} \left( \frac{1}{w} \right)^{\frac{1}{r-1}} dx \right)^{\frac{\alpha(r-1)}{s}}
\]
for all balls $B$ with $\rho B \subset \Omega$ and all closed forms $c$. Substituting (4.5) into (4.4) we obtain
\[
\left( \int_B |du|^s w^\alpha dx \right)^{\frac{1}{s}} \\
\leq C_3 \text{diam}(B)^{-1} |B| \left( \frac{m-1}{mt} \right) \|w\|_{1,B}^{\frac{s}{m}} \left( \frac{1}{w} \right) \left( \frac{1}{1+\alpha(r-1)} \right) \|u - c\|_{m,\rho B} \left( \int_{\rho B} |u - c|^s w^\alpha dx \right)^{\frac{1}{s}}.
\]
Now $w \in A_r(\Omega)$ yields
\[
\|w\|_{1,B}^{\frac{s}{m}} \left( \frac{1}{w} \right) \left( \frac{1}{1+\alpha(r-1)} \right) \|u - c\|_{m,\rho B} \left( \int_{\rho B} |u - c|^s w^\alpha dx \right)^{\frac{1}{s}} \\
\leq C_4 |B|^{\frac{\alpha}{s}}.
\]
Combining (4.7) and (4.6) we find that

\[
\left( \int_B |du|^s w^\alpha dx \right)^{\frac{1}{s}} \leq \frac{C_5}{\text{diam}(B)} \left( \int_{\rho B} |u-c|^s w^\alpha dx \right)^{\frac{1}{s}}
\]

(4.8)

for all balls \(B\) with \(\rho B \subset \Omega\) and all closed forms \(c\). We have proved that (4.1) is true if 

\(0 < \alpha < 1\).

For the case \(\alpha = 1\), by Lemma 2.3 there exist constants \(\beta > 1\) and \(C_6 > 0\) such that

\[
\|w\|_{\beta,B} \leq C_6 \frac{|B|^{1-\beta}}{\beta} \|w\|_{1,B} \quad (4.9)
\]

for any cube or any ball \(B \subset \mathbb{R}^n\). Choose \(t = \frac{s\beta}{\beta - 1}\). Then \(1 < s < t\) and \(\beta = \frac{t}{t-s}\). Since

\(\frac{1}{s} = \frac{1}{t} + \frac{t-s}{st}\), by Lemma 2.2, Theorem C and (4.9) we have

\[
\left( \int_B |du|^s w dx \right)^{\frac{1}{s}} = \left( \int_B (|du|^{\frac{1}{t}})^s dx \right)^{\frac{1}{s}} \leq \left( \int_B |du|^{t} dx \right)^{\frac{1}{t}} \left( \int_B \left( w^{\frac{1}{t}} \right)^{st} dx \right)^{\frac{t-s}{st}} \leq C_7 \|du\|_{t,B} \|w\|_{\beta,B} \leq C_7 \|du\|_{t,B} \|w\|_{1,B} \leq C_7 \|u-c\|_{t,\sigma B} \|w\|_{\beta,B} \leq C_7 \|u-c\|_{t,\sigma B} \|w\|_{1,B} \leq C_7 \|u-c\|_{t,\sigma B} \|w\|_{1,B} \leq C_7 \|u-c\|_{t,\sigma B} \]

which is similar to (4.2). Now, choosing \(m = \frac{s}{r}\) and repeating the same procedure as the case \(0 < \alpha < 1\), we can also obtain (4.1) if \(\alpha = 1\). This ends the proof of Theorem 4.1.

Note that the parameter \(\alpha\) in Theorem 4.1 is any real number with \(0 < \alpha < 1\). Therefore, we can have different versions of the Caccioppoli-type inequality by choosing \(\alpha\) to be different values. For example, setting \(t = 1 - \alpha\) in Theorem 4.1 we obtain the following result.

**Corollary 4.2.** Let \(u \in D'(\Omega, \Lambda^l)\) \((l = 0, 1, \ldots, n)\) be an \(A\)-harmonic tensor in a domain \(\Omega \subset \mathbb{R}^n\) and \(\rho > 1\). Assume that \(1 < s < \infty\) is a fixed exponent associated with the \(A\)-harmonic equation and \(w \in A_r(\Omega)\) for some \(r > 1\). Then there exists a constant \(C > 0\), independent of \(u\), such that

\[
\left( \int_B |du|^s w^{-t}d\mu \right)^{\frac{1}{s}} \leq \frac{C}{\text{diam}(B)} \left( \int_{\rho B} |u-c|^s w^{-t}d\mu \right)^{\frac{1}{s}}
\]

(4.10)

for all balls \(B\) with \(\rho B \subset \Omega\) and all closed forms \(c\). Here \(t\) is any real number with \(0 \leq t < 1\) and \(d\mu = w(x) dx\).

Choosing \(\alpha = \frac{1}{r}\) in Theorem 4.1 we have the following result.
**Corollary 4.3.** Let \( u \in D'(\Omega, \Lambda^l) \) \((l = 0, 1, \ldots, n)\) be an \( A \)-harmonic tensor in a domain \( \Omega \subset \mathbb{R}^n \) and \( \rho > 1 \). Assume that \( 1 < s < \infty \) is a fixed exponent associated with the \( A \)-harmonic equation and \( w \in A_r(\Omega) \) for some \( r > 1 \). Then there exists a constant \( C > 0 \), independent of \( u \), such that

\[
\left( \int_B |du|^s w^{\frac{1}{r}} dx \right)^{\frac{1}{s}} \leq \frac{C}{\text{diam}(B)} \left( \int_{\rho B} |u-c|^s w^{\frac{1}{r}} dx \right)^{\frac{1}{s}} \tag{4.11}
\]

for all balls \( B \) with \( \rho B \subset \Omega \) and all closed forms \( c \).

If we choose \( \alpha = \frac{1}{s} \) in Theorem 4.1, then \( 0 < \alpha < 1 \) since \( 1 < s < \infty \). Thus, Theorem 4.1 reduces to the following symmetric version.

**Corollary 4.4.** Let \( u \in D'(\Omega, \Lambda^l) \) \((l = 0, 1, \ldots, n)\) be an \( A \)-harmonic tensor in a domain \( \Omega \subset \mathbb{R}^n \) and \( \rho > 1 \). Assume that \( 1 < s < \infty \) is a fixed exponent associated with the \( A \)-harmonic equation and \( w \in A_r(\Omega) \) for some \( r > 1 \). Then there exists a constant \( C > 0 \), independent of \( u \), such that

\[
\left( \int_B |du|^s w^{\frac{1}{r}} dx \right)^{\frac{1}{s}} \leq \frac{C}{\text{diam}(B)} \left( \int_{\rho B} |u-c|^s w^{\frac{1}{r}} dx \right)^{\frac{1}{s}} \tag{4.12}
\]

for all balls \( B \) with \( \rho B \subset \Omega \) and all closed forms \( c \).

If we choose \( \alpha = 1 \) in Theorem 4.1, we have the following result.

**Corollary 4.5.** Let \( u \in D'(\Omega, \Lambda^l) \) \((l = 0, 1, \ldots, n)\) be an \( A \)-harmonic tensor in a domain \( \Omega \subset \mathbb{R}^n \) and \( \rho > 1 \). Assume that \( 1 < s < \infty \) is a fixed exponent associated with the \( A \)-harmonic equation and \( w \in A_r(\Omega) \) for some \( r > 1 \). Then there exists a constant \( C > 0 \), independent of \( u \), such that

\[
\|du\|_{s, B, w} \leq C \text{diam}(B)^{-1} \|u-c\|_{s, \rho B, w} \tag{4.13}
\]

or

\[
\left( \int_B |du|^s d\mu \right)^{\frac{1}{s}} \leq \frac{C}{\text{diam}(B)} \left( \int_{\rho B} |u-c|^s d\mu \right)^{\frac{1}{s}} \tag{4.14}
\]

for all balls \( B \) with \( \rho B \subset \Omega \) and all closed forms \( c \).

Finally, we prove the following global \( A_r(\Omega) \)-weighted Caccioppoli-type estimate for \( A \)-harmonic tensors.

**Theorem 4.6.** Let \( u \in D'(\Omega, \Lambda^l) \) \((l = 0, 1, \ldots, n)\) be an \( A \)-harmonic tensor in a bounded domain \( \Omega \subset \mathbb{R}^n \) which has a finite open cover \( \mathcal{V} = \{B_1, B_2, \ldots, B_m\} \) consisting of open balls. Assume that \( 1 < s < \infty \) is a fixed exponent associated with the \( A \)-harmonic equation and \( w \in A_r(\cup_i B_i) \) for some \( r > 1 \). Then there exists a constant \( C > 0 \), independent of \( u \), such that

\[
\left( \int_{\Omega} |du|^s w^\alpha dx \right)^{\frac{1}{s}} \leq \frac{C}{\text{diam}(\Omega)} \left( \int_{\Omega} |u-c|^s w^\alpha dx \right)^{\frac{1}{s}} \tag{4.15}
\]
for all closed forms $c$ and any constant $\alpha$ with $0 < \alpha \leq 1$.

**Proof.** Let $V = \{B_1, B_2, \ldots, B_m\}$ be an open cover of the bounded domain $\Omega \subset \mathbb{R}^n$ and $d_i = \text{diam}(B_i) > 0 \ (i = 1, 2, \ldots, m)$. Assume that $d = \min\{d_1, d_2, \ldots, d_m\}$. Since $\Omega$ is bounded, then there exists a constant $C_1 > 0$ such that

$$\frac{1}{d} \leq \frac{C_1}{\text{diam}(\Omega)}. \quad (4.16)$$

Using (4.16) and Theorem 4.1, we have

$$\left(\int_\Omega |du|^s w^\alpha dx\right)^{\frac{1}{s}} \leq \sum_{B \in V} \left(\int_B |du|^s w^\alpha dx\right)^{\frac{1}{s}} \leq \sum_{B \in V} \frac{C_2}{\text{diam}(B)} \left(\int_{\rho B} |u - c|^s w^\alpha dx\right)^{\frac{1}{s}} \leq \sum_{B \in V} \frac{C_2}{d} \left(\int_{\rho B} |u - c|^s w^\alpha dx\right)^{\frac{1}{s}} \leq \sum_{B \in V} \frac{C_3}{\text{diam}(\Omega)} \left(\int_\Omega |u - c|^s w^\alpha dx\right)^{\frac{1}{s}} \leq \frac{C_4}{\text{diam}(\Omega)} \left(\int_\Omega |u - c|^s w^\alpha dx\right)^{\frac{1}{s}}.$$  

Hence (4.15) follows. The proof of Theorem 4.6 has been completed.

**Remark.** Choosing $\alpha$ to be some special values in (4.15), we shall have some corresponding global results.

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**References**


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