Abstract. In view of closed graph theorems in case of maps defined by operator-valued matrices $L_\varphi$-spaces were recently introduced by two of the present authors as a generalization of separable $FK(X)$-spaces. In this paper we study the class of $L_\varphi$-spaces and a few closely related classes of sequence spaces. It is shown that an analogue of Kalton's closed graph theorem holds for matrix mappings if we consider $L_\varphi$-spaces as range spaces, and paralleling a result of Qiu we prove that the class of $L_\varphi$-spaces is the best-possible choice here. As a consequence we show that for any $L_\varphi$-space $E$ every matrix domain $E_A$ is again an $L_\varphi$-space.

Keywords: Matrix mappings, closed graph theorems, $L_\varphi$-spaces, $L_r$-spaces

1. Introduction

Let $E$ and $F$ be locally convex spaces and suppose that $E$ is a Mackey space, the space $(E',\sigma(E',E))$ is sequentially complete and $F$ is separable and $B_r$-complete. Then Kalton's closed graph theorem \cite{Kalton} states that every closed linear map $T : E \to F$ is continuous. Subsequently, Qiu \cite{Qiu} has identified the maximal class of range spaces $F$ in this result, calling its elements $L_r$-spaces.

Kalton's theorem was successfully applied in classical summability theory to obtain inclusion theorems for $K$-spaces that are important in connection with Mazur-Orlicz-type theorems (cf. \cite{Mazur-Orlicz}). In these applications $F$ is a convergence domain $c_A$ of some matrix $A$, which is always a separable Fréchet space. However, if one tries to extend these results to operator-valued matrices one encounters the problem that convergence domains are no longer separable in general. In fact, they need not even be $L_r$-spaces \cite{Boos-Grosse-Erdmann-Leiger}.
Thus a new idea was needed. Now, in summability theory one usually deals with matrix mappings between sequence spaces, which ordinarily are particular closed mappings. In a recent paper two of the present authors were able to show that if we only consider matrix mappings, then a Kalton-type result obtains for all spaces $F$ from a new class of spaces, which they call $L_\varphi$-spaces (see [8: Theorem 4.2]). As desired, this class is large enough to contain all convergence domains of operator-valued matrices, so that one can now deduce inclusion theorems for such matrices [8: Theorem 4.4].

In this paper we study the class of $L_\varphi$-spaces and a few closely related classes of sequence spaces. We show that, indeed, Kalton's theorem and Qiu's characterization hold for $L_\varphi$-spaces if closed mappings are replaced by matrix mappings. It is also shown that for every $L_\varphi$-space $E$ any matrix domain $E_A$ is again an $L_\varphi$-space, answering a question in [8]. Similar results are proved for the other classes of sequence spaces considered here. For further investigations into $L_\varphi$-spaces see [6].

2. Notations and preliminaries

Throughout this paper we assume that $(X, \tau_X)$ and $(Y, \tau_Y)$ are (locally convex) Fréchet spaces. A sequence space (over $X$) is a subspace of the space $\omega(X)$ of all sequences $x = (x_k)$ in $X$. In particular, $c(X)$ and $\varphi(X)$ denote the spaces of convergent and finite sequences in $X$, respectively. The $\beta$-dual $E^\beta$ of a sequence space $E$ over $X$ is defined as

$$E^\beta = \{ (A_k) \in \omega(X') \mid \forall (x_k) \in E : \sum_k A_k(x_k) \text{ converges} \}.$$ 

Now suppose that the sequence space $E$ over $X$ is endowed with a locally convex topology $\tau$. Then $E$ is called a $K(X)$-space if the inclusion map $i : E \to \omega(X)$ is continuous, where $\omega(X)$ carries the product topology. If, in addition, $(E, \tau)$ is a Fréchet (Banach) space, then $E$ is called an $FK(X)$-space ($BK(X)$-space). A $K(X)$-space $E$ is called an $AK$-space ($SAK$-space) if $(x_1, \ldots, x_n, 0, \ldots) \to x$ (weakly) in $E$ as $n \to \infty$ for all $x = (x_k) \in E$. If $E$ is a $K(X)$-space, then every element $(A_k) \in E^\beta$ defines a linear functional on $E$ via $(x_k) \to \sum_k A_k(x_k)$. Hence, as usual, we can consider $E^\beta$ as a subspace of $E^*\cap E^*$, the algebraic dual of $E$. In particular we have $\varphi(X') \subset E^*\cap E^*$.

Let $A = (A_{nk})$ be a matrix with entries $A_{nk} \in B(X, Y)$, i.e., continuous linear operators $A_{nk} : X \to Y$. $A$ is called row-finite if each sequence $(A_{nk})_k (n \in \mathbb{N})$ is finite. For a sequence space $E$ over $Y$ the matrix domain $E_A$ is defined as

$$E_A = \{ x \in \omega(X) \mid \forall n \in \mathbb{N} : \sum_k A_{nk}(x_k) \text{ converges and } \left( \sum_k A_{nk}(x_k) \right)_n \in E \}.$$ 

Here, the convergence of $\sum_k A_{nk}(x_k)$ is taken in the topology $\tau_Y$. If, instead, we only require convergence with respect to $\sigma(Y, Y')$, then the corresponding sequence space is called a weak matrix domain, denoted by $E_{A_w}$. For any $x \in E_{A_w}$ we put $Ax := (\sum_k A_{nk}(x_k))_n$. If $F$ is a sequence space over $X$ with $F \subset E_A (F \subset E_{A_w})$, then the
mapping $A : F \to E, x \mapsto Ax$, is called a (weak) matrix mapping. The space $\omega(Y)_A$ is an $FK(X)$-space by [5: Theorem 2.14], and the matrix domain $E_A$ becomes an $FK(X)$-space when it is endowed with the strongest topology that makes the matrix mappings $A : E_A \to E, x \mapsto Ax$ and $i : E_A \to \omega(Y)_A, x \mapsto x$ continuous [1: Proposition 2.4].

The terminology from the theory of locally convex spaces is standard. We follow Wilansky [12]. For the theory of $FK(X)$-spaces and operator-valued matrix domains we refer to [1] and [5].

3. $L_\varphi$–$K$–spaces and some related $K$–spaces

Let $(E, \tau)$ be a locally convex space with topological dual $E'$ and algebraic dual $E^*$. For any subspace $S$ of $E^*$, $S < E^*$, we use the notations

$$S := \{ g \in E^* \mid \exists (g_n) \text{ in } S : g_n \to g (\sigma(E^*, E)) \},$$

$$S := \bigcap \{ V < E^* \mid S \subset V = \overline{V} \},$$

$$S^j := S \cap E' \quad \text{and} \quad S^{j+1} := S^j = S \cap E' \quad (j \in \mathbb{N}).$$

Following J. Qiu [11] we define $E$ to be an $L_\tau$–space if $E' \subset S$ for any $\sigma(E', E)$-dense subspace $S$ of $E'$.

In case of $K(X)$–spaces $E$ we note that $\varphi(X')$ is $\sigma(E', E)$–dense in $E'$ [8: Theorem 3.4] and introduce the following notations (see also [8]).

**Definition and Remarks 3.1.** Let $E$ be a $K(X)$–space and $j \in \mathbb{N}$. $E$ is called an

- $L_\varphi$–space if $E' \subset \overline{\varphi(X')}$
- $L_\varphi(j)$–space if $E' \subset \overline{\varphi(X')^j}$
- $L_\beta(j)$–space if $E' \subset \overline{E'^j}$.

In [8] $L_\varphi(1)$–spaces and $L_\beta(1)$–spaces are called spaces having $\varphi$–sequentially dense dual and $\beta$–sequentially dense dual, respectively.

$E$ is an $L_\varphi$–space if and only if $E' \subset \overline{E^\beta}$ since $E^\beta \subset \overline{\varphi(X')}$. In fact, we even have $E^\beta \subset \overline{\varphi(X')}$. The above definitions depend only on the dual pair $(E, E')$ and not on the particular topology compatible with this dual pair. Obviously, for each $j \in \mathbb{N}$ we have

$$L_\varphi(j)$–space $\Rightarrow L_\beta(j)$–space $\Rightarrow L_\varphi(j+1)$–space $\Rightarrow L_\varphi$–space $\Leftarrow L_\tau$–space.
Remarks 3.2. Let $E$ be any sequence space over $X$ and let $H$ with $\varphi(X') < H < \overline{E'}$ be given.

(a) Then $(E, \tau(E, H))$ is an $L_\varphi$-space. (The proof of the Inclusion Theorem in [7] shows us that we may be interested in $L_\varphi$-spaces $(E, \tau(E, H))$ where $H$ is a very small subspace of $\overline{E'}$ containing $\varphi(X')$.)

(b) The statement in (a) remains true for any topology $\tau$ (instead of $\tau(E, H)$) that is compatible with the dual pair $(E, H)$.

(c) Obviously, $\tau(E, \overline{E'})$ is the strongest locally convex topology $\tau$ such that $(E, \tau)$ is an $L_\varphi$-space.

(d) If $j \in \mathbb{N}$ and $\tau$ is any topology that is compatible with the dual pair $(E, H)$ such that $(E, \tau)$ is an $L_\varphi(j)$-space ($L_\varphi(j)$-space), then $(E, \tau(E, H))$ is an $L_\varphi(j+1)$-space ($L_\varphi(j+1)$-space).

Examples 3.3. (a) Each separable $F(K(X))$-space, more generally each $\text{subWCG}$-$F(K(X))$-space, is an $L_\varphi$-space (see [8: Theorem 3.3]). Here, a $\text{subWCG}$-space is a (topological) subspace of a weakly compactly generated locally convex space.

(b) Every $SAK$-$K(X)$-space, in particular every $AK$-$K(X)$-space, is an $L_\varphi(1)$-space.

(c) The $BK(m)$-space $c(m)$ is an $L_\varphi(1)$-space, however, in general it is not separable and no $L_\varphi$-space. (See [8: Example 3.13/(b)].)

(d) Based on an example of P. Erdős and G. Piranian [9] in [8: Example 3.12] a regular (real-valued) matrix $A$ is given such that the domain $c_A$ is an $L_\varphi(1)$-space but no $L_\varphi$-space. In Remark 4.2 below we will give an example of an $L_\varphi(2)$-space that is no $L_\varphi(1)$-space. We do not know if the $L_\varphi(2)$-spaces and the $L_\varphi(2)$-spaces coincide.

The following result will be needed in the next section. For sake of brevity we put $\overline{S} = S$ (not to be confused with the polar of $S'$).

**Proposition 3.4.** Let $E$ and $F$ be locally convex spaces, $U < E^*$, $S < F^*$ and $i, j \in \mathbb{N}_0$. Let $T : E \to F$ be a continuous linear mapping such that

$$f \circ T \in \overline{U^i} \quad \text{whenever} \quad f \in \overline{S^i}.$$ 

Then

$$g \circ T \in \overline{U^{i+j}} \quad \text{whenever} \quad g \in \overline{S^j}.$$ 

**Proof.** We can assume $j > 0$. Let $g \in \overline{S^j}$. Then there are elements $f_{\nu_{i+1}, \ldots, \nu_{i+j}} \in S \cap F'$ for $\nu_{i+1}, \ldots, \nu_{i+j} \in \mathbb{N}$ such that:

(a) For $i + 1 \leq \rho < i + j$ and all $\nu_{i+1}, \ldots, \nu_{i+j} \in \mathbb{N}$ the mappings

$$y \to \lim_{\nu_{i+1} \to \nu_{i+1}} \ldots \lim_{\nu_{i+j} \to \nu_{i+j}} f_{\nu_{i+1}, \ldots, \nu_{i+j}}(y) \quad (y \in F)$$
exist and belong to $F'$. 

(b) For all $y \in F$ we have

$$g(y) = \lim_{\nu_{i+1}} \lim_{\nu_{i+2}} \ldots f_{\nu_{i+1}, \ldots, \nu_{i+j}}(y).$$

From our assumption we know that $f_{\nu_{i+1}, \ldots, \nu_{i+j}} \circ T \in \overline{U}$ for each $\nu_{i+1}, \ldots, \nu_{i+j} \in \mathbb{N}$. This implies that there are elements $g_{\nu_{i+1}, \ldots, \nu_{i+j}} \in U \cap E'$ for $\nu_{i+1}, \ldots, \nu_{i+j} \in \mathbb{N}$ such that:

(c) For $1 \leq \sigma < i$ and all $\nu_{i+1}, \ldots, \nu_{i+j} \in \mathbb{N}$ the mappings

$$x \rightarrow \lim_{\nu_{i+1}} \ldots \lim_{\nu_{i+j}} g_{\nu_{i+1}, \ldots, \nu_{i+j}}(x) \quad (x \in E)$$

exist and belong to $E''$.

(d) For all $x \in E$ and $\nu_{i+1}, \ldots, \nu_{i+j} \in \mathbb{N}$ we have

$$(f_{\nu_{i+1}, \ldots, \nu_{i+j}} \circ T)(x) = \lim_{\nu_{i+1}} \ldots \lim_{\nu_{i+j}} g_{\nu_{i+1}, \ldots, \nu_{i+j}}(x).$$

We thus have found elements $g_{\nu_{i+1}, \ldots, \nu_{i+j}} \in U \cap E'$ for $\nu_{i+1}, \ldots, \nu_{i+j} \in \mathbb{N}$ with the following properties:

(a') For $1 \leq \rho < i + j$ and all $\nu_{\rho+1}, \ldots, \nu_{i+j} \in \mathbb{N}$ the mappings

$$x \rightarrow \lim_{\nu_{\rho+1}} \ldots \lim_{\nu_{i+j}} g_{\nu_{\rho+1}, \ldots, \nu_{i+j}}(x) \quad (x \in E)$$

exist and belong to $E''$ (this is just (c) in case $\rho < i$; for $\rho = i$ it follows from (d) and for $\rho > i$ from (a) if we note that $T$ is continuous).

(b') For all $x \in E$ we have

$$(g \circ T)(x) = \lim_{\nu_{i+1}} \ldots \lim_{\nu_{i+j}} g_{\nu_{i+1}, \ldots, \nu_{i+j}}(x)$$

(this follows from (b) and (d)).

But (a') and (b') together imply that $g \circ T \in \overline{U}$. 

**Remark 3.5.** Using the adjoint $T' : F' \rightarrow E'$ of the mapping $T$, the assertion of the proposition can be put more concisely as

$$T' \left( \overline{S} \right) \subset \overline{U} \quad \text{implies} \quad T' \left( \overline{S'} \right) \subset \overline{U}^{i+j}.$$

4. **Domains of operator–valued matrices**

From [8: Theorems 3.9 and 3.10] it is known that the domain $c(Y)_A$ of an operator–valued matrix $A$ is an $L_\beta(1)$–space, and that $E_A$ is an $L_\sigma$–space whenever $E$ is an $L_\beta(1)$–space. Here we are going to improve these results.
Theorem 4.1. Let $E$ be a $K(Y)$-space, $A = (A_{nk})$ a matrix with $A_{nk} \in B(X,Y)$ and let $j \in \mathbb{N}$.

(a) If $E$ is an $L_{\varphi}(j)$-space, then $E_A$ is an $L_{\varphi}(j)$-space.

(b) If $E$ is an $L_\beta(j)$-space, then $E_A$ is an $L_\beta(j+1)$-space.

Suppose that in addition $A$ is row-finite. Then:

(a') If $E$ is an $L_{\varphi}(j)$-space, then $E_A$ is an $L_{\varphi}(j)$-space.

(b') If $E$ is an $L_\beta(j)$-space, then $E_A$ is an $L_{\varphi}(j+1)$-space.

Special case (see [8: Theorem 3.9]): $c(Y)_A$ is an $L_1(1)$-space, and even an $L_\infty(1)$-space if $A$ is row-finite.

Remark 4.2. Example 3.3/(d) tells us that, in general, we cannot replace ' $L_\beta(j)$-space' by ' $L_{\varphi}(j)$-space' in statement (a). Assertion (a') is obviously best-possible, while in statement (b') we cannot replace ' $L_{\varphi}(j+1)$-space' by ' $L_\beta(j)$-space' in general: In [8: Example 3.14] there is an example of a (real-valued) row-finite matrix $A$ and an $L_\beta(1)$-space $E$ such that the domain $E_A$ is no $L_\beta(1)$-space. (From statement (b') above we see that it is an $L_{\varphi}(2)$-space.) We do not know if one can replace ' $L_\beta(j+1)$-space' in statement (b) by ' $L_{\varphi}(j+1)$-space'.

Proof of Theorem 4.1. Let $E$ be a $K(Y)$-space, and let $f \in E_A'$ be given. Then we may choose elements $g \in E'$ and $h \in \omega(Y)_A = \omega(Y)'_A$ with $f = g \circ A + h$ (see [1: Proposition 2.10] and [5: Theorem 2.14/(b)]). Since $E_A \subseteq \omega(Y)_A$, we have $h \in E_A^0 \subseteq \varphi(X)' \subseteq E_A^0$ for all $j \in \mathbb{N}$. Hence in order to prove the various statements of the theorem we need only show that $g \circ A$ belongs to $E_A^0$, $E_A^j$, $\varphi(X)'$ and $\varphi(X)'^j$, respectively. To this end we apply Proposition 3.4 to the mapping $A : E_A \rightarrow E$.

(a) Let $E$ be an $L_{\varphi}(j)$-space. Then $g \in E' \subseteq \varphi(Y)'$. Here we choose $U = E_A^0$, $S = \varphi(Y')$ and $i = 0$. If $\Phi = (\Phi_n)_{n=1}^{\infty} \in \varphi(Y)'$, then we have for $x \in E_A$

$$
(\Phi \circ A)(x) = \sum_{n=1}^{\infty} \Phi_n \left( \sum_{k=1}^{\infty} A_{nk}(x_k) \right) = \sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \Phi_n \circ A_{nk} \right)(x_k)
$$

so that $\Phi \circ A \in E_A^0$. Hence the hypothesis of Proposition 3.4 holds, so that $g \circ A \in E_A^0$, as desired.

(b) Let $E$ be an $L_\beta(j)$-space. Then $g \in E' \subseteq E_A^j$. Here we choose $U = E_A^0$, $S = E^0$ and $i = 1$. If $\Phi = (\Phi_n) \in E^0$, then we have for $x \in E_A$

$$
(\Phi \circ A)(x) = \lim_{m \to \infty} \sum_{n=1}^{m} \Phi_n \left( \sum_{k=1}^{\infty} A_{nk}(x_k) \right) = \lim_{m \to \infty} \sum_{k=1}^{\infty} \left( \sum_{n=1}^{m} \Phi_n \circ A_{nk} \right)(x_k)
$$

so that $\Phi \circ A \in E_A^j$. Proposition 3.4 implies that $g \circ A \in E_A^j$.

Now suppose that $A$ is row-finite.
(a') Let $E$ be an $L_\varphi(j)$-space. Then $g \in E' \subset \varphi(Y')$. Here we choose $U = \varphi(X')$, $S = \varphi(Y')$ and $i = 0$. If $\Phi = (\Phi_n)_{n=1}^{N} \in \varphi(Y')$, then we have for $z \in E_A$

$$
(\Phi \circ A)(z) = \sum_{k=1}^{\infty} \left( \sum_{n=1}^{N} \Phi_n \circ A_{nk} \right)(x_k)
$$

and hence $\Phi \circ A \in \varphi(X')$. Now Proposition 3.4 implies that $g \circ A \in \varphi(X')$.

(b') This follows from statement (a') since every $L_\varphi(j)$-space is also an $L_\varphi(j+1)$-space.

5. Matrix maps into $L_\varphi-K$-spaces

The aim of this section is to show that the class of $L_\varphi$-spaces is the complete analogue of Qiu's $L_r$-spaces if closed linear mappings are replaced by matrix mappings. We also prove that the matrix domain $E_A$ of an operator-valued matrix is an $L_\varphi$-space whenever $E$ is an $L_\varphi$-space. This result may be considered as a generalization of the classical fact that the matrix domain $E_A$ of a scalar-valued matrix is separable if $E$ is a separable $FK$-space.

Our first result is the analogue for matrix mappings of Qiu's extension of Kalton's closed graph theorem. It generalizes the results in Theorem 4.2 and Theorem 4.4/(a) $\Rightarrow$ (b) of [8].

**Theorem 5.1.** Let $E$ be a $K(X)$-space and $F$ a $K(Y)$-space. If $E$ is a Mackey space, $(E', \sigma(E', E))$ is sequentially complete and $F$ is an $L_\varphi$-space, then every (weak) matrix mapping $A : E \to F$ is continuous.

**Proof.** We put

$$
D^*_A := (A')^{-1}(E') = \{ f \in F^* : f \circ A \in E' \}
$$

and $D_A := D^*_A \cap F'$. If we can show that $D_A = F'$, then $A$ is weakly continuous hence continuous as $E$ is a Mackey space.

To that end let $f \in F^*$ and $(f_n)$ in $F^*$ with $f_n \circ A \in E'$ and $f_n \to f$ in $(F^*, \sigma(F^*, F'))$ be given. Then we have $f_n \circ A \to f \circ A$ in $(E^*, \sigma(E^*, E))$. Since $(E', \sigma(E', E))$ is sequentially complete, this shows that $f \circ A \in E'$, so that $f \in D^*_A$. Thus $D^*_A$ is $\sigma(F^*, F')$-sequentially closed, which implies that $D^*_A \subset D^*_A$, hence $D^*_A \cap F' = D_A$.

We next show that $\varphi(Y') \subset D_A$. For this it suffices to prove that for each $g \in Y'$ and $n \in \mathbb{N}$ the mapping $x \mapsto g(\sum_{k=1}^{\infty} A_{nk}(x_k))$ belongs to $E'$. But since we have

$$
g \left( \sum_{k=1}^{\infty} A_{nk}(x_k) \right) = \lim_{m} \sum_{k=1}^{m} (g \circ A_{nk})(x_k)
$$

for all $x \in E$, this follows from the weak sequential completeness of $E'$. 

In conclusion, \( \varphi(Y') \subset D_A \) and the fact that \( F \) is an \( L_\varphi \)-space imply that

\[ F' = \varphi(Y') \cap F' \subset D_A \cap F' = D_A, \]

which had to be shown.

**Remark 5.2.** The proof shows that the theorem remains true for any linear mapping \( A = (A_n) : E \to F \) with the property that \( \varphi(Y') \subset D_A \), which is equivalent to the continuity of each mapping \( A_n : E \to Y \) \((n \in \mathbb{N})\).

The next result is the analogue to Qiu's characterization of \( L_r \)-spaces [11]. It shows that the class of \( L_\varphi \)-spaces is the maximal class of range spaces in Theorem 5.1.

**Theorem 5.3.** Let \( F \) be a \( K(X) \)-space. Then the following statements are equivalent:

(a) \( F \) is an \( L_\varphi \)-space.

(b) For each \( K(X) \)-space \( E \) that is a Mackey space such that \((E',\sigma(E',E)) \) is sequentially complete every matrix mapping \( A : E \to F \) is continuous.

**Proof.** The implication (a) \( \Rightarrow \) (b) is contained in Theorem 5.1. The converse implication follows immediately from the following remark.

**Remark 5.4.** Let \( F \) be a \( K(X) \)-space. If the inclusion map

\[ i : \left( F, \tau(\varphi(Y')) \right) \to F \]

is continuous, then \( F \) is an \( L_\varphi \)-space. (Namely, in this situation we have \( F' \subset F^\beta = \varphi(Y') \).)

Using the last remark we can now obtain a permanence result for \( L_\varphi \)-spaces under the formation of matrix domains, answering a question in [8].

**Theorem 5.5.** Let \( A = (A_{nk}) \) be a matrix with \( A_{nk} \in B(X,Y) \). If \( E \) is an \( L_\varphi \)-\( K(Y) \)-space, then \( E_A \) is an \( L_\varphi \)-\( K(X) \)-space.

**Proof.** By Remark 5.4 we have to prove the continuity of

\[ i : \left( E_A, \tau\left( E_A, \overline{E_A}^{\beta} \right) \right) \to E_A, \]

which is equivalent to the continuity of the inclusion map

\[ i_\omega : \left( E_A, \tau\left( E_A, \overline{E_A}^{\beta} \right) \right) \to \omega(Y)_A \]

and of the map

\[ A : \left( E_A, \tau\left( E_A, \overline{E_A}^{\beta} \right) \right) \to E, \quad x \to Ax. \]

However, since in both cases the range space is an \( L_\varphi \)-space (note that \( \omega(Y)_A \) is an \( AK \)-space by [5: Theorem 2.14]), this is an immediate corollary of Theorem 5.1.
References


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