A weak type bound for a singular integral

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Abstract. A weak type $(1, 1)$ estimate is established for the first order $d$-commutator introduced by Christ and Journé, in dimension $d \geq 2$.

1. Introduction

Let $K$ be regular Calderón–Zygmund convolution kernel on $\mathbb{R}^d$, $d \geq 2$, i.e. $K \in \mathcal{S}'$, is locally bounded in $\mathbb{R}^d \setminus \{0\}$ and satisfies

\begin{equation}
|K(x)| \leq A |x|^{-d} \quad x \neq 0,
\end{equation}

and, for some $\epsilon \in (0, 1],

\begin{equation}
|K(x+h) - K(x)| \leq A |h|^\epsilon |x|^{-d-\epsilon} \quad \text{if } |x| > 2|h|;
\end{equation}

moreover

$$\|\hat{K}\|_\infty \leq A < \infty.$$ 

Let $a \in L^\infty(\mathbb{R}^d)$. The so-called $d$-commutator $T \equiv T[a]$ of first order associated with $K$ and $a$ is defined for Schwartz functions $f$ by

$$T[a]f(x) = p.v. \int K(x-y) \int_0^1 a(sx + (1-s)y) \, ds \, f(y) \, dy.$$ 

In dimensions $d \geq 2$ this definition yields a rough analog of the Calderón commutator [1] in one dimension. Christ and Journé [3] proved that $T$ and higher order versions extend to bounded operators on $L^p(\mathbb{R}^d)$, for $1 < p < \infty$. We prove that the first order $d$-commutator is also of weak type $(1, 1)$.

Theorem 1.1. There is $C_d < \infty$ so that for any $f \in L^1(\mathbb{R}^d)$ and any $a \in L^\infty(\mathbb{R}^d),

$$\sup_{\lambda > 0} \lambda \text{meas}\left(\{x \in \mathbb{R}^d : |T[a]f(x)| > \lambda\}\right) \leq C_d A^{\frac{1}{\epsilon}} \log(\frac{2}{\epsilon}) \|a\|_\infty \|f\|_{L^1(\mathbb{R}^d)}.$$ 

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In dimension two this result has recently been established by Grafakos and Honzík [6] (assuming \( \varepsilon = 1 \)). Their approach relies on a method developed in [2], [4] and [7] for proving a weak type \((1,1)\) bound for rough singular convolution operators. A dyadic decomposition \( T[a] = \sum T_j \) is used on the kernel side, and the argument relies on the fact that in dimension two the kernels of the operators \( T_j T_i \) have certain Hölder continuity properties. This argument is no longer valid for higher dimension. It is conceivable that for \( d \geq 3 \) one might be able to develop the more complicated iterated \( T^j T^i \) arguments introduced by Christ and Rubio de Francia [4] and further extended by Tao [11], but this route would lead to substantial technical difficulties and we shall not pursue it. Our approach is different and relies on an idea introduced in [8]. An orthogonality argument for a microlocal decomposition of the operator is used. The implementation of this idea in the present setting is more complicated in the convolution case as the Christ–Journé operators can be viewed as an amalgam of operators of generalized convolution type (for which there is a suitable calculus of wavefront sets) and operators of multiplication with a rough function.

**Notation.** We write \( E_1 \lesssim E_2 \) to indicate that \( E_1 \leq C E_2 \) for some "constant" \( C \) that may depend on \( d \). We also use the notation \( \lesssim_N \) to indicate dependence on other parameters \( N \). We denote by \( \hat{f} \) or \( \mathcal{F} f \) the Fourier transform of \( f \), defined for Schwartz functions by \( \hat{f}(\xi) = \int f(y) e^{-i(y,\xi)} dy \).

**This paper.** In §2 we outline the proof of Theorem 1.1 with the three technical propositions 2.2, 2.3, 2.4 proved in §3, §4, §5, respectively. In §6 we shall mention some open problems.

### 2. Decompositions and auxiliary estimates

We may assume that \( A \leq 1, \|a\|_{\infty} \leq 1 \) and write \( T = T[a] \). Fix \( f \in L^1(\mathbb{R}^d) \).

We use the standard Calderón–Zygmund decomposition of \( f \) at height \( \lambda \) (see [10]). Then

\[
  f = g + b = g + \sum_{Q \in \Omega_\lambda} b_Q
\]

where \( \|g\|_{\infty} \leq \lambda, \|g\|_1 \lesssim \|f\|_1 \), each \( b_Q \) is supported in a dyadic cube \( Q \) with sidelength \( 2^{L(Q)} \) and center \( y_Q \), and \( \Omega_\lambda \) is a family of dyadic cubes with disjoint interiors. Moreover \( \|b_Q\|_1 \lesssim \lambda |Q| \) for each \( Q \in \Omega_\lambda \) and \( \sum_{Q \in \Omega_\lambda} |Q| \lesssim \lambda^{-1} \|f\|_1 \).

For each \( Q \) let \( Q^* \) be the dilate of \( Q \) with same center and \( L(Q^*) = L(Q) + 10 \), and let \( E = \bigcup_{Q \in \Omega_\lambda} Q^* \). Then also

\[
  \text{meas}(E) \lesssim \lambda^{-1} \|f\|_1.
\]

Finally, for each \( Q \), the mean value of \( b_Q \) vanishes:

\[
  \int b_Q(y) \, dy = 0.
\]
Since $T$ is bounded on $L^2$ (cf. [3]) we have, as in standard Calderón–Zygmund theory,

$$\|Tg\|_2^2 \leq \|T\|_{L^2 \to L^2} \|g\|_2 \lesssim \|g\|_1 \|g\|_\infty \lesssim \lambda \|g\|_1,$$

the estimate for the good function $g$. By Chebyshev’s inequality,

$$\left| \{ x \in \mathbb{R}^d : |Tg(x)| > \lambda / 10 \} \right| \leq 100 \lambda^{-2} \|Tg\|_2^2 \lesssim \lambda^{-1} \|g\|_1 \lesssim \lambda^{-1} \|f\|_1.$$

We use a dyadic decomposition of the kernel. Let $\varphi$ be a radial $C^\infty$ function, so that $\varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq 6/5$. Let $K_j(x) = (\varphi(2^{-j}x) - \varphi(2^{-j+1}x))K(x)$ so that $K = \sum K_j$ in the sense of distributions on $\mathbb{R}^d \setminus \{0\}$ and $K_j$ is supported in the annulus $\{ x : 2^{j-1} \leq |x| \leq \frac{6}{5} 2^j \}$. Let $T_j$ be the integral operator with Schwartz kernel

$$K_j(x-y) \int_0^1 a(sx + (1-s)y) \, ds.$$

For $m \in \mathbb{Z}$ let

$$B_m = \sum_{Q \in \mathcal{Q}_\lambda \atop L(Q) = m} b_Q.$$

Observe that for each $j$ and $m$ the function $T_j B_m$ belongs to $L^1$, and that $\text{supp}(T_j B_m) \subset E$, $m \geq j$.

Moreover, for each $n$,

$$\sum_j \|T_j B_{j-n}\|_1 \lesssim \|f\|_1$$

and thus, if

$$n(\epsilon) = 10^{10} \epsilon d^{-1} \log_2(2\epsilon^{-1})$$

we have by Chebyshev’s inequality

$$\text{meas} \left( \{ x \in \mathbb{R}^d : \sum_{0 < n \leq n(\epsilon)} \sum_j |T_j B_{j-n}(x)| > 10/10 \} \right) \lesssim \epsilon^{-1} \log(2\epsilon^{-1}) \lambda^{-1} \|f\|_1. \quad (2.1)$$

It thus suffices to show that $\sum_{n>n(\epsilon)} \left( \sum_j T_j B_{j-n} \right)$ converges in the topology of $(L^1 + L^2)(\mathbb{R}^d \setminus E)$ and satisfies the inequality

$$\text{meas} \left( \{ x \in \mathbb{R}^d \setminus E : \sum_{n>n(\epsilon)} \sum_j |T_j B_{j-n}(x)| > 4\lambda / 5 \} \right) \lesssim \lambda^{-1} \|f\|_1. \quad (2.2)$$
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Finer decompositions

We first slightly modify the kernel $K_j$ and subtract an acceptable error term which is small in $L^1$. In what follows assume $n > n(\varepsilon)$ as defined above. Let

$$\ell(n) = [2 \log_2(n)] + 2 \quad \text{and} \quad \ell_\varepsilon(n) = [2\varepsilon^{-1} \log_2 n] + 2.$$  

Let $\Phi$ be a radial $C^\infty_0$ function supported in $\{|x| \leq 1\}$, and satisfying $\int \Phi(x) dx = 1$. Let $\Phi_m(x) = 2^{-md} \Phi(2^{-m}x)$. Define

$$K^n_j = K_j \ast \Phi_{\ell_\varepsilon(n)}.$$  

Then $K^n_j$ is supported in $\{|x|: 2^{-j-2} \leq |x| \leq 2^{j+2}\}$, and, by the regularity assumption (1.2),

$$\|K_j - K^n_j\|_1 \lesssim 2^{-(j-\ell_\varepsilon(n))d} \int_{|h| \leq 2^{j-1-\ell_\varepsilon(n)}} |K_j(x) - K_j(x-h)| \, dx \, dh$$  

(2.4)

$$\lesssim 2^{-\ell_\varepsilon(n)} \lesssim n^{-2}.$$  

By differentiation and (1.1)

$$|\partial^\alpha K^n_j(x)| \leq C_\alpha 2^{-jd} 2^{(\ell_\varepsilon(n) - j)|\alpha|}.$$  

Let $\vartheta_n \in C^\infty(\mathbb{R})$ be supported in $(n^{-2}, 1 - n^{-2})$, such that $\vartheta_n(s) = 1$ for $s \in [2n^{-2}, 1 - 2n^{-2}]$, and such that the derivatives of $\vartheta_n$ satisfy the natural estimates

$$\|\vartheta^{(N)}_n\|_\infty \leq C_N n^{2N}.$$  

We then let $T^n_j$ be the integral operator with Schwartz kernel

$$K^n_j(x-y) \int \vartheta_n(s) a(sx + (1-s)y) \, ds.$$  

The following lemma is an immediate consequence of estimate (2.4) and the support property of $\vartheta_n$.

**Lemma 2.1.** The operator $T_j - T^n_j$ is bounded on $L^1$, with operator norm

$$\|T_j - T^n_j\|_{L^1 \rightarrow L^1} \lesssim n^{-2}.$$  

Lemma 2.1 implies

$$\text{meas}\left( \left\{ x : \sum_{n>n(\varepsilon)} \left| \sum_j (T_j B_{j-n}(x) - T^n_j B_{j-n}(x)) \right| > \lambda/10 \right\} \right)$$  

$$\leq 10 \lambda^{-1} \left\| \sum_{n>n(\varepsilon)} \sum_j |T_j B_{j-n} - T^n_j B_{j-n}| \right\|_1$$  

$$\lesssim \lambda^{-1} \sum_{n \geq 1} n^{-2} \sum_j \|B_{j-n}\|_1 \lesssim \lambda^{-1} \|f\|_1.$$  

(2.5)
and therefore it is enough to show

\begin{equation}
\text{meas}\left( \left\{ x : \sum_{n>n(x)} \sum_j |T^n_j B_{j-n}(x)| > \frac{2}{10} \lambda \right\} \right) \lesssim \lambda^{-1} \|f\|_1.
\end{equation}

For the proof of (2.7) we subtract various regular or small terms from the operators \(T^n_j\). Let \( \ell(n) \) be as in (2.3) and denote by \( P_m \) the convolution operator with convolution kernel \( \Phi_m \) (defined following (2.3)). We have:

**Proposition 2.2.** For \( n > 1 \),

\[ \|P_{j-n+\ell(n)}T^n_j B_{j-n}\|_1 \lesssim n^{-2} \log n \|B_{j-n}\|_1. \]

The proposition will be proved in §3. It yields

\[ \text{meas}\left( \left\{ x \in \mathbb{R}^d \setminus E : \sum_{n>n(x)} \left| \sum_j P_{j-n+\ell(n)}T^n_j B_{j-n}(x) \right| > \lambda/10 \right\} \right) \]

\[ \lesssim 10^{-1} \sum_{n>n(x)} \sum_j \|P_{j-n+\ell(n)}T^n_j B_{j-n}\|_1 \]

\[ \lesssim \lambda^{-1} \sum_{n>1} n^{-2} \log n \sum_j \|B_{j-n}\|_1 \lesssim \lambda^{-1} \|f\|_1 \]

and thus we need to consider the term

\begin{equation}
\sum_{n>n(x)} \sum_j (I - P_{j-n+\ell(n)})T^n_j B_{j-n}(x)
\end{equation}

and estimate the measure of the set where \(|(2.8)| > 3\lambda/5\). We will have to exploit the fact that the integral \( \int_0^1 a(sx + (1-s)y) ds \) smooths the rough function \( a \) in the direction parallel to \( x - y \), and use a microlocal decomposition which we now describe.

Let \( 1/10 < \gamma < 9/10 \) (say \( \gamma = 1/2 \)), and let \( \Theta_n \) be a set of unit vectors such that if \( \nu \neq \nu' \) and \( \nu, \nu' \in \Theta_n \), then \( |
u - \nu'| \geq 2^{-4-n\gamma} \), and assume that \( \Theta_n \) is *maximal* with respect to this property. Note that

\[ \text{card}(\Theta_n) \lesssim 2^{n\gamma(d-1)}. \]

For each \( \nu \) we may choose a function \( \tilde{\chi}_{n,\nu} \) on \( C^\infty(S^{d-1}) \) with the property that \( \tilde{\chi}_{n,\nu}(x) \geq 0 \), \( \tilde{\chi}_{n,\nu}(\theta) = 1 \) if \( |	heta - \nu| \leq 2^{-3-n\gamma} \), \( \tilde{\chi}_{n,\nu}(\theta) = 0 \) if \( |	heta - \nu| > 2^{-2-n\gamma} \), and such that for each \( M \in \mathbb{N} \) the functions \( 2^{-n\gamma M} \tilde{\chi}_{n,\nu} \) form a bounded family in \( C^M(S^{d-1}) \). For each \( \theta \) there is at least one \( \nu \) such that \( \tilde{\chi}_{n,\nu}(\theta) = 1 \), by the maximality assumption. Moreover, by the separatedness assumption the number of \( \nu \in \Theta_n \) for which \( \tilde{\chi}_{n,\nu}(\theta) \neq 0 \) is bounded above, uniformly in \( \theta \) and \( n \). Define, for \( \nu \in \Theta_n \),

\[ \chi_{n,\nu}(x) = \frac{\tilde{\chi}_{n,\nu}(x/|x|)}{\sum_{\nu' \in \Theta_n} \tilde{\chi}_{n,\nu'}(x/|x|)}. \]
Then $\sum_{\nu \in \Theta_n} \chi_{n,\nu}(x) = 1$ for every $x \in \mathbb{R}^d \setminus \{0\}$ and by homogeneity we have the following estimates for multi-indices $\alpha$ and $x \neq 0$,

$$|\langle (\nu, \nabla) \rangle^M \chi_{n,\nu}(x)| \leq C M |x|^{-M},$$

$$|\partial^\alpha \chi_{n,\nu}(x)| \leq C_n 2^{\gamma |\alpha|} |x|^{-|\alpha|}.$$  

Let $K_j^{n,\nu}(x) = K_j^{\nu}(x) \chi_{n,\nu}(x)$ and let $T_j^{n,\nu}$ be the operator with Schwartz kernel

$$K_j^{n,\nu}(x-y) \int \vartheta_n(s) a(sx + (1-s)y) \, ds.$$  

We then have

$$T_j^n = \sum_{\nu \in \Theta_n} T_j^{n,\nu}.$$  

Let $\phi \in C^\infty(\mathbb{R})$ so that $\phi(u) = 1$ for $|u| < 1/2$ and $\phi(u) = 0$ for $|u| \geq 1$ and define the singular convolution operator $\mathcal{S}_{n,\nu}$ by

$$\widehat{\mathcal{S}_{n,\nu} f}(\xi) = \phi(2^{\gamma} n^{-5} (\nu, \xi/|\xi|)) \widehat{f}(\xi).$$  

The terms involving $\langle I - \mathcal{S}_{n,\nu} \rangle T_j^{n,\nu}$ can be dealt with by $L^1$ estimates. In §4 we shall prove:

**Proposition 2.3.** For $n > n(\varepsilon)$ and $\nu \in \Theta_n$,

$$\left\| \sum_j \langle I - P_{j-n+\ell(n)} \rangle (I - \mathcal{S}_{n,\nu}) T_j^{n,\nu} B_{j-n} \right\|_1 \lesssim n^{-2} 2^{-n\gamma(d-1)} \|f\|_1.$$  

For the rougher terms involving $\mathcal{S}_{n,\nu} T_j^{n,\nu}$ we shall use a weak orthogonality argument from [8] to prove the following $L^2$ estimate.

**Proposition 2.4.** For $n > n(\varepsilon)$,

$$\left\| \sum_{\nu \in \Theta_n} \sum_j \langle I - P_{j-n+\ell(n)} \rangle \mathcal{S}_{n,\nu} T_j^{n,\nu} B_{j-n} \right\|_2^2 \lesssim 2^{-n\gamma n^5} \lambda \|f\|_1.$$  

Given these two propositions we can finish the outline of the proof of Theorem 1.1. Namely by Chebyshev’s inequality,

$$\text{meas} \left( \left\{ x : \left| \sum_{n>n(\varepsilon)} \sum_j \langle I - P_{j-n+\ell(n)} \rangle T_j^n B_{j-n}(x) \right| > \frac{4}{3} \lambda \right\} \right) \lesssim 5\lambda^{-1} \left\| \sum_{n>n(\varepsilon)} \sum_{\nu \in \Theta_n} \sum_j \langle I - P_{j-n+\ell(n)} \rangle (I - \mathcal{S}_{n,\nu}) T_j^{n,\nu} B_{j-n} \right\|_1$$

$$+ 25\lambda^{-2} \left\| \sum_{n>n(\varepsilon)} \sum_{\nu \in \Theta_n} \sum_j \langle I - P_{j-n+\ell(n)} \rangle \mathcal{S}_{n,\nu} T_j^{n,\nu} B_{j-n} \right\|_2^2$$

and by Propositions 2.3 and 2.4 and Minkowski’s inequality this is bounded by

$$C \lambda^{-1} \|f\|_1 \left( \sum_n n^{-2} 2^{-n\gamma(d-1)} \text{card}(\Theta_n) + \sum_n 2^{-n\gamma n^5} \right) \lesssim \lambda^{-1} \|f\|_1.$$
3. Proof of Proposition 2.2

Let $Q \in \Omega_{j}$ with $L(Q) = j - n$. We apply Fubini’s theorem and write

$$P_{j-n+\ell(n)}^n T^*_j b_Q(x) = \int \vartheta_n(s) \int b_Q(y)$$

$$\times \left[ \int \Phi_{j-n+\ell(n)}(x-w) K^n_j (w-y) a(sw + (1-s)y) dw \right] dy \, ds.$$ 

Changing variables $z = w + \frac{1-s}{s} y$ we get

$$P_{j-n+\ell(n)}^n T^*_j b_Q(x) = \int \vartheta_n(s) \int a(sz) \int A_{j,n}^{x,z,s}(y) b_Q(y) dy \, dz \, ds,$$

where

$$A_{j,n}^{x,z,s}(y) = \Phi_{j-n+\ell(n)}(x-z + \frac{1-s}{s} y) K^n_j (z-y/s).$$

We expand $A_{j,n}^{x,z,s}(y)$ about the center $y_Q$ of $Q$ and in view of the cancellation of $b_Q$ we may write

$$\left| P_{j-n+\ell(n)}^n T^*_j b_Q(x) \right| \leq \int \int |\vartheta_n(s) a(sz)| \left| \int (A_{j,n}^{x,z,s}(y) - A_{j,n}^{x,z,s}(y_Q)) b_Q(y) dy \right| dz \, ds.$$

Using

$$A_{j,n}^{x,z,s}(y) - A_{j,n}^{x,z,s}(y_Q) = \langle y - y_Q, \int_0^1 \nabla A_{j,n}^{x,z,s}(y_Q + \sigma(y-y_Q)) d\sigma \rangle$$

one obtains after applying Fubini’s theorem

$$\| P_{j-n+\ell(n)}^n T^*_j b_Q(x) \|_1 \leq \text{diam}(Q) \int_0^1 \int |\vartheta_n(s)|$$

$$\times \left[ \| \nabla \Phi_{j-n+\ell(n)} \|_1 \frac{1-s}{s} \int |b_Q(y)| \int |K^n_j (z - \frac{y_Q + \sigma(y-y_Q)}{s})| dz \, dy \
+ \| \Phi_{j-n+\ell(n)} \|_1 \int |b_Q(y)| \int \frac{1-s}{s} \| \nabla K^n_j (z - \frac{y_Q + \sigma(y-y_Q)}{s}) \| dz \, dy \right] ds \, d\sigma.$$

Now use $\| \nabla K^n_j \|_1 \lesssim 2^{-j+\ell(n)}$ and $\int_0^1 |\vartheta_n(s)| s^{-1} ds \lesssim \log n$. Since $\text{diam}(Q) \lesssim 2^{j-n}$ we obtain

$$\| P_{j-n+\ell(n)}^n T^*_j b_Q \|_1 \lesssim \log n \left[ 2^{-\ell(n)} + 2^{\ell(n)-n} \right] \|b_Q\|_1 \lesssim n \log n \|b_Q\|_1.$$ 

Finally we sum over all $Q \in \Omega_{j}$ with $L(Q) = j - n$ to obtain the asserted bound. \qed
4. Proof of Proposition 2.3

Let $Q \in \mathfrak{Q}$ with $L(Q) = j - n$, and let $y_Q$ be the center of $Q$. Fix a unit vector $\nu$, and let $\pi^\perp_\nu$ be the projection to the orthogonal complement of $\nu$, i.e. $\pi^\perp_\nu(x) = x - \langle x, \nu \rangle \nu$. In view of the support properties of the kernel it suffices to show that, for $n > n(\epsilon)$,

\[
(4.1) \quad \left\| (I - P_{j - n + \ell(n)}) (I - \mathfrak{S}_{n, \nu}) T_j^{n, \nu} b_Q \right\|_1 \leq n^{-2} 2^{-n\gamma(d-1)} \| b_Q \|_1,
\]

under the additional assumption that the support of $\alpha$ is contained in

\[
\{ y : | \langle y - y_Q, \nu \rangle | \leq 2^{j+4} d, | \pi^\perp_\nu (y - y_Q) | \leq 2^{j+4-n\gamma} d \}.
\]

Note that with this hypothesis

\[
(4.2) \quad \| \tilde{\alpha} \|_\infty \lesssim 2^{j-d-n\gamma(d-1)}.
\]

We introduce a frequency decomposition of $\alpha$. Let $\varphi$ be a radial $C^\infty$ function as in §2, but now defined in $\xi$-space, so that $\varphi(\xi) = 1$ for $|\xi| \leq 1$ and $\varphi(\xi) = 0$ for $|\xi| \geq 6/5$. Define $\beta_k(\xi) = \varphi(2^k \xi) - \varphi(2^{k+1} \xi)$; then $\beta_k$ is supported in $\{ \xi : 2^{-k-1} \leq |\xi| \leq 2^{2^{-k}} \}$. Let $\tilde{\beta}$ be a radial $C^\infty$ function so that $\tilde{\beta}$ is supported in $\{ \xi : 1/3 \leq |\xi| \leq 3/2 \}$ and $\tilde{\beta}(\xi) = 1$ for $1/2 \leq |\xi| \leq 6/5$, and define $\tilde{\beta}_k(\xi) = \tilde{\beta}(2^k \xi)$. Then $\beta_k \tilde{\beta}_k = \beta_k$. Define convolution operators $V_k$, $\Lambda_k$ and $\tilde{\Lambda}_k$ with Fourier multipliers $\varphi(2^k \cdot)$, $\beta_k$ and $\tilde{\beta}_k$, respectively; then $\Lambda_k \tilde{\Lambda}_k = \Lambda_k$ and, for every $m \in \mathbb{Z}$, the identity operator is decomposed as $I = \Lambda_m + \sum_{k < m} \Lambda_k$.

For fixed $y \in Q$ we define an operator $\mathcal{K}^{n, \nu}_{j, y}$ acting on $\alpha$ by

\[
\mathcal{K}^{n, \nu}_{j, y} [a](x) = K_j^{n, \nu}(x - y) \int \partial_n(s) a(sx + (1 - s)y) \, ds
\]

so that

\[
(4.3) \quad T_j^{n, \nu} b_Q(x) = \int b_Q(y) \mathcal{K}^{n, \nu}_{j, y} [a](x) \, dy.
\]

We use dyadic frequency decompositions and split

\[
(4.4) \quad (I - \mathfrak{S}_{n, \nu}) (I - P_{j - n + \ell(n)}) T_j^{n, \nu} b_Q = \sum_{k_1} \Lambda_{k_1} (I - \mathfrak{S}_{n, \nu}) \tilde{\Lambda}_{k_1} (I - P_{j - n + \ell(n)}) \int b_Q(y) \mathcal{K}^{n, \nu}_{j, y} [a] \, dy
\]

and then further split in (4.4)

\[
(4.5) \quad \alpha = V_{j - n + \ell(n)} \alpha + \sum_{k_2 < j - n + \ell(n)} \Lambda_{k_2} \alpha.
\]

We prove three lemmata with various bounds for the terms in (4.4) and (4.5).
Lemma 4.1.

\[ \left\| \int b_Q(y) K^{n,\nu}_{j,y}[V_{j-n+\ell(n)}a] \, dy \right\|_1 \lesssim n^{-2} 2^{-n\gamma(d-1)} \| b_Q \|_1. \]

Proof. We use the cancellation of \( b_Q \) to estimate the left-hand side by

\[ \int |b_Q(y)| \int \left| K^{n,\nu}_{j,y}[V_{j-n+\ell(n)}a](x) - K^{n,\nu}_{j,y}[V_{j-n+\ell(n)}a](x) \right| \, dx \, dy. \]

For \( y \in Q \) we may estimate

\[ \int \left| K^{n,\nu}_{j,y}[V_{j-n+\ell(n)}a](x) - K^{n,\nu}_{j,y}[V_{j-n+\ell(n)}a](x) \right| \, dx \leq E_1(y) + E_2(y), \]

where

\[ E_1(y) = \| V_{j-n+\ell(n)}a \|_\infty \int \left| K^{n,\nu}_j(x - y) - K^{n,\nu}_j(x - y_Q) \right| \, dx \]

and, abbreviating

\[ \Gamma^{Q}_{j-n+\ell(n)}(x,y,z) \]

\[ = \int_0^1 \langle y - y_Q, \nabla F[\varphi(2^{j-n+\ell(n)})](sx + (1 - s)(y_Q + \sigma(y - y_Q)) - z) \rangle \, ds, \]

\( E_2 \) is given by

\[ E_2(y) = \int \left| K^{n,\nu}_j(x - y_Q) \right| \int |\varphi_n(s)| \int |\Gamma^{Q}_{j-n+\ell(n)}(x,y,z)| \, dz \, ds \, dx. \]

Now by (2.5), and since \( |\varphi_{n,x}\nu(x)| \lesssim 2^{n\gamma}|x|^{-1} \) we get

\[ |E_1(y)| \leq |y - y_Q| \| \nabla K^{n,\nu}_j \|_1 \lesssim 2^{j-n} [2^{\ell(n) - j} + 2^{n\gamma - j}] 2^{-n\gamma(d-1)}. \]

Notice that for \( n > n(\varepsilon) \) and \( \gamma > 1/10 \) we have \( 2^{\ell(n)} \lesssim 2^{n\gamma} \) and thus we see that

\[ |E_1(y)| \lesssim 2^{-n\gamma(d-1)}n^{-2}. \]

Moreover,

\[ |E_2(y)| \lesssim \| K^{n,\nu}_j \|_1 |y - y_Q| \left\| \nabla F^{-1}[\varphi(2^{j-n+\ell(n)})] \right\|_1 \lesssim 2^{-n\gamma(d-1)} 2^{j-n} 2^{n\gamma(d-1)} \lesssim 2^{-n\gamma(d-1)}n^{-2}. \]

Integrating in \( y \), we get

\[ \int \left( |E_1(y)| + |E_2(y)| \right) |b_Q(y)| \, dy \lesssim 2^{-n\gamma(d-1)}n^{-2} \| b_Q \|_1, \]

and the assertion follows. \( \square \)
Lemma 4.2. Let $y \in Q$ and $a$ be as in (4.2).

(i) Let $k_1 > k_2 + \ell(n) + 10$. Then

$$
\|A_{k_1} \mathcal{K}_{j,y}^n[a_{k_2}]\|_1 \leq C_N 2^{-n\gamma(d-1)} \min\{1, n^{2d+2N} 2^{2(k_2-j+n\gamma)N}\}.
$$

(ii) Let $k_1 < k_2 - 10$. Then

$$
\|A_{k_1} \mathcal{K}_{j,y}^n[a_{k_2}]\|_1 + \|A_{k_1} \mathcal{K}_{j,y}^n[V_{k_2}a]\|_1 
\leq C_N 2^{-n\gamma(d-1)} \min\{1, 2^{n\gamma} 2^{(k_1-k_2)d} 2^{2(k_1-j+n\gamma)N}\}.
$$

Proof. Clearly $\|\mathcal{K}_{j,y}^n[a]\|_1 \lesssim 2^{-n\gamma(d-1)}\|a\|_\infty$, and since the operators $\Lambda_k$ and $V_k$ are uniformly bounded we get the bound $O(2^{-n\gamma(d-1)})$ in (i) and (ii). We seek to prove the two other bounds for $\Lambda_{k_1}
\mathcal{K}_{j,y}^n[a_{k_2}]$ in the two cases $k_1 < k_2 - 10$ and $k_1 > k_2 + \ell(n) + 10$. In (ii) the corresponding estimate for $\Lambda_{k_1}
\mathcal{K}_{j,y}^n[V_{k_2}a]$ is entirely analogous and will be omitted.

We use the Fourier inversion formula for $a$ and for the convolution kernel of $\Lambda_{k_1}$: write

$$
\Lambda_{k_1} \mathcal{K}_{j,y}^n[a_{k_2}](x) = \frac{1}{(2\pi)^{2d}} \int \vartheta_n(s) \int \hat{\beta}_{k_2}(\xi)\hat{\beta}_{k_2}(\eta)\hat{a}(\eta) 
\times \left[ \int_w e^{i(x-w,\xi)+(s\omega+(1-s)w,\eta)} K_j^n(w-y) \, dw \right] d\xi \, d\eta \, ds,
$$

and integrate by parts with respect to $w$ and $\xi$. The integral can then be rewritten as

$$
\frac{1}{(2\pi)^{2d}} \int \vartheta_n(s) \int \beta_{k_2}(\eta)\hat{a}(\eta) \int \left[ \frac{e^{i(x-w,\xi)+(s\omega+(1-s)w,\eta))}}{(1 + 2^{-2k_1} |x-w|^{2}) |k|} \right] d\xi \, d\eta \, ds,
$$

and we choose $N_1 = [d/2] + 1$. Note that for $s \in \text{supp}(\vartheta_n)$,

$$
|\xi - sn| \gtrsim C(k_1, k_2, n):= \begin{cases} 2^{-k_2-\ell(n)} & \text{if } k_1 > k_2 + \ell(n) + 10, \\
2^{-k_2-2} & \text{if } k_1 < k_2 - 10. \end{cases}
$$

Now $(2^{-k_2} \partial_k)^{N_3} \beta_{k_1} = O(1)$ and a computation yields

$$
|\left( I - 2^{-2k_1} \Delta_x \right)^{N_1} [\beta_{k_1}(\xi) |\xi - sn|^{-2N_2}]| \lesssim [C(k_1, k_2, n)]^{-N_2}.
$$

Moreover

$$
\|(-\Delta_w)^{N_2} K_j^n\|_1 \lesssim 2^{-2N_2} (2^{2N_2n\gamma} + 2^{2N_2\ell(n)}) 2^{-n\gamma(d-1)} 
\lesssim 2^{-n\gamma(d-1)} 2^{2N_2(n\gamma-j)}.
$$

1 Thanks to Xudong Lai who pointed out an inaccuracy in the original version of this and another formula.
We integrate in $\eta$ and use that the measure of the support of $\beta_{k_2}$ is $O(2^{-k_2d})$. Then we integrate in $x$ and $\xi$ and use that
\[
\int_{\supp(\beta_{k_1})} \int (1 + 2^{-2k_1}|x - w|^2)^{-N_1} \, dx \, d\xi = O(1).
\]
Using (4.2) we then get
\[
\|\Lambda_k \mathcal{X}^{n,\nu}_{j,\gamma}[\Lambda_{k_2}a]\|_1 \lesssim_{N_2} 2^{-k_2d} \|\widehat{g}\|_\infty \|(-\Delta)^{N_2} K_j^{n,\nu}\|_1 \left[\mathcal{C}(k_1, k_2, n)\right]^{-2N_2}
\]
\[
\lesssim_{N_2} 2^{2(d(n-\nu(j-2))2(2N_2-d)(k_2-j+\ell(n)+n\gamma)} \quad \text{if } k_1 > k_2 + \ell(n) + 10,
\]
\[
\lesssim_{N_2} 2^{-n\gamma(d-2)}2(2N_2-d)(k_2-j+n\gamma) \quad \text{if } k_1 < k_2 - 10.
\]
If we put $N = 2N_2 - d$ this gives the asserted bound for $\|\Lambda_k \mathcal{X}^{n,\nu}_{j,\gamma}[\Lambda_{k_2}a]\|_1$. For $k_1 < k_2 - 10$ the corresponding expression with $\Lambda_k$ replaced by $V_{k_2}$ is estimated in exactly the same way.

**Lemma 4.3.** Let $k_2 - 10 \leq k_1 \leq k_2 + \ell(n) + 10$. Then
\[
\|\Lambda_k (I - \mathcal{G}^{n,\nu}) \mathcal{X}^{n,\nu}_{j,\gamma}[\Lambda_{k_2}a]\|_1 \leq CN_2 2^{-n\gamma(d-1)} \min\{1, n^{(N+d)/\varepsilon} 2^{(d+3)n\gamma 2(k_1-j+n\gamma)N}\}
\]
for every $y \in Q$.

**Proof.** We may again assume that (4.2) holds. Define the convolution operator $S_{n,\nu}$ by
\[
\widehat{S_{n,\nu}g}(\eta) = \phi(2^n\nu n^{-2}(\nu, \eta/|\eta|)) \widehat{g}(\eta)
\]
and split $a = S^{n,\nu}a + (I - S^{n,\nu})a$. We shall prove the two estimates
\[
(4.6) \quad \|\Lambda_k (I - \mathcal{G}^{n,\nu}) \mathcal{X}^{n,\nu}_{j,\gamma}[\Lambda_{k_2}S^{n,\nu}a]\|_1 \leq CN_2 n^{(2\varepsilon - 4)(N+d)} 2^{4n\gamma} 2(k_1-k_2)d 2^{k_1-j+n\gamma)N},
\]
and
\[
(4.7) \quad \|\Lambda_k (I - \mathcal{G}^{n,\nu}) \mathcal{X}^{n,\nu}_{j,\gamma}[\Lambda_{k_2}(I - S^{n,\nu})a]\|_1 \leq CN_2 n^{-5d} 2^{4n\gamma} 2(k_2-j+n\gamma)N.
\]
These imply the somewhat weaker bound asserted in the lemma.

**Proof of (4.6).** Set
\[
b_{k_1, n, \nu}(\xi) = \beta_{k_1}(\xi) (1 - \phi(2^{n\gamma} n^{-5}(\nu, \xi/|\xi|)))
\]
and write
\[
(2\pi)^2d \Lambda_k (I - \mathcal{G}^{n,\nu}) \mathcal{X}^{n,\nu}_{j,\gamma}[\Lambda_{k_2}S^{n,\nu}a](x)
\]
\[
= \int \hat{g}(s) \int b_{k_1, n, \nu}(\xi) \beta_{k_2}(\eta) \phi(2^{n\gamma} n^{-5}(\nu, \eta/|\eta|)) \widehat{a}(\eta)
\]
\[
\times \left[ \int e^{i(x-w, \xi)}(s w + (1-s)y, \eta) K_j^{n,\nu}(w - y) \, dw \right] d\xi d\eta ds.
\]
If \((\xi, \eta)\) is in the support of the amplitude then for \(n > 10^{10}\)
\[
|\langle \xi - s\eta, \nu \rangle| \geq |\xi||\langle \xi, \nu \rangle| - |\eta||\langle \eta, \nu \rangle|
\geq |\xi| (2^{-n\gamma - 1} n^5 - 2^{k_1 - k_2 + 2} 2^{-n\gamma} n^2)
\geq |\xi| 2^{-n\gamma - 1} (n^5 - 8 \cdot 2^{(n+10)\mu^2}) \geq 2^{-k_1 - n\gamma} n^5.
\]
(4.8)
Now we can integrate by parts as in the proof of Lemma 4.2, except that we use the directional derivative \(\langle \nu, \nabla_w \rangle\) instead of \(\Delta_w\). The above integral is then estimated by
\[
\int \int \int \int |\beta_{k_2}(\eta)| |\hat{a}(\eta)| \left| \phi(2^{n\gamma} n^{-2} \langle \nu, \frac{\eta}{|\eta|} \rangle) \right|
\times \frac{|(I - 2^{-2k_1} \Delta_\eta)^{N_1}[\frac{b_{k_1,n,\nu}(\xi)}{(\xi - s\eta, \nu)^{N_2}}]|}{(1 - 2^{-2k_1} |x - w|^2)^{N_1}} [\langle \nu, \nabla_w \rangle^{N_2} \mathcal{K}_j^{n,\nu}(w - y)] d\xi d\eta ds.
\]
Observe that
\[
|\partial_x^{N_1} b_{k_1,n,\nu}(\xi) | \leq C_{N_1} (2^{n\gamma} n^{-5})^{N_3} 2^{k_1 N_3}
\]
and thus
(4.9) \[
|(I - 2^{-2k_1} \Delta_\eta)^{N_1}[\frac{b_{k_1,n,\nu}(\xi)}{(\xi - s\eta, \nu)^{N_2}}]| \leq C_{N_1} (2^{n\gamma} n^{-5})^{2N_1} (2^{-(k_1 + n\gamma)} n^5)^{-N_2}.
\]
Moreover,
\[
[\langle \nu, \nabla_w \rangle^{N_2} \mathcal{K}_j^{n,\nu}] \leq C_{N_2} 2^{(e(n)-j)N_2} 2^{-n\gamma(d-1)}.
\]
We assume \(2N_1 > d\), integrate in \(x\) and \(\xi\), and use (4.8). Then we obtain
\[
\|\Lambda_{k_1} (I - \mathcal{S}^{n,\nu}) \mathcal{K}_j^{n,\nu} [\mathcal{A}_{k_2} \mathcal{S}^{n,\nu} a] \|_1 \leq C_{N_1, N_2} (2^{n\gamma} n^{-5})^{2N_1} \|\hat{a}\|_{\infty} 2^{-k_1 d} \frac{2^{(e(n)-j)N_2} 2^{-n\gamma(d-1)}}{(2^{-k_1 - n\gamma} n^5)^{N_2}}.
\]
We use (4.2) and that the support of \(\eta \rightarrow \beta_{k_2}(\eta)\) has measure \(O(2^{-k_2 d})\). Thus the expression in the previous displayed inequality can be crudely estimated by
\[
C_{N_1, N_2} n^{(2^{-\mu} - 1)N_2 - 10N_1} 2^{n\gamma(2N_1 - d+2)} 2^{(k_1 - k_2) d} 2^{(k_1 - j + n\gamma)} (N_2 - d)
\]
and, if we chose the integer \(N_1 \in \{(d+1)/2, (d+2)/2\}\) and \(N = N_2 - d\), we obtain (4.6).

Proof of (4.7). Set
\[
\tilde{b}_{k_2,n,\nu}(\eta) = \beta_{k_2}(\eta) (1 - \phi(2^{n\gamma} n^{-2} \langle \nu, \eta/|\eta| \rangle))
\]
and write
\[
(2\pi)^d \Lambda_{k_1} (I - \mathcal{S}^{n,\nu}) \mathcal{K}_j^{n,\nu} [\Lambda_{k_2} (I - \mathcal{S}^{n,\nu}) a](x)
= \int K_j^{n,\nu}(w - y) \int b_{k_1,n,\nu}(\xi) \tilde{b}_{k_2,n,\nu}(\eta) \hat{a}(\eta)
\times \left[ \int \vartheta_n(s) e^{i(x-w, \xi) + iyw + (1-s)y, w)} ds \right] d\xi d\eta dw.
\]
Now if \( w - y \in \operatorname{supp}(K_{j}^{n,\nu}) \) then \( \left| \frac{w - y}{|w - y|} - \nu \right| \leq 2^{-n\gamma} \) and if \( \eta \in \operatorname{supp}(\tilde{b}_{k_{2},n,\nu}) \) we get

\[
|\langle w - y, \eta \rangle| \geq |w - y| (\langle \nu, \eta \rangle - |\eta| 2^{-n\gamma}) \geq |w - y| |\eta| 2^{-n\gamma} \left( \frac{1}{2} n^{2} - 1 \right)
\]

and hence

\[
(4.10) \quad |\langle w - y, \eta \rangle| \geq 2^{j-k_{2} - n\gamma - 4} n^{2}.
\]

Integration by parts with respect to \( s \) yields

\[
(2\pi)^{d} \Lambda_{k_{1}} (I - \mathcal{S}^{n,\nu}) \mathcal{K}_{j,y}^{n,\nu} [\Lambda_{k_{2}} (I - S^{n,\nu}) a](x) = \int K_{j}^{n,\nu}(w - y) \int \tilde{a}(\eta) \tilde{b}_{k_{2},n,\nu}(\eta) \frac{\mathcal{N}}{1 + 2 - 2k_{1}|x - w|^{2}} N_{1} \left[ \int \mathcal{N}^{j}(s) \hat{a}_{y,\eta}^{(N_{3})}(w - y, \eta) \right] d\xi d\eta dw.
\]

We apply this with \( N_{1} > d/2 \) and, using (4.2), (4.9), and (4.10), obtain

\[
\|\Lambda_{k_{1}} (I - \mathcal{S}^{n,\nu}) \mathcal{K}_{j,y}^{n,\nu} \Lambda_{k_{2}} (I - S^{n,\nu}) a\|_{1} \lesssim_{N_{1},N_{3}} n^{-2 - 10N_{1}} (2n^{2})^{2N_{1} - 2N_{3}} \left( \frac{2^{j-k_{2} - n\gamma - 4} n^{2}}{\| \mathcal{N}^{j} \|_{1}} \right) \| \hat{a}\|_{\infty}.
\]

Inequality (4.7) follows if we choose \( N = N_{3} - d \) and \( N_{1} \in \{(d + 1)/2, (d + 2)/2\} \).

**Proof of Proposition 2.3, conclusion.** Let, for fixed \( n, \nu \) and \( j \), and for a fixed cube \( Q \in \Omega_{\Lambda} \) with \( L(Q) = j - n \),

\[
I_{k_{1}} = \tilde{\Lambda}_{k_{1}} (I - P_{j-n+\ell(n)}) \Lambda_{k_{1}} (I - \mathcal{S}_{n,\nu}) \left[ \int b_{Q}(y) \mathcal{N}_{j,y}^{n,\nu} [V_{j-n+\ell(n)} a] \, dy \right],
\]

and

\[
\tilde{I}_{k_{1},k_{2}} = \tilde{\Lambda}_{k_{1}} (I - P_{j-n+\ell(n)}) \Lambda_{k_{1}} (I - \mathcal{S}_{n,\nu}) \left[ \int b_{Q}(y) \mathcal{N}_{j,y}^{n,\nu} [\Lambda_{k_{2}} a](x) \, dy \right].
\]

By (4.4) and (4.5) it is enough to show that

\[
(4.11) \quad \sum_{k_{1}} \| I_{k_{1}} \|_{1} + \sum_{k_{1}} \sum_{k_{2} < j-n+\ell(n)} \| \tilde{I}_{k_{1},k_{2}} \|_{1} \lesssim n^{-2} 2^{-\gamma(n-1)} \| b_{Q} \|_{1}.
\]

We have

\[
(4.12) \quad \| \Lambda_{k_{1}} (I - \mathcal{S}_{n,\nu}) \|_{L^{1} \rightarrow L^{1}} \lesssim C
\]
uniformly in $n$, $\nu$ and $k_1$, and using the support and cancellation properties of the kernel of $I - P_{j-n+\ell(n)}$ we also have

\begin{equation}
\|\tilde{A}_{k_1}(I - P_{j-n+\ell(n)})\|_{L^1 \to L^1} \lesssim \min\{1, 2^{j-n+\ell(n)-k_1}\}.
\end{equation}

Lemma 4.1 together with (4.13) and (4.12) immediately gives

\begin{equation}
\sum_{k_1 \geq j-n+\ell(n)-10} \|I_{k_1}\|_1 \lesssim n^{-2} 2^{-\gamma n(d-1)} \|b_Q\|_1.
\end{equation}

It remains to verify that the other terms satisfy better bounds, namely

\begin{equation}
\sum_{k_1 < j-n+\ell(n)-10} \|I_{k_1}\|_1 + \sum_{k_1} \sum_{k_2 < j-n+\ell(n)} \|II_{k_1,k_2}\|_1 \lesssim C_N n^{A_1} n^{2^{n(n-1)}N} \|b_Q\|_1
\end{equation}

for all $N$, and suitable $A_1 \leq 10d/\epsilon$ and $A_2 \leq 10$. Choose $N = 100d$. Taking into account that $\gamma \leq 9/10$ one can check that the bound in (4.15) is dominated by $C n^{-2} 2^{-n\gamma(d-1)} \|b_Q\|_1$ for all $n$ with $n^{-1} \log n \leq 10^{-4} \epsilon/d$, which is satisfied for $n > n(\epsilon)$.

For the terms involving $I_{k_1}$, with $k_1 \geq j - n + \ell(n) + 10$ we get by the second estimate in part (ii) of Lemma 4.2, with $k_2 = j - n + \ell(n)$,

\begin{equation}
\sum_{k_1 < j-n+\ell(n)-10} \|I_{k_1}\|_1 
\lesssim n^{-2\gamma(d-2)} \sum_{k_1 < j-n+\ell(n)-10} 2^{(k_1-j-n+\ell(n))d} 2^{(k_1-j+n\gamma)N} \|b_Q\|_1
\end{equation}

Next consider $\sum_{k_1,k_2} \|II_{k_1,k_2}\|_1$ where the $k_2$-summation is extended over $k_2 < j - n + \ell(n)$. For $k_1 \geq j - n + \ell - 10$ we can sum a geometric series in $k_1$, with a uniform bound, due to (4.13). By Lemma 4.2, part (i)

\begin{equation}
\sum \|II_{k_1,k_2}\|_1 \lesssim 2^{-n\gamma(d-2)} n^{2d+2N} \sum_{k_2 < j-n+\ell(n)} 2^{(k_2-j+n\gamma)N} \|b_Q\|_1
\end{equation}

and by Lemma 4.3

\begin{equation}
\sum \|II_{k_1,k_2}\|_1 \lesssim \|b_Q\|_1 \ell(n) n^{2(N+d)/\epsilon} 2^{4n\gamma} \sum_{k_2 \leq j-n+2\ell(n)+10} 2^{(k_2-j+n\gamma)N}
\end{equation}

\begin{equation}
\lesssim \|b_Q\|_1 \log(n) n^{2(N+d)(\epsilon^{-1}+2)} 2^{n(n-1)N}.
\end{equation}
The case $k_2 > k_1 + 10$ does not occur when $k_1 \geq j - n + \ell(n) - 10$ because of the restriction $k_2 < j - n + \ell(n)$. Thus in all cases of (4.15) which involve the restriction $k_1 \geq j - n + \ell(n) - 10$ we obtain the required estimate.

Now sum the terms $\|II_{k_1,k_2}\|_1$ with $k_1 < j - n + \ell(n) - 10$. By Lemma 4.2, part (i),
\[
\sum_{(k_1,k_2)} \|II_{k_1,k_2}\|_1 \lesssim n^{2d+2N} 2^{-n\gamma(d-2)} \sum_{(k_1,k_2)} 2^{(k_2-j+n\gamma)N} \|b_Q\|_1 \lesssim n^{2d+2N} 2^{-n\gamma(d-2)} 2^{n(\gamma-1)N} \|b_Q\|_1 ;
\]
by Lemma 4.2, part (ii),
\[
\sum_{(k_1,k_2)} \|II_{k_1,k_2}\|_1 \lesssim 2^{-n\gamma(d-2)} \sum_{k_1 < j-n+\ell(n)-10} 2^{(k_1-j+n\gamma)N} \sum_{k_2 > k_1+10} 2^{(k_1-k_2)d} \|b_Q\|_1 ,
\]
and finally, by Lemma 4.3,
\[
\sum_{(k_1,k_2)} \|II_{k_1,k_2}\|_1 \lesssim \log(n)n^{2(N+d)/\varepsilon} 2^{4n\gamma} \sum_{j \leq j-n+\ell(n)} 2^{(k_1-j+n\gamma)N} \|b_Q\|_1 \lesssim n^{2(N+d)(\varepsilon^{-1}+1)} 2^{4n\gamma} 2^{n(\gamma-1)N} \|b_Q\|_1 .
\]
This finishes the proof of (4.15). \(\square\)

5. Proof of Proposition 2.4

We use a slightly modified version of an argument in [8]. The main observation is that, for fixed $n > 0$, we have
\[
(5.1) \quad \sup_{\xi \neq 0} \sum_{\nu \in \Theta_n} |\phi(2^n\gamma n^{-5}\nu; \xi/|\xi|)| \lesssim 2^{n\gamma(d-2)} n^5 .
\]
To see this it suffices, by homogeneity, to take the supremum over all $\xi \in S^{d-1}$. Now if $|\xi| = 1$ and $\phi(2^n\gamma n^{-5}\langle \theta, \xi \rangle) \neq 0$ then the distance of $\nu$ to the hyperplane $\xi$ is at most $C n^{5\gamma} 2^{-n\gamma}$ and since the vectors in $\Theta_n$ are $c2^{-\gamma n}$-separated there are $O(2^{n\gamma(d-2)} n^5)$ such vectors, hence (5.1) holds.

From (5.1) it follows that
\[
\left\| \sum_{\nu \in \Theta_n} \mathcal{E}_{n,\nu} \sum_j (I - P_{j-n+\ell(n)}) T^{n,\nu}_{j} B_{j-n} \right\|_2^2 \lesssim 2^{n\gamma(d-2)} n^5 \sum_{\nu \in \Theta_n} \left\| \sum_j (I - P_{j-n+\ell(n)}) T^{n,\nu}_{j} B_{j-n} \right\|_2^2 .
\]
and since \( \#\Theta_n \lesssim 2^{n\gamma(d-1)} \) the asserted inequality is a consequence of

\[
(5.2) \quad \left\| \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \right\|_2^2 \lesssim 2^{-2n\gamma(d-1)} \lambda \|f\|_1
\]

for each \( \nu \in \Theta_n \).

For the proof of (5.2) the cancellation of \( B_{j-n} \) plays no role. Let

\[
H_j^{n,\nu}(x) = 2^{-jd} \chi_{\tau_j^{n,\nu}}(x).
\]

where

\[
\tau_j^{n,\nu} = \{ x : |\langle x, \nu \rangle| \leq 2^{j+2}, |x - \langle x, \nu \rangle| \leq 2^{j+2-\gamma n} \}.
\]

Then from (1.1) we get

\[
\left| (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n}(x) \right| \lesssim H_j^{n,\nu} \ast |B_{j-n}(x)|.
\]

Therefore

\[
\left\| \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \right\|_2 \lesssim 2 \sum_j \int |B_{j-n}(x)| \sum_{i \leq j} H_i^{n,\nu} \ast H_i^{n,\nu} \ast |B_{i-n}(x)| \, dx.
\]

Observe that \( \|H_i^{n,\nu}\|_1 \lesssim 2^{-id} \text{meas}(\tau_i^{n,\nu}) \lesssim 2^{-n\gamma(d-1)} \) and thus

\[
H_j^{n,\nu} \ast H_i^{n,\nu}(x) \lesssim 2^{-n\gamma(d-1)} 2^{-jd} \chi_{\tau_j^{n,\nu}}(x)
\]

where \( \tau_j^{n,\nu} \) is the double of \( \tau_j^{n,\nu} \). Hence, for each \( x \in \mathbb{R}^d \) and \( j \in \mathbb{Z} \),

\[
\sum_{i \leq j} H_j^{n,\nu} \ast H_i^{n,\nu} \ast |B_{i-n}(x)| \lesssim 2^{-n\gamma(d-1)} 2^{-jd} \sum_{i \leq j} \int_{x+\tau_i^{n,\nu}} |B_{i-n}(y)| \, dy
\]

\[
\lesssim 2^{-n\gamma(d-1)} 2^{-jd} \sum_{i \leq j} \sum_{Q \in \Omega_{\lambda} : L(Q) = i-n, Q \cap (x+\tau_i^{n,\nu}) \neq \emptyset} \int |b_Q(x)| \, dx
\]

\[
\lesssim 2^{-n\gamma(d-1)} 2^{-jd} \lambda \text{meas}(\tau_j^{n,\nu}) \lesssim 2^{-2n\gamma(d-1)} \lambda.
\]

Here we have used \( \|b_Q\|_1 \lesssim \lambda |Q| \), and the disjointness of the interiors of the cubes \( Q \) in \( \Omega_{\lambda} \). Thus we get the estimate

\[
\left\| \sum_j (I - P_{j-n+\ell(n)}) T_j^{n,\nu} B_{j-n} \right\|_2^2 \lesssim 2^{-2n\gamma(d-1)} \lambda \sum_j \|B_{j-n}\|_1,
\]

which yields (5.2). \( \square \)
6. Open problems

6.1. Principal value integrals

Let

\[ T_r f(x) = \int_{|x-y|>r} K(x-y) \int_0^1 a(sx + (1-s)y) \, ds \, f(y) \, dy. \]

Our proof shows that the operators \( T_r \) are of weak type \((1,1)\), with uniform bounds; moreover, for \( f \in L^1 \), \( T_r f \) converges in measure to \( Tf \) where \( T \) is weak type \((1,1)\). However it is currently open whether the principal value \( \lim_{r \to 0} T_r f(x) \) exists for almost every \( x \in \mathbb{R}^d \). By Stein’s theorem [9] this is equivalent to the open question whether the maximal singular integral \( \sup_{r>0} |T_r f| \) defines an operator of weak type \((1,1)\).

6.2. Principal value integrals for rough singular convolution operators

The question analogous to 6.1 is open for classical singular integral operators with rough convolution kernel \( \Omega(y/|y|)|y|^{-d} \) where \( \Omega \in L \log L(S^{d-1}) \), \( d \geq 2 \) and \( \int_{S^{d-1}} \Omega(\theta) \, d\sigma = 0 \). These operators are known to be of weak type \((1,1)\), [8], but the a.e. existence of the principal value integrals is open even for \( \Omega \in L^\infty(S^{d-1}) \).

6.3. Christ–Journé operators

Let \( F \in C^\infty(\mathbb{R}) \), let \( K \) be a Calderón–Zygmund convolution kernel, and let \( a \in L^\infty(\mathbb{R}^d) \). Christ and Journé [3] showed that the operator defined for \( f \in C_0^\infty(\mathbb{R}^d) \) by

\[ T f(x) = p.v. \int F \left( \int_0^1 a(sx + (1-s)y) \, ds \right) K(x-y) \, f(y) \, dy \]

extends to a bounded operator on \( L^p(\mathbb{R}^d) \), \( 1 < p < \infty \). It would be interesting to get the weak type \((1,1)\) inequality for nonlinear \( F \), in dimension \( d \geq 2 \).

References


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