On ovals on Riemann surfaces

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Abstract. We prove that $k$ ($k \geq 9$) non-conjugate symmetries of a Riemann surface of genus $g$ have at most $2g - 2 + 2^{r-3}(9 - k)$ ovals in total, where $r$ is the smallest positive integer for which $k \leq 2^{r-1}$. Furthermore we prove that for arbitrary $k \geq 9$ this bound is sharp for infinitely many values of $g$.

1. Introduction.

Let $X$ be a compact Riemann surface of genus $g \geq 2$. By a symmetry of $X$ we mean, in this paper, an antiholomorphic involution $\sigma$ which has fixed points. A surface admitting a symmetry is said to be symmetric. The principal motivation for the study of symmetric Riemann surfaces comes from the theory of algebraic curves. A compact Riemann surface $X$ corresponds to a smooth complex projective algebraic curve and symmetries, non-conjugate in the group Aut$^\pm(X)$ of all automorphisms of $X$, give rise to non-isomorphic over the reals, real models of the curve. A classical theorem of Harnack [8] states that the set $F(\sigma)$ of fixed points of $\sigma$ consists of $\|\sigma\|$ in range $1 \leq \|\sigma\| \leq g + 1$ disjoint simple closed curves to which, following Hilbert’s terminology, we shall refer to as the ovals of $\sigma$. The number of ovals of a symmetry equals the number of connected components of the corresponding real model.

In this paper we are looking for the maximal number $\omega(g, k)$ of ovals that $k$ non-conjugate symmetries of a Riemann surface $X$ of genus $g$ may admit. This question was investigated at the end of seventies by S.
M. Natanzon in [11], [12] and [13] who proved many results concerning low values of $k$. In particular, he proved that \( \omega(g, k) \leq 2g + 2^{k-1} \) for $2 \leq k \leq 4$ and that this bound is attained respectively for every $g$ congruent to 1 modulo $2^{k-2}$. However the problem of finding the bound for $\omega(g, k)$ for $k \geq 5$ has not been solved up to now. Results concerning surfaces of even $g$, which by [6] have at most 4 non-conjugate symmetries with fixed points, have been recently obtained in [7].

Recently this question was taken up by Singerman [17] who showed that for arbitrary $k$ there exist infinitely many values of $g$ for which there exists a Riemann surface of genus $g$ having $k$ non-conjugate symmetries and $M_k = 2g + 2^{k-3}(9 - k) - 2$ ovals in total and he conjectured that this is the best bound. From the recent paper of Natanzon [14] it follows that this indeed is the case in the special situation of separable symmetries. Observe that for $k = 3$ and 4 the Singerman and Natanzon bounds coincide without this additional assumption.

Here we show that for $k \geq 9$, \( \omega(g, k) \leq 2g - 2 + 2r^{-3}(9 - k) \), where $r$ is the smallest positive integer for which $k \leq 2r^{-1}$. Furthermore we prove that for arbitrary $k \geq 9$ this bound is sharp for infinitely many values of $g$. In particular there are no $k > 9$ for which Singerman’s conjecture is true. It is true for $k = 9$ and probably true for $5 \leq k \leq 8$.

2. Preliminaries.

The results announced in the previous section will be proved using combinatorial techniques based on Fuchsian and NEC groups. The basic results concerning these matter can be found in [3]. However for the reader’s convenience we point out some of the most important concepts and results.

The starting point in a combinatorial study of compact Riemann surfaces of genus $g \geq 2$ is the Riemann uniformization theorem by which each such surface can be represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of some Fuchsian surface group $\Gamma$. Furthermore having a surface $X$ so represented its group of automorphisms can be represented as $\Delta/\Gamma$ for another Fuchsian group $\Delta$. Now the orbit space of $X$ under the action of some symmetry $\sigma$ has a structure of Klein surface and the point is that the counterpart of these results for Klein surfaces also holds (see [10] and [15]), where NEC groups play the role of Fuchsian groups.

The algebraic structure of an NEC group $\Lambda$ is determined by its
signature ([9], [18]) which is a symbol of the form
\[
(g'; \pm; [m_1, \ldots, m_r]; \{C_1, \ldots, C_k\}),
\]
where the numbers \(m_i \geq 2\) are called the proper periods, \(C_i\) are the \(s_i\)-uples \((n_{i1}, \ldots, n_{is_i})\) called the period cycles, the numbers \(n_{ij} \geq 2\) are the link periods and \(g' \geq 0\) is said to be the orbit genus of \(\Lambda\). A surface NEC group is an NEC group with only empty period cycles and without proper periods, i.e., an NEC group with signature \((g'; \pm; [-], \{(-), \ldots, (-)\})\), a Fuchsian group can be regarded as an NEC group with signature \((g'; +; [m_1, \ldots, m_r]; \{-\})\) and finally a Fuchsian surface group is a Fuchsian group with signature \((g'; +; [-]; \{-\})\).

A group \(\Lambda\) with signature (1) has a presentation with canonical generators
\[
x_i, \quad 1 \leq i \leq r, \quad e_i, c_{ij}, \quad 1 \leq i \leq k, \quad 0 \leq j \leq s_i,
\]
and
\[
a_i, b_i \text{ or } d_i, \quad 1 \leq i \leq g',
\]
and relations
\[
x_i^{m_i}, \quad 1 \leq i \leq r, \quad c_{ij}^2, \quad (c_{ij-1} c_{ij})^{n_{ij}}, \quad c_{i0} e_i^{-1} c_{is_i} e_i,
\]
with \(1 \leq i \leq k, \quad 0 \leq j \leq s_i\), and
\[
x_1 \cdots x_r e_1 \cdots e_k a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_{g'} b_{g'} a_{g'}^{-1} b_{g'}^{-1},
\]
or
\[
x_1 \cdots x_r e_1 \cdots e_k d_1^2 \cdots d_{g'}^2,
\]
according as the sign is + or −.

Finally the hyperbolic area of an arbitrary fundamental region of an NEC group \(\Lambda\) with signature (1) equals
\[
(2) \quad \mu(\Lambda) = 2\pi \left(\varepsilon g' - 2 + k + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right)\right),
\]
where \(\varepsilon = 2\) if there is a “+” sign and \(\varepsilon = 1\) otherwise. If \(\Gamma\) is a subgroup of finite index in \(\Lambda\), then it is an NEC group itself and we have the Hurwitz-Riemann formula
\[
(3) \quad [\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}.
\]
3. Centralizers, conjugacy classes and some combinatorics.

A group $G$ is said to be *abstractly orientable* if it admits an epimorphism $\alpha : G \rightarrow \mathbb{Z}_2 = \{\pm 1\}$ which will be called an *abstract orientation* of $G$. An element $g$ of $G$ is said to be *orientation preserving* (respectively *orientation reversing*) subject to the orientation $\alpha$ if $\alpha(g) = +1$ (respectively $\alpha(g) = -1$). Examples of orientable groups are provided by proper NEC groups and groups $\text{Aut}^+(X)$ of all automorphisms of symmetric Riemann surfaces $X$. The first lemma of this section is an immediate consequence of Sylow theorems.

**Lemma 3.1.** Let $2^n$ be the biggest power of 2 that divides the order of an abstractly oriented finite group $G$. Then $G$ has at most $2^{n-1}$ conjugacy classes of orientation reversing elements of order 2.

**Proof.** Indeed let $S$ be a Sylow subgroup of $G$. Then each conjugacy class has a representative in $S$. So the lemma follows since $\text{Ker}\alpha|_S$, which consists of orientation preserving elements is a subgroup of $S$ of index 2.

**Lemma 3.2.** Let $G$ be a finite group and let $y_1, y_2$ be two elements of order 2 whose product has order $n$. Then the order of the centralizer $C(G, y_i)$ of $y_i$ in $G$ does not exceed $2|G|/n$ for $i = 1, 2$.

**Proof.** Let $H$ be the group generated by $y_1$ and $y_2$ and observe first that $C(H, y_i) = \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ according as $n$ is odd or even. Fix a system $X$ of representatives for the cosets of $G/H$. Then each element $g$ of $G$ can be represented as $g = y x$ for some $y \in H$ and $x \in X$ uniquely determined. Now assume that both $g = y x$ and $g' = y' x \in C(G, y_i)$. Then $H \ni y'y^{-1} = g'g^{-1} \in C(G, y_i)$. Thus $y'y^{-1} \in C(H, y_i)$ and so the lemma follows.

Finally in this section we prove the following elementary combinatorial lemma that we shall need in the sequel.

**Lemma 3.3.** Assume that $k, k \geq 3$ labels are used to label $s$ points situated on a circle in such a way that no two consecutive points have the same label. Then at least $k - 1$ points have neighbours with distinct labels.
PROOF. We shall prove the lemma by induction on $s$. Observe first that $s \geq k$ and that the cases $s = 3$ and $s = 4$ are trivial. So assume that $s \geq 5$. There is nothing to prove if no point has neighbours with the same label; here $s$ points have neighbours with distinct labels. So assume that there are three consecutive points $i - 1, i, i + 1$, say with labels $1, k$ and $1$ respectively and consider the induced configuration of $s - 2$ points $1, \ldots, i - 1, i + 2, \ldots, s$.

Assume first that some of these points have label $k$. Then by the inductive hypothesis $t \geq k - 1$ points have neighbours with distinct labels. If, in the new configuration, the point $i - 1$ has neighbours with the same label then in the former configuration these $t$ points have neighbours with distinct labels whilst if $i - 1$ has neighbours with distinct labels then in the former configuration $t - 1$ of these points and one among $i - 1$ and $i + 1$ has neighbours with distinct labels.

If none of the points $1, \ldots, i - 1, i + 2, \ldots, s$ has label $k$ then we have a configuration of $s - 2$ points on circle labeled by $k - 1$ labels. For $k = 3$, $s$ is even and we see that $i - 1$ and $i + 1$ have neighbours with distinct labels. So assume that $k > 3$. Then by the inductive hypothesis, $k - 2$ of these points have distinct labels. So the assertion follows since in this case these points and $i + 1$ have neighbours with distinct labels in the former configuration.

4. Symmetries of Riemann surfaces and their ovals.

Let $\text{Aut}^+(X)$ be the group of orientation preserving automorphisms of a compact Riemann surface $X$ represented as $\mathcal{H}/\Gamma$. Then $\text{Aut}^+(X) = \Delta/\Gamma$ for some Fuchsian group $\Delta$ which is the normalizer of $\Gamma$ in $\text{PSL}(2, \mathbb{R})$. Now, $X$ is symmetric if and only if there exists an NEC group $\Lambda$ containing $\Delta$ as a subgroup of index 2 and $\Gamma$ as a normal subgroup. In such case $G = \Lambda/\Gamma = \text{Aut}^+(X)$ is the group of all automorphisms of $X$, including those that reverse its orientation. Let $\theta : \Lambda \rightarrow G$ be the canonical projection. A symmetry of $X$ is an element $\sigma \in \text{Aut}^+(X) \backslash \text{Aut}^+(X)$ of order 2. Let us denote by $\langle \sigma \rangle$ the group generated by $\sigma$ and represent it as $\Gamma_\sigma / \Gamma$ for some NEC subgroup $\Gamma_\sigma$ of $\Lambda$. Then the orbit space $X / \langle \sigma \rangle \cong \mathcal{H} / \Gamma_\sigma$ is a Klein surface whose boundary coincides with $\text{Fix}(\sigma)$. So $\| \sigma \|$ is the number of period cycles of the signature of $\Gamma_\sigma$. Given a system of canonical generators of $\Lambda$, let $\{ c_i : i \in I \}$ be a set of representatives for the conjugacy classes of reflections in $\Lambda$. 


With these notations, a symmetry $\sigma$ of $X$ with non-empty set of fixed points is conjugate to $\theta(c_j)$ for some $j \in I$ and it was shown in [4] (see also [5]) that it has

\[ \| \sigma \| = \sum [C(\theta(A), \theta(c_i)) : \theta(C(A, c_i))] \]

ovals, where the sum is taken over all elements $i$ of $I$ for which $\theta(c_i)$ is conjugate to $\sigma$. The index $w_i = w_i^X = [C(\theta(A), \theta(c_i)) : \theta(C(A, c_i))]$ will be called a contribution of $c_i$ to $\| \sigma \|$.

Now let $\| X \|$ be the sum of all $\| \sigma \|$, where $\sigma$ is running over all conjugacy classes of symmetries of $X$. From (4) it follows immediately that

\[ \| X \| = \sum_{i \in I} [C(\theta(A), \theta(c_i)) : \theta(C(A, c_i))] . \]

In this context $w_i$ will be called a contribution of $c_i$ to $\| X \|$ or we shall say simply that $c_i$ contributes to $X$ with $w_i$ ovals.

Singerman [16] proved that the centralizer $C(A, c_j)$ of a canonical reflection $c_j$ in an NEC group $A$ is

\[ \langle c_j \rangle \times \langle c_j \rangle = \mathbb{Z}_2 \times \mathbb{Z} \]

if $c_j$ corresponds to an empty period cycle and

\[ \langle c_0 \rangle \times \langle (c_0 c_1)^{n_1/2} \rangle \times \langle (c_0 c_1 c_2)^{n_2/2} \rangle = \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2) \]

or

\[ \langle c_j \rangle \times \langle (c_{j-1} c_j)^{n_j/2} \rangle \times \langle (c_{j+1} c_j)^{n_{j+1}/2} \rangle = \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2) \]

if $c_j$ corresponds to a period cycle $(n_1, \ldots, n_s)$ with even link periods, where $j = 0$ or $j \neq 0$ respectively. We are ready to state and prove the main result of the paper.

**Theorem 4.1.** Let $\sigma_1, \ldots, \sigma_k$ be non-conjugate symmetries of a Riemann surface $X$ of genus $g \geq 2$ for which $G = Aut^+(X)$ is a 2-group. Then $\| \sigma_1 \| + \cdots + \| \sigma_k \| \leq 2 g - 2 + (9 - k) |G|/8$.

**Proof.** Let $X = \mathcal{H}/\Gamma$ and $G = \Lambda/\Gamma$. Assume that $\Lambda$ has signature of a general form

\[ (g'; \pm; [m_1, \ldots, m_r]; \{C_1, \ldots, C_m, (-), \ldots, (-)\}) , \]
where \( C_i = (n_{i1}, \ldots , n_{is}) \) and denote \( s = s_1 + \cdots + s_m \). Observe that every link period is a power of 2. Let \( \theta : \Lambda \to G \) be the canonical epimorphism.

Assume first that none of \( \sigma_1, \ldots , \sigma_k \) is central. Then \( |C(G, \sigma_i)| \leq |G|/2 \) for \( i \leq k \). So any canonical reflection \( c \) corresponding to an empty period cycle contributes to \( \|X\| \) with at most \( |G|/4 \) ovals, by (6) and (5) whilst a reflection corresponding to a non-empty period cycle contribute to \( \|X\| \) with at most \( |G|/8 \) ovals by (5) and (7) or (8). So \( \|X\| \leq (2l + s)|G|/8 \). On the other hand \( g - 1 \geq (4l + 4m - 8 + s)|G|/8 \) by the Hurwitz-Riemann formula as \( \mu(A) \geq 2\pi (l + m - 2 + s/4) \). Thus since \( k \leq l + s \) we obtain \( 6l + 8m + s > 7 + k \) since for \( m = 0 \) we have \( l \geq k \geq 9 \). Consequently

\[
\|X\| \leq (2s + 8l + 8m - 16) \frac{|G|}{8} + (16 - 6l - 8m - s) \frac{|G|}{8} \\
\leq 2g - 2 + (9 - k) \frac{|G|}{8}.
\]

So we can assume that some of the symmetries in question, say \( z \), is a central element of \( G \). Furthermore we can assume that \( l = 0 \) and \( m = 1 \). Observe first that \( m \neq 0 \). Indeed if \( m = 0 \) then as above we prove that \( \|X\| \leq l |G|/2 \) and \( 2g - 2 \geq |G|(l - 2) \). So

\[
\|X\| \leq l \frac{|G|}{2} = |G|(l - 2) + (4 - l) \frac{|G|}{2} \\
\leq 2g - 2 + (16 - 4l) \frac{|G|}{8} \\
< 2g - 2 + (9 - k) \frac{|G|}{8}
\]

since \( 4l - k > 7 \) as \( l \geq k \geq 9 \). Thus we can assume that \( m > 0 \) because otherwise the theorem holds.

We can assume that \( \theta(c_{10}) \neq z \). If \( l \neq 0 \) consider an NEC group \( \Lambda' \) with signature

\[
(g'; \pm [m_1, \ldots , m_r]; \{(2, 2, 2, n_{11}, \ldots , n_{1s}), C_2, \ldots , C_m, (-), (\cdots , (-))\}).
\]

For the sake of technical simplicity, we denote in the same way as in the group \( \Lambda \) some of the canonical generators of \( \Lambda' \); namely those generators
which correspond to “pieces” of the signature of $\Lambda$ in the signature of $\Lambda'$ and for the sake of terminological convenience we shall refer to these generators of $\Lambda'$ as old generators. To be more precise, this means here in the case of the signatures (9) and (10) that the hyperbolic generators of $\Lambda'$ are $a_1, b_1, \ldots, a_g', b_g'$ or $d_1, \ldots, d_g'$ according to whether the sign is $+$ or $-$, the elliptic generators are $x_1, \ldots, x_r$, generators corresponding to the first nonempty period cycle are $e_1, e_0', c_1', c_2', c_3, c_{10}, c_{11}, \ldots, c_{18}$, the generators corresponding to the remaining nonempty period cycles are $e_i, c_{10}, c_{11}, \ldots, c_{18}$, whilst generators corresponding to empty period cycles are $e_{m+1}, e_{m+1}, \ldots, e_{m+l-1}, e_{m+l-1}$. Furthermore according to this convention $e_0', c_1', c_2'$ and $c_3'$, are new generators whilst the remaining are old ones. We shall consider separately two cases

\[ a) \ \theta(c_{m+i}) \neq z, \quad b) \ \theta(c_{m+i}) = z. \]

Case a). Here we define $\theta': \Lambda' \to G$ on all old canonical generators but $e_1$ by $\theta$ and we put $\theta'(e_1) = \theta(e_1 \cdots e_{m+i}) \theta(e_2 \cdots e_{m+i-1})^{-1}$, $\theta'(e_0) = \theta'(c_3 \cdots c_1') = \theta'(c_3') = \theta'(c_5') = z$, and $\theta'(c_2') = \theta(c_{m+i})$. Then, using results of [3, Chapter 2], it is not difficult to see that $\Gamma' = \text{Ker } \theta'$ is a Fuchsian surface group. Indeed, by Theorem 2.2.4, its signature has no proper periods, by Theorem 2.3.3, it has no link periods, and finally, by Theorem 2.1.3, its sign is $+$. Let $X' = \mathcal{H}/\Gamma'$. As $\mu(\Lambda) = \mu(\Lambda')$ we see that $X$ and $X'$ have the same genus. We shall show that $\|X'\| \geq \|X\|$. As the images under $\theta'$ of all old, except $c_{10}$, canonical reflections corresponding to nonempty period cycles and their neighbours are the same as their images under $\theta$ we see, by (5) and (7) or (8), that each of these reflections contributes to $X'$ with the same number of ovals as to $X$. Similarly, by (6) and (5), old reflections corresponding to empty period cycles contribute to $X'$ with the same number of ovals as to $X$. So we have to show that $c_{10}, c_0', c_1', c_2'$ and $c_3'$ contribute all together to $X'$ with at least as many ovals as $c_{m+i}$ and $c_{10}$ contribute to $X$.

Let $w_{10}$ be the contribution of $c_{10}$ to $\|X\|$. Then $c_{10}$ contributes to $X'$ with $w_{10}$ or $w_{10}/2$ ovals according to whether $\theta(c_{10} c_{11})^{n_{11}/2} = z$ or not. Similarly $c_0'$ contributes to $X'$ with $w_{10}$ or $w_{10}/2$ ovals according to whether $\theta(c_{18} c_{11})^{n_{11}/2} = z$ or not. Consequently reflections $c_{10}$ and $c_0'$ contribute to $\theta(c_{10})$ at least the same number of ovals as $c_{10}$ to $\theta'(c_{10})$.

Assume now, that $c_{m+i}$ had contributed with $k$ ovals to $\theta(c_{m+i})$. Then $c_0'$ contributes to the new surface $X'$ also with $k$ ovals if $\theta(c_{m+i}) \neq 1$ and in this case we are done since the new surface has at least the same number of ovals as the former one. If $\theta(c_{m+i}) = 1$ then $c_0'$ contribute to $X'$ with $k/2$ ovals. Let $n'$ and $n''$ be the orders of $\theta'(c_0') \theta'(c_2')$
and $\theta'(c_2) \theta'(c_{10})$ respectively and let $n = \max \{n', n''\}$. Then the centralizer of $\theta(c_{m+1})$ had order not bigger than $2 |G|/n$ by the Lemma 3.2 and so $c_{m+1}$ had contributed to the former surface at most with $|G|/n$ ovals, i.e., $k \leq |G|/n$ whilst now $c_1'$ and $c_3'$ contribute to $z$ with $|G|/4n' + |G|/4n'' \geq |G|/2n \geq k/2$ ovals on the new surface $X'$. So indeed $\|X'\| \geq \|X\|$.

Case b). If $\theta(c_{m+1}) = z$ then we define $\theta' : \Lambda' \to G$ on all old canonical generators and on $c_0'$ as for the case $\theta(c_{m+1}) \neq z$ and we put $\theta'(c_1') = \theta'(c_2') = \theta(c_{m+1})$, and $\theta'(c_2') = \theta(c_{10})$. Again, using results of [3, Chapter 2] one can prove that $\Gamma' = \text{Ker} \theta'$ is a Fuchsian surface group and by the Hurwitz-Riemann formula $X' = \mathcal{H}/\Gamma'$ is a Riemann surface of genus $g$. We shall show that $\|X'\| \geq \|X\|$. Also here all old canonical reflections but $c_{10}$ contribute to $X'$ with the same number of ovals as to $X$. The new reflection $c_2'$ contributes to $X'$ with no less ovals than $c_{10}$ to $X$. Here $c_{m+1}$ had contributed to $\theta(c_{m+1})$ with $|G|/4$ or $|G|/2$ ovals according as $\theta(c_{m+1}) \neq 1$ or $\theta(c_{m+1}) = 1$. In the first case we see that $\|X'\| \geq \|X\|$ as $c_2'$ contribute to $X'$ with $|G|/4$ ovals also. If $\theta(c_{m+1}) = 1$, then $\theta'(c_1') = \theta(c_{10})$. So in this case $\theta'(c_0') = \theta(c_{10})$ and therefore $c_1'$ and $c_3'$ contribute to $X'$ with $|G|/4$ ovals each. Hence again $\|X'\| \geq \|X\|$.

Thus we can assume that $\Lambda$ has no empty period cycles, i.e., it has signature

$$\text{(11)} \quad (g'; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{m1}, \ldots, n_{ms_m})\}).$$

Now we shall see that, actually we can assume that $m = 1$, i.e., $\Lambda$ has just one period cycle. For, observe that we can assume that $\theta(c_{1s_1}) \neq z$ and $\theta(c_{20}) \neq z$. Let $\Lambda'$ be an NEC group with signature

$$\text{(12)} \quad (g'; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}, 2, 2, n_{21}, \ldots, n_{2s_2}, 2, 2),
\quad C_3, \ldots, C_s\}).$$

Here the reflections corresponding to the first period cycle are

$$c_{10}, \ldots, c_{1s_1}, c_0', c_{20}, \ldots, c_{2s_2}, c_1', c_2'$$

and also here $\mu(\Lambda) = \mu(\Lambda')$. We define $\theta' : \Lambda' \to G$ on all old canonical generators but $c_1$ as before i.e., by $\theta$ and we put $\theta'(c_1) = \theta(c_1) \theta(c_2)$. Furthermore we define $\theta'(c_0') = \theta'(c_1') = z$ and $\theta'(c_0') = \theta'(c_1) \theta'(c_{10}) \theta'(c_1')$. Once more, using results of [3, Chapter 2], we
see that $\Gamma' = \operatorname{Ker} \theta'$ is a Fuchsian surface group. Then $X' = \mathcal{H}/\Gamma'$ is a Riemann surface of genus $g$. In a similar way, we can prove that $\|X'\| \geq \|X\|$. Indeed all old canonical reflections, but $c_{i0}$ and $c_{i2}$ contribute to $X'$ with the same number of ovals as to $X$.

Let $w_i^X$ be the contribution of $c_{i0}$ to $\|X\|$ and let $l_i$ be the order of the centralizer of $\theta(c_{i0})$ for $i = 1, 2$. Then $w_i^X = l_i/4 k_i$, where $k_i$ is the order of $\theta(c_{i0} c_{i1})^{n_{i1}/2} \theta(c_{i0}^{-1} c_{i, s_1} c_{i, s_2})^{n_{i2}/2} c_{i2}$. In particular we see that $w_i^X \leq l_i/4$. On the other hand, as $\theta'(c_{i0} c_{i1})^{n_{i1}/2} \theta'(c_{i0}^{-1} c_{i} c_{i2})$ and $\theta'(c_{i1} c_{i} c_{i2})^{n_{i2}/2} \theta'(c_{i1} c_{i2})$ have order 2 we see that $c_{i0}$ and $c_{i1}$ contribute to $X'$ with no less ovals than $c_{i0}$ to $X$. Similarly $c_{i2}$ and $c_{i2}$ contribute to $X'$ with no less ovals than $c_{i2}$ to $X$. So we see that indeed $\|X'\| \geq \|X\|$.

So at last we arrive at the case of an NEC group $\Lambda$ with signature

$$(g'; \pm; [m_1, \ldots, m_r]; \{(n_1, \ldots, n_s)\}).$$

Let $c_0, \ldots, c_s$ denote the corresponding canonical reflections. Observe that $s \leq 8 (g-1)/|G| + 4$.

We can assume that $\theta(c_0)$ is a central symmetry of $X$ and so in particular $\theta(c_0) = \theta(c_s)$. Consider $c_0, c_1, \ldots, c_{s-1}$ as $s$ points on a circle labelled by $\theta(c_0), \theta(c_1), \ldots, \theta(c_{s-1})$ respectively. By the Lemma 3.3, at least for $k - 1$ numbers in range $0 \leq i_1 < \cdots < i_{k-1} \leq s - 1$, $\theta(c_{i_{k-1}+1}) = \theta(c_{i_{k-1}+1})$, where the indices are taken modulo $s$.

Now if $n_{i_i} > 2$ or $n_{i_{i+1}} > 2$ then $\theta(c_{i_i})$ is not central and so $|C(G, \theta(c_{i_i}))| \leq |G|/2$. Therefore $c_{i_i}$ contributes to the corresponding surface $X$ with at most $|G|/8$ ovals. If $n_{i_i} = n_{i_{i+1}} = 2$ then $|\theta(C(\Lambda, c_{i_i}))| \geq 8$ and thus also now $c_{i_i}$ contributes to $X$ with at most $|G|/8$ ovals. The remaining canonical reflections contribute to $X$ with no more than $|G|/4$ ovals. So

$$\|X\| \leq (k - 1) \frac{|G|}{8} + (s - k + 1) \frac{|G|}{4}$$

$$= s \frac{|G|}{4} + (1 - k) \frac{|G|}{8}$$

$$\leq 2 g - 2 + |G| + (1 - k) \frac{|G|}{8}$$

$$= 2 g - 2 + (9 - k) \frac{|G|}{8}.$$ 

This completes the proof.
Corollary 4.2. Let $\sigma_1, \ldots, \sigma_k$, where $k \geq 9$ be non-conjugate symmetries of a Riemann surface $X$ of genus $g \geq 2$. Then $\|\sigma_1\| + \cdots + \|\sigma_k\| \leq 2g - 2 + 2^{r-3}(9 - k)$, where $r$ is the smallest positive integer for which $k \leq 2^{r-1}$.

Proof. As we are looking for the ovals of these symmetries and conjugate symmetries have the same number of ovals we can assume, using Sylow theorem, that they generate a 2-subgroup $G$ of Aut$^\pm(X)$. Let $X = \mathcal{H}/\Gamma$ and $G = \Lambda/\Gamma$. Assume that $\Lambda$ has signature $(9)$. Then, as $s + l \geq k \geq 9$, we see, by [2] (see also [3, Theorem 2.4.7]), that its signature is maximal. So by [3, Theorem 5.1.2] there exists a maximal NEC group $\Lambda'$ and algebraic isomorphism $\varphi : \Lambda \rightarrow \Lambda'$. Let $X' = \mathcal{H}/\Gamma'$, where $\Gamma' = \varphi(\Gamma)$. Then $\text{Aut}^\pm(X') = \Lambda'/\Gamma'$ and $\varphi$ induces an isomorphism $\bar{\varphi} : \Lambda/\Gamma \rightarrow \Lambda'/\Gamma'$. Now $\bar{\varphi}(\sigma_1), \ldots, \bar{\varphi}(\sigma_k)$ are non-conjugate symmetries of $X'$. Furthermore if $\langle \sigma_i \rangle = \Lambda_i/\Gamma$, then $\|\sigma_i\|$ is the number of empty period cycles of $\Lambda_i$. So $\|\sigma_i\| = \|\bar{\varphi}(\sigma_i)\|$ since $\langle \bar{\varphi}(\sigma_i) \rangle = \varphi(\Lambda_i)/\Gamma'$. Furthermore $\|X\| \leq \|X'\|$ and $G \cong \text{Aut}^\pm(X')$ is a 2-group. Then by Theorem 4.1, $\|X'\| \leq 2g - 2 + (9 - k) |G|/8$ and by Lemma 3.1, $|G| \geq 2^r$. Hence the Corollary follows.

The next theorem shows that the bound obtained in Corollary 4.2 is sharp.

Theorem 4.3. Let $k \geq 9$ be an arbitrary integer and let $r$ be the smallest positive integer for which $k \leq 2^{r-1}$. Then for arbitrary $g = 2^{r-2}t + 1$, where $t \geq k - 3$ there exists a Riemann surface $X$ of genus $g$ having $k$ non-conjugate symmetries which have $2g - 2 + 2^{r-3}(9 - k)$ ovals in total.

Proof. Let $G = \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 = \langle z_1 \rangle \oplus \cdots \oplus \langle z_r \rangle$ and let $\Lambda$ be a maximal NEC-group with signature $(0; +; [-]; \{(2, 2^s, 2)\})$, where $s = (g - 1)/2^{r-2} + 2 \geq k - 1$. Let $\{a_1, \ldots, a_{2^{r-1}}\}$ be all elements of order 2 in $G$ which have odd length in $z_1, \ldots, z_r$ and assume that $a_1, \ldots, a_r$ generate $G$. Then since $r$ is the minimal integer such that $k \leq 2^{r-1}$ we have $k \geq r$ and so the assignment

$$\theta(e) = 1, \quad \text{and} \quad \theta(a_i) = \begin{cases} a_1, & \text{for } i = 2j, \ 0 \leq j \leq s, \\ a_{2j+2}, & \text{for } i = 2j + 1, \ 0 \leq j \leq k - 2, \\ a_k, & \text{for } i = 2j + 1, \ k - 1 \leq j \leq s - 1, \end{cases}$$

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defines an epimorphism \( \theta : \Lambda \rightarrow G \) for which \( \Gamma = \text{Ker} \theta \) is a surface group and \( X = \mathcal{H}/\Gamma \) is a Riemann surface having \( k \) non-conjugate symmetries with fixed points.

We see that \( c_{2j} \), for \( 0 \leq j \leq k - 2 \) contribute to \( a_1 \) with \( 2^{r-3} \) ovals whilst the remaining \( 2s - k + 1 \) non-conjugate canonical reflections of \( \Lambda \) contribute to the corresponding surface with \( 2^{r-2} \) ovals. As a result

\[
\|\sigma_1\| + \cdots + \|\sigma_k\| = 2^{r-3}(k - 1) + 2^{r-2}(2s - k + 1) \\
= 2^{r-1}s + 2^{r-3}(1 - k) \\
= 2g - 2 + 2^r + 2^{r-3}(1 - k) \\
= 2g - 2 + 2^{r-3}(9 - k).
\]

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