Cohomology Mod 2 of the Classifying Space of $Spin^c(n)$

By

Masana HARADA* and Akira KONO**

In this paper we determine the mod 2 cohomology ring and the integral cohomology ring of the classifying space of the compact, connected Lie group $Spin^c(n)$, which is a subgroup of the group of units in the complex Clifford algebra $C_n \otimes C$ (see [1]). The group $Spin^c(n)$ is very important for the orientations in the KO-theory. We also determine (the mod 2 reduction of) the Chern classes of the complex spin representations and the Hopf algebra structure of the mod 2 cohomology ring of $Spin^c(n)$.

The first section is devoted to studying an ideal of a polynomial ring over $F_2$ which is associated to a symplectic bilinear form on a $F_2$ vector space and whose variety of geometric points is the union of the maximal isotropic subspaces rational over $F_2$. We show that the generators of the ideal form a regular sequence and we determine the decomposition of the ideal into prime ideals. These algebraic-geometric results are applied in the second and third sections to compute the mod 2 and integral cohomology ring of $BSpin^c(n)$ and determine the Chern classes of the spin representation of $Spin^c(n)$. In the last section we compute the Steenrod operations and the coproducts of the mod 2 cohomology of $Spin^c(n)$.

Throughout the paper $H^*(X)$ denotes the mod 2 cohomology ring.

1. Let $V$ be an $n$-dimensional vector space over $F_2$, $V^*$ its dual, $S(V^*)$ the symmetric algebra over $V^*$ and $B$ a symplectic bilinear form on $V$. Let $h'$ be the codimension of a $B$-isotropic subspace of maximum dimension. Consider the following sequence of homogeneous
elements of length $h'$ in $S(V^*)$:

$$B(x, x^2), \ldots, B(x, x^{2^h}).$$

Let $Q$ be a universal field of $F_2$, $V_Q = V \otimes Q$, $J'$ the ideal of $S(V^*)$ generated by (1.1) and $\text{Var} J'$ the variety of zeros in $V_Q$. First we prove the following:

**Theorem 1.2.** $\text{Var} J' = \bigcup W$ where $W$ ranges over the maximal $B$-isotropic subspaces of $V$.

**Proof.** Using the identity

$$B(x^{2^i}, x^{2^j}) = B(x, x^{2^j-i}) x^i (i \leq j),$$

one see for an element $x \in V_Q$, that $x \in \text{Var} J'$ if and only if the $Q$-subspace

$$N_x = \Omega x + \Omega x^2 + \cdots + \Omega x^{2^h}$$

of $V_Q$ is $B$-isotropic. To prove the theorem we must therefore show that $x \in \text{Var} J'$ if and only if $N_x$ is stable under the Frobenius, which is shown by computing the dimension of maximal $B$-isotropic subspaces of $V_Q$ as was done in the proof of Theorem 2.4 of [5].

**Corollary 1.3.** The sequence (1.1) is a regular sequence.

**Remark 1.4.** All maximal $B$-isotropic subspaces of $V$ are of the same dimension $n - h'$.

Counting the number of the maximal $B$-isotropic subspaces, we can prove the following by Bezout's theorem (see Section 3 of [5]):

**Theorem 1.5.** The ideal $J'$ has a prime decomposition $J' = \cap p_W$, where $W$ ranges over all maximal $B$-isotropic subspaces and $p_W = \text{Ker} \{S(V^*) \rightarrow S(W^*)\}$.

Let $Q$ be a quadratic form on $V$. Then $B(x, y) = Q(x + y) + Q(x) + Q(y)$ is a symplectic bilinear form. Let $h$ be the codimension of a $Q$-isotropic subspace of maximum dimension. Then we can easily get the following:
Lemma 1.6. \( h = h' + e \) where \( e = 0, 1, \) or \( 1 \) depending on \( Q \) is real, complex, or quaternion respectively.

See Section 3 of [5].

2. First consider the central extension
\[
0 \rightarrow S^1 \xrightarrow{i} \tilde{V}^c \xrightarrow{\pi} V \rightarrow 1
\]
where \( V \) is an elementary abelian 2-group. As is well known that (2.1) is classified by an element \( b \in H^2(BV; \mathbb{Z}) \). Let \( \rho \) be the mod 2 reduction and \( B' = \rho(b) \). Since \( \text{Im} \rho = \text{Im} Sq^1 \), there is an element \( Q \in H^2(BV) \) such that \( B' = Sq^1 Q \). Note that \( H^2(BV) \) is isomorphic to \( S(V^*) \) and so \( Q \) is a quadratic form and \( B' = B(x, x^2) \). Let \( W \) be a maximal \( B \)-isotropic subspace. Then \( \tilde{W}^c = \pi^{-1}(W) = W \times S^1 \) since \( \rho \) is a monomorphism. Let \( \chi: \tilde{W}^c \rightarrow S^1 \) be a complex representation of \( \tilde{W}^c \) whose restriction to \( S^1 \) is the standard representation \( \varepsilon \) and let \( \Delta \) be the representation of \( \tilde{V}^c \) obtained by inducing \( \chi \) from \( \tilde{W}^c \) to \( \tilde{V}^c \). Then \( \Delta \) has dimension \( 2h' \) and \( i^*(\Delta) = 2h' \varepsilon \). Now we can prove the following:

Theorem 2.2. As an algebra \( H^*(BV^c) \) is isomorphic to \( S(V^*) \otimes F_2[e] \), where \( J' \) is the ideal generated by \( B(x, x^2), \ldots, B(x, x^{h'}) \) and \( e \in H^{h'+1}(BV^c) \) is the Euler class of \( \Delta \).

Proof. Consider the Serre spectral sequence for the fibering
\[
BS^1 \xrightarrow{i} \tilde{V}^c \xrightarrow{\pi} BV
\]
\[
E_1^{s,t} = H^s(BV; H^t(BS^1)) \Rightarrow E_{\infty} = \text{Gr}(H^*(BV^c)).
\]
Let \( z \) be a generator of \( H^2(BS^1) \) so that \( H^*(BS^1) = F_2[z] \). The element \( z \) is transgressive with \( \tau(z) = B' = B(x, x^2) \). Therefore
\[
\tau(z^k) = \tau(S_k z) = S_k B(x, x^2) = B(x, x^{k+1})
\]
where \( S_k = Sq^{x^k} \ldots Sq^{x^2} \). Since \( B(x, x^2), \ldots, B(x, x^{h'}) \) is a regular sequence by Corollary 1.3, we can easily get
\[
E_{2h'+2} = S(V^*) \otimes F_2[z^{2h'}].
\]
On the other hand \( i^*(\varepsilon) = z^{2h'} \) since \( i^*(\Delta) = 2h' \varepsilon \). Hence \( E_{\infty} = E_{2h'+2} \) and we get the theorem.

Theorem 2.3. The homomorphism \( H^*(BV^c) \rightarrow \prod_{w} H^*(BW^c) \) is injective.
ive, where the product is taken over all maximal $B$-isotropic subspaces of $V$.

**Proof.** Recall the fact that $\ker \{H^*(B\mathcal{V}_c) \to H^*(B\mathcal{W}_c)\}$ is equal to $(p_w/J') \otimes F_2[e]$. Therefore Theorem 2.3 follows from Theorem 1.5.

**Remark 2.4.** We can determine the Chern classes of $\mathcal{A}$ using Theorem 2.3.

3. First recall the fact that the extension $0 \to \mathbb{Z}/2 \to \text{Spin}(n) \to SO(n) \to 1$ is classified by $\omega_2 \in H^2(BSO(n))$ and the extension

\[0 \to S^1 \to \text{Spin}^c(n) \xrightarrow{\tau} SO(n) \to 1\]

is classified by $b' \in H^3(BSO(n);\mathbb{Z})$ where $p(b') = \omega_3 = Sq^1 \omega_2$ since $\text{Spin}^c(n) = \text{Spin}(n) \times_{\mathbb{Z}/2} S^1$. Let $V$ be the diagonal matrices in $SO(n)$, $j : V \to SO(n)$ the inclusion and $\mathcal{V} = \pi^{-1}(V)$. Then the extension $0 \to S^1 \to \mathcal{V} \to \mathcal{W} \to 1$ is classified by $b = j^*(b')$ and so $Q = j^*(\omega_2)$ and $B' = j^*(\omega_3)$ in Section 2. Now using Table 6.2 of [5] and Lemma 1.6, we have the following:

**Lemma 3.2.** Let $h'$ be the codimension of a $B$-isotropic subspace of maximum dimension where $B$ is the associated bilinear form of $Q = j^*(\omega_2)$. Then $h' = \left\lfloor \frac{n-1}{2} \right\rfloor$ and $2^h$ is equal to the dimension of the complex spin representation of $\text{Spin}^c(n)$.

Consider the following commutative diagram:

\[
\begin{array}{ccc}
SO(n)/V & \xrightarrow{j^*} & B\mathcal{V}_c \xrightarrow{\tilde{j}} B\text{Spin}^c(n) \\
\| & & \| \\
SO(n)/V & \xrightarrow{j} & BV \xrightarrow{i} BSO(n)
\end{array}
\]

where the horizontal lines are fiberings. Since the Serre spectral sequence for the fibering $SO(n)/V \to BV \to BSO(n)$ collapses, the Serre spectral sequence for $SO(n)/V \to B\mathcal{V}_c \to B\text{Spin}^c(n)$ also collapses. Therefore we have the following:

**Lemma 3.3.** $H^*(BV)$ is a free module over $H^*(BSO(n))$ and $H^*(B\mathcal{V}_c)$ is a free module over $H^*(B\text{Spin}^c(n))$.

Since $j^*(S_i \omega_3) = S_i j^*(\omega_3) = S_i B(x, x^2) = B(x, x^{i+1})$, we have that
$f^*(w_3), \ldots, f^*(S_{h'-1}w_3)$ form a regular sequence. We have therefore the following by Lemma 3.3:

**Lemma 3.4.** The sequence $w_3, \ldots, S_{h'-1}w_3$ is a regular sequence.

Put $V_0 = \{ x \in V \mid B(x, y) = 0 \text{ for all } y \in V \}$. Then $\dim V_0 = 0$ if $n$ is odd and $\dim V_0 = 1$ if $n$ is even. There is a unique spin representation $A_{2m+1}$ if $n = 2m+1$ and there are two spin representations $A_{2m}$ if $n = 2m$. Consider the following orthogonal decomposition:

$$V = W_1 \oplus W_1' \oplus V_0.$$  

Then $W = W_1 \oplus V_0$. Put $\tilde{W}_s = \pi^{-1}(W)$ then $\tilde{W}_s = (W_1 \oplus V_0) \times S^1$ and $A |_{\pi^*} = (\text{reg } W_1) \otimes \theta \otimes e$

where $\text{reg } W_1$ is the regular representation, $\dim \theta = 1$ and $\theta$ is trivial (resp. non trivial) if $A = A_{2m}$ (resp. if $A = A_{2m+1}$). Therefore $i^*(A) = 2^i$. In the Serre spectral sequence for $BS^3 \to BSpin^c(n) \to BSO(n)$, $z$ is transgressive with $\tau(z) = w_3$. Now we have the following:

**Theorem 3.5.** As an algebra $H^*(BSpin^c(n))$ is isomorphic to $H^*(BSO(n))/J' \otimes \mathbb{F}_2[e]$, where $J'$ is the ideal generated by $w_3, S_1w_3, \ldots, S_{h'-1}w_3$ and $e$ is the Euler class of the complex spin representation $A$.

This follows by computing the Serre spectral sequence for $BS^3 \to BSpin^c(n) \to BSO(n)$ as was done in the proof of Theorem 2.2.

Now we determine the Chern classes of $A$. By Theorem 2.3 and Lemma 3.3, we need only determine $c_i(A |_{\pi^*})$. By a similar method to that of [5], we have the following (cf. Section 5 of [5]):

**Theorem 3.6.** (1) The classes $c_i(A_{2m})$ for $i < 2^{h'}$ are independent of $\pm$.
(2) $c_i(A) = 0$ for $i \neq 2^j$, $2^{h'} - 2^j$, $(j = 0, 1, \ldots, h')$.
(3) The sequence $\{c_i(A) ; i = 2^{h'}, 2^{h'} - 2^j, (j = 0, 1, \ldots, h' - 1)\}$ is a regular sequence in $H^*(BSpin^c(n))$.

On the other hand for the integral cohomology we can prove the following:
Theorem 3.7. The torsion elements of $H^*(BSpin^e(n); \mathbf{Z})$ are of order 2.

This follows by computing the $Sq^1$-cohomology of $H^*(BSpin^e(n))$ as was done in [4].

Remark 3.8. The natural map $H^*(BSpin^e(n); \mathbf{Z}) \to H^*(BSpin^e(n)) \times H^*(BSpin^e(n); \mathbf{R})$ is injective (see [4]).

4. Let $s = s(n)$ be the integer given by $2^{t-1} < n \leq 2^t$. Define $x_j \in H^j(Spin^e(n))$ by $\sigma(\pi^*(w_{j+1}))$ where $\sigma$ is the cohomology suspension. Note that $w_j = 0$ if $j > n$. By Theorem 3.5, as an algebra $H^*(BSpin^e(n))$ is isomorphic to

$$F_2[w'_j; 2 \leq j \leq n, j \neq 2^{t'} + 1 \ (j' \geq 1)]/(r)$$

for $* \leq 2^t + 1$, where

$$r = \sum_{i=1}^{s-1} w'_i w'_{2^{t-i} - 1} + \text{higher}$$

and $w'_j = \pi^*(w_j)$ ($w'_j$ is decomposable if $j = 2^{t'} + 1$). ($S_k w_3 = \sum_{i=0}^{k} w_i w_{2^i + 1 - i}$ mod $\tilde{H}^*(BO)^3$ can be shown by induction on $k$ using the Wu's formula (see 15.7 of [2])).

On the other hand by Theorem 1.1 of [3], there exists $a \in H^{2^{t-1}}(Spin^e(n))$ which is transgressive with respect to $Spin^e(n) \to Spin^e(n)/T \to BT$, where $T$ is a maximal torus, so that

$$H^*(Spin^e(n)) = \mathcal{A}(x_j; 1 \leq j < n, j \neq 2^{t'} (j' \geq 1)) \otimes \mathcal{A}(a)$$

where $\mathcal{A}(\ldots)$ means that $(\ldots)$ is a simple system of generators. Since $\phi(x_j) = 0$ by definitions where $\phi$ denotes the reduced coproduct of $H^*(Spin^e(n))$,

$$\phi(a) = \sum_{i+j = 2^t-1} \alpha_i x_{2^i} \otimes x_{2^j} \quad (\alpha_i \in F_2)$$

by Theorem 2.2 of [3] (see also Lemma 3.4 of [3]). Then by the Rothenberg–Steenrod spectral sequence ([6]) and (4.1), $\alpha_i = 1$ for any $i$. Then we have (see the proof of Theorem 3.2 of [3]):

Theorem 4.4. (1) In (4.2) $\phi(x_j) = 0$ and $\phi(a) = \sum_{i+j = 2^t-1} x_{2^i} \otimes x_{2^j}$
(x_j=0 \text{ if } j=2^{i'}(j'\geq 1)).

(2) \quad Sq^i x_j = \binom{i}{j} x_{j+i} (x_{2j}=x_j), \quad Sq^i a = \sum_{i+j^2=1} x_{2i} x_{2j} \text{ and } Sq^i a = 0 \text{ for } i \geq 2
\quad (a^2=0).

References
