A Classification of Factors, II

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Abstract

The algebraic invariant $r_0^M(M')$ of a factor $M$, introduced in an earlier paper and called the asymptotic ratio set, is shown to be closed for any factor $M$. As a consequence, this set must be one of the following sets: (i) the empty set, (ii) {0}, (iii) {1}, (iv) a one parameter family of sets $\{0, x^n; n=0, \pm 1, \ldots\}$, $0 < x < 1$, (v) all non-negative reals, (vi) {0, 1}.

§1. Introduction

In an earlier paper [1], we introduced an algebraic invariant $r_\omega(M)$ for a factor $M$. It is the set of all $x$, $0 \leq x < \infty$, such that $M$ is algebraically isomorphic to $M \otimes R_\omega$. Here $R_\omega$ is the type $I_\omega$ factor, $R_1$ is the hyperfinite type $II_1$ factor, and $R_x = R_{x^{-1}}$ for $0 < x < 1$ is a type III factor given by definition 3.10 of [1].

In [1], it is shown that $r_\omega(M) - \{0\}$ is either empty or a multiplicative group. Furthermore, for the case where $M$ is an infinite tensor product of type I factors, $r_\omega(M)$ is shown to be closed. However, this was not known in [1] for arbitrary $M$.

In this note, we show that $r_\omega(M)$ is closed for any factor $M$. The method of proof is already indicated in section 6 of [1], but new additional technique here is the use of weak clustering property, which is obtained by the crucial lemma 2.4.

§2. Lemmas

Lemma 2.1. Let $R_\ell$ be mutually commuting factors such that
$R = (\bigcup_i R_i)^{''}$ is a factor. Let $D$ be a finite set of unit vectors. Given $\epsilon$, there exists an $N$ such that

\begin{equation}
|\langle \psi, Q\phi \rangle - \langle \psi, \phi \rangle \langle \phi, Q\phi \rangle| < \epsilon
\end{equation}

for any $i > N$, $Q \in R_i$, $\|Q\| = 1$, $\psi \in D$, $\phi \in D$.

**Proof.** (cf. [2]) Since $R$ is a factor, the von Neumann algebra generated by $\bigcup_i R_i$ and $R'$ is the set of all bounded operators. Thus the self adjoint elements of the $*$ algebra generated by $\bigcup_i R_i$ and $R'$ are strongly dense among all self adjoint operators. In particular for the one dimensional projection $P(\phi)$ associated with a vector $\phi$, there exists a self adjoint $P'$ in \{ $\bigcup_i R_i \bigcup R'$ \} $''$ for some finite $N$ such that $P'$ is in the following strong neighbourhood of $P(\phi)$:

\begin{equation}
\{ A; \ |\langle P(\phi) - A \psi \rangle| < \epsilon/2, \forall \psi \in D \}.
\end{equation}

Then for any $Q \in R_i$, $i > N$, $\|Q\| = 1$, we have $[Q, P'] = 0$ and

\begin{align*}
|\langle \psi, Q\phi \rangle - \langle \psi, \phi \rangle \langle \phi, Q\phi \rangle| \\
= |\langle \psi, Q\{P(\phi) - P'\}\phi \rangle| + |\langle (P' - P(\phi))\psi, Q\phi \rangle| < \epsilon.
\end{align*}

**Definition 2.2.** Let $M$ be a type $I_\infty$ factor with a matrix unit $u_{kl}$, $k, l = 1, \ldots, n$ and $R$ be a factor containing $M$. For any $Q \in R$, define

\begin{equation}
\tau_{\psi}(M)Q = \sum_{j=1}^{n} u_{ji}Qu_{ij}.
\end{equation}

**Lemma 2.3.** Let $R$ be a factor and $M$ be a type $I_\infty$ factor in $R'$. For $Q \in (M \cup R)'$, $\tau_{\psi}(M)Q$ is in $R$, $\|\tau_{\psi}(M)Q\| \leq \|Q\|$ and

\begin{equation}
Q = \sum_{k, l} u_{kl}(\tau_{\psi}(M)Q).
\end{equation}

Furthermore, let $\psi$ be a unit vector and

\begin{equation}
Q' = \sum_{k, l} u_{kl}(\psi, \tau_{\psi}(M)Q\psi) \in M.
\end{equation}

Then $\|Q'\| \leq \|Q\|$.

**Proof.** Since $M$ is type $I_\infty$, it is possible to identify the Hilbert space $H$ with a tensor product $H_i \otimes H_i$, $M$ with $B(H_i) \otimes 1$ and $u_{kl}$ with $u_{kl} \otimes 1$, where $H_i$ is spanned by an orthonormal basis $\varphi_1, \ldots, \varphi_n$,
$u_{ki}v_j = \delta_{ij}v_k$ and $\mathcal{B}(H_1)$ denotes the set of all bounded operators on $H_1$. $M'$ is then $1 \otimes \mathcal{B}(H_2)$, in which $R$ is contained.

The equality (2.4) follows from (2.3) and $u_{ki}u_{ij} = \delta_{ij}u_{ki}$, $\sum u_{ki} = 1$. If $Q$ is in the $*$ algebra generated by $M$ and $R$, then it is of the form (2.4) where $\tau_{kl}(M)Q$ is in $R$. Therefore $\tau_{kl}(M)Q \in R$ holds also for the weak closure of such $Q$, namely for all $Q$ in $(M \cup R)'$. The norm of $\tau_{kl}(M)Q$ can be estimated by

$$\|\tau_{kl}(M)Q\| \leq \sup_{\|\psi\| = 1} |\langle \varphi_1 \otimes \psi^1, \tau_{kl}(M)Q \{\varphi_1 \otimes \psi^2\}\rangle|$$

because $\tau_{kl}(M)Q \in 1 \otimes \mathcal{B}(H_2)$. The right hand side is majorized by

$$\sup_{\|\psi\| = 1} |\langle \varphi_1 \otimes \psi^1, u_{kl}Qu_{12} \{\varphi_1 \otimes \psi^2\}\rangle| \leq \|u_{kl}Qu_{12}\| \leq \|Q\|.$$

A unit vector $\psi$ defines a density matrix $\rho$ in $\mathcal{B}(H_2)$ ($\rho \geq 0$, $tr \rho = 1$) through the relation

$$\langle \psi, (1 \otimes \hat{Q}) \psi \rangle = tr \rho \hat{Q}, \quad \hat{Q} \in \mathcal{B}(H_2).$$

For any unit vector $\varphi_1$ and $\varphi_2$ in $H_1$, we have

$$|\langle \varphi_1, Q' \varphi_2 \rangle| = |tr \{\varphi_1 \otimes \rho\}(u \otimes \mathbf{1})Q| \leq \|\rho\| \|\varphi\| = \|Q\|$$

where $\rho_1$ is the one dimensional projection on $\varphi_2$ and $u$ is an isometric operator with one dimensional range, bringing $\varphi_1$ onto $\varphi_2$. Therefore

$$\|Q'\| \leq \|Q\|.$$

**Lemma 2.4.** Let $R_i$ be mutually commuting factors such that $R = (\cup R_i)'''$ is a factor. Let $M$ be a type $I_n$ ($n < \infty$) factor contained in $R'$. Let $D$ be a finite sets of unit vectors such that the inequality (2.1) holds for any $Q \in M$, $\|Q\| = 1$, $\varphi \in D$, $\varphi \in D$. Given $\epsilon' > 0$. Then there exists an $N$ such that

$$|\langle \varphi, Q\varphi \rangle - \langle \varphi, \varphi \rangle (\varphi, Q\varphi)\rangle| < \epsilon + \epsilon'$$

for any $i > N$, $Q \in (M \cup R)'''$, $\|Q\| = 1$, $\varphi \in D$, $\varphi \in D$.

**Proof.** Let $u_{kl}$, $k, l = 1, \cdots, n$ be a matrix unit for $M$. Let $P(\varphi)$ be the one dimensional projection associated with each $\varphi \in D$. Find sufficiently large $N(\varphi)$ for each $\varphi \in D$ such that there exists a selfadjoint $P'(\varphi)$ belonging to
and satisfying

\begin{align}
(2.7) \quad & \| P'(\phi) - P(\phi) \| \psi' < \epsilon'' \\
(2.8) \quad & \| P'(\phi) - P(\phi) \| u_{ik} \psi' < \epsilon''
\end{align}

for all $\psi \in D$ and $l, k = 1, \ldots, n$. Here

\begin{align}
(2.9) \quad & \epsilon'' = 2^{-1}(1 + n^2)^{-1} \epsilon.
\end{align}

Let $N = \max_{\psi \in D} N(\phi)$ and $Q \in (M \cup R_i)'$, $i > N$, $\| Q \| = 1$. We then have the following inequalities, which proves (2.6):

\begin{align*}
| (\psi, Q \phi) - (\psi, \phi) (\phi, Q \phi) | \\
\leq | (\psi, Q \{ P(\phi) - P'(\phi) \} \phi) | \\
+ \sum_{k,l} | (\{ P'(\phi) - P(\phi) \} u_{ik} \psi, Q_{il} \phi) | \\
+ | (\psi, Q' \phi) - (\psi, \phi) (\phi, Q' \phi) | \\
+ \sum_{k,l} | (\{ P(\phi) - P'(\phi) \} u_{ik} \phi, Q_{il} \phi) (\psi, \phi) | \\
+ | (\phi, Q \{ P'(\phi) - P(\phi) \} \phi) (\psi, \phi) | \\
\leq \| Q \| \epsilon' + \sum_{k,l} \| Q_{il} \| \epsilon'' + \| Q' \| \epsilon + \sum_{k,l} \| Q_{il} \| \epsilon'' + \| Q \| \epsilon'' \\
\leq 2(1 + n^2) \epsilon'' + \epsilon = \epsilon' + \epsilon.
\end{align*}

Here we have used the notation and result of the previous lemma in which $\psi_0 = \phi$ and denoted $\tau_{\psi_0}(M)Q$ simply by $Q_{\psi_0}$.

**Definition 2.5.** A unit vector $\psi$ is **pure** for a type I factor $M$ if $\varphi_\psi(Q) = (\psi, Q \psi)$, $Q \in M$ is a pure state on $M$.

If $H = H_1 \otimes H_2$, $M = \mathcal{B}(H_1) \otimes 1$, then $\psi$ is pure if and only if $\psi = \psi_1 \otimes \psi_2$ for some $\psi_1 \in H_1$, $\psi_2 \in H_2$.

**Lemma 2.6.** Let $H = H_3 \otimes H_1$, $H_3 = \otimes_{\mu \neq \nu} (H_\nu, \varphi_\nu)$, $R_\nu = 1_\nu \otimes (\mathcal{B}(H_\nu) \otimes (\otimes_{\mu \neq \nu} 1_\mu))$. Let $M$ be a type I factor in $\mathcal{B}(H_2) \otimes 1$ and let a unit vector $\psi$ be pure for $M$. Given $\epsilon > 0$. Then there exists an $N$ and a unit vector $\psi_\epsilon$ such that $\psi_\epsilon$ is pure for $M$ as well as for $R_\nu$ for any $\nu > N$, $\| \psi - \psi_\epsilon \| < \epsilon$ and $\varphi_\psi$ is the same as the vector state corresponding to $\varphi_\psi$ for each $R_\nu$, $\nu > N$. 

Proof. Since $M$ is type I, we may identify $H_i$ with $H_{i1} \otimes H_{i2}$, $M$ with $\mathcal{B}(H_i) \otimes 1$. Since $\Psi$ is pure for $M$, it can be identified with $\Psi_{\epsilon_1} \otimes \Psi'$ where $\Psi' \in H_{i2} \otimes H_3$. For given $\epsilon > 0$, there exists $\Psi'_\epsilon$ of the form

$$\sum_{i=1}^k \Psi_i \otimes \Psi'_i', \Psi_i \in H_{i2}, (\Psi_i, \Psi_j) = \delta_{ij}, \sum ||\Psi'_i||^2 = 1$$

such that

$$||\Psi' - \Psi'_\epsilon|| < \epsilon/2.$$  

By lemma 2.7 of [1], there exists an $N$ and $\psi''_i$ for each $i$ such that

$$\psi''_i = \psi''''_i \otimes \bigotimes_{\nu > N} (\otimes \Omega_{\nu}),$$

$$||\psi'_i - \psi''_i|| < \epsilon/(2k),$$

$$||\psi''_i|| = ||\psi'_i||.$$

Then the vector

$$\Psi_\epsilon = \sum_{i=1}^k \Psi_{\epsilon_1} \otimes \Psi_i \otimes \psi''_i' \otimes \bigotimes_{\nu > N} (\otimes \Omega_{\nu})$$

has all the required properties.

§3. Theorem

Theorem 3.1. The asymptotic ratio set $r_\infty(M)$ for any factor $M$ is closed.

Proof. If $x \neq 0$ and $x_1$ is in $r_\infty(M)$, then $R \sim R \otimes R \sim R \otimes R \sim R$ shows that $0 \in r_\infty(M)$ and $1 \in r_\infty(M)$. Thus we consider the case where $x_\infty \in r_\infty(M)$, $\lim x_\infty = x$, $0 < x_\infty < 1$, $0 < x < 1$ and prove that $x \in r_\infty(M)$; i.e. $M \sim M \otimes R$.

First fix a countable sequence of unit vectors $\psi_n$, $n=1,2,\ldots$ which are dense in the unit sphere of $H$ and let $D_n = \{\psi_1, \ldots, \psi_n\}$. Let $\epsilon_0 > 0, \epsilon = \sum \epsilon_n < \infty$. We shall now construct by a mathematical induction on $n$ a sequence of mutually commuting type I factors $M_n$ in $R$, and $N_n$ in $R'$, and a sequence of unit vectors $x_n$, $n=1,2,\ldots$, such that (1) $x_n$ is pure for each $(M_n \cup N_n)'', m \leq n$, (2) the vector state $\varphi_n$ for each $M_n$, $m \leq n$ has a spectrum $((1 + x_n)^{-1}, x_n(1 + x_n)^{-1})$, (3) $||x_n - x_{n-1}|| < \epsilon_n$ ($n \geq 2$) and (4)

$$|| (\psi, Q\phi) - (\psi, \phi) (\phi, Q\phi) || < \sum_{\alpha=0}^{x-m} \epsilon_{n+\alpha}.$$
for any \( Q \) in \( \bigcup_{n=0}^{k-1} (M_{n+\alpha} \cup N_{n+\alpha})'' \), \( \|Q\| = 1 \), \( \psi \in D_\nu \), \( \phi \in D_\mu \), \( m \leq n \).

For \( n = 0 \), we do not have any object to construct. Now suppose \( M_\nu, N_\nu \) and \( z_\nu \) are constructed for \( n < k \) satisfying all the requirements related to \( M_\nu, N_\nu, z_\nu \). We then want to construct \( M_k, N_k \) and \( z_k \).

Let \( M^{(41)} = (\bigcup M_\nu)'' \), \( M^{(42)} = (\bigcap M_\nu)'' \), \( N^{(41)} = (\bigcup N_\nu)'' \), \( N^{(42)} = (\bigcap N_\nu)'' \). Since \( M_\nu \) and \( N_\nu \) are finite type I factors, we may identify \( H \) with \( H^{(41)} \otimes H^{(42)} \); \( M^{(41)}, M^{(42)}, N^{(41)}, N^{(42)} \) with \( \hat{M}^{(41)} \otimes 1, 1 \otimes \hat{M}^{(42)}, \hat{N}^{(41)} \otimes 1, 1 \otimes \hat{N}^{(42)} \); and \( (M^{(41)} \cup N^{(41)})'' \) with \( \mathcal{B}(H^{(41)}) \otimes 1 \). By using (2.4), it is easily shown that \( M \) and \( M' \) are identified with \( \hat{M}^{(41)} \otimes \hat{M}^{(42)} \) and \( \hat{N}^{(41)} \otimes \hat{N}^{(42)} \). Since \( M \) is type III \( (x_\nu = 1, 0 \in r_M(M)) \), \( M \) is spatially isomorphic to \( \hat{M}^{(42)} \). Since \( z_{k-1} \) is pure for each \( M_\nu \cup N_\nu \), \( n < k \), it is pure for \( \hat{M}^{(41)} \otimes \hat{N}^{(41)} \) and can be identified with \( \psi^{(41)} \otimes \psi^{(42)} \), \( \|\psi^{(41)}\| = \|\psi^{(42)}\| = 1 \).

We now use the information that \( M \) is isomorphic to \( M \otimes R_\nu \) where \( R_\nu = \bigotimes R_\nu \) on \( H_\nu = \bigotimes (H_0, \mathcal{A}_0) \). Let \( R_\nu \) be \( 1 \bigotimes \hat{R}_\nu \bigotimes (\bigotimes 1, \nu) \). By lemma 2.6, there exist an \( N_\nu \) and a unit vector \( \psi^{(43)} \) on \( H^{(42)} \) such that \( \|\psi^{(43)} - \psi^{(42)}\| < \epsilon_\nu \), \( \psi^{(43)} \) is pure for every \( (R_\nu \cup S_\nu)'' \), with \( \nu > N_1 \), and the vector state \( \varphi^{(43)} \) for \( (R_\nu \cup S_\nu)'' \), \( \nu > N_1 \) is the same as \( \varphi_{\nu+1}^{(1)} \) for \( (\hat{R}_\nu \cup (\bigotimes 1, \nu))'' \). We then set \( x_\nu = \psi^{(41)} \otimes \psi^{(42)} \). (If \( k = 1 \), take \( x_\nu = \psi^{(43)} = \psi \otimes (\bigotimes 1, \nu) \) for any \( \|\psi\| = 1 \).) The conditions (1), (2), (3) are automatically satisfied for \( M_\nu = R_\nu, N_\nu = S_\nu \), any \( \nu > N_1 \).

By lemma 2.1, there exists an \( N_2 \) such that

\[
(3.2) \quad | (\psi, Q\phi) - (\psi, \phi) (\phi, Q\phi) | < \epsilon_\nu
\]

for any \( Q \subset R_\nu \), \( \nu > N_2 \), \( \|Q\| = 1 \), \( \psi \in D_\nu \), \( \phi \in D_\mu \).

By lemma 2.4, there exists an \( N_3^\nu \) for each \( n < k \) such that

\[
(3.3) \quad | (\psi, Q\phi) - (\psi, \phi) (\phi, Q\phi) | < \sum_{a=\nu}^{k-1} \epsilon_a + \epsilon_\nu
\]

for any \( \nu > N_3^\nu \), \( Q \in \bigcup_{a=\nu}^{k-1} (M_a \cup N_a)'' \cup (R_a \cup S_a)'' \), \( \|Q\| = 1 \), \( \psi \in D_\nu \), \( \phi \in D_\mu \).

We then set \( M_\nu = R_\nu, N_\nu = S_\nu \) for some \( \nu \) larger than \( \max (N_1, N_2, N_3^\nu, \ldots, N_{k-1}^\nu) \). The required properties are now all satisfied.
By the property (3) and \(\sum \epsilon_n < \infty\), the unit vectors \(z_n\) form a Cauchy sequence. Let \(z\) be its strong limit. Then \(z\) is a unit vector, pure for each \((M_n \cup N_n)''\) and the vector state \(\varphi_z\) on \(M_n\) has the spectrum \(((1 + x_n)^{-1}, x_n(1 + x_n)^{-1})\). Let

\[
R = (\bigcup M_n)'', \quad S = (\bigcup N_n)'',
\]

\[
H_0 = [(R \cup S)'']^w
\]

where \(w\) denotes the closure. The properties of \(z\) imply that the restrictions of \(R\) and \(S\) to \(H_0\) and the space \(H_0\) are unitarily equivalent to \(\bigotimes R_n\), \(\bigotimes R_n^*\) and \(\bigotimes (H_n, \Omega_n)\) where \(\dim H_n = 4\), \(\text{Sp}(\Omega_n/R_n) = \text{Sp}(\Omega_n/R_n^*) = ((1 + x_n)^{-1}, x_n(1 + x_n)^{-1})\). Thus \((R \mid H_0) \sim (S \mid H_0) \sim \bigotimes R_n\), where \(R \mid H_0\) denotes the restriction of \(R\) to \(H_0\).

Next we use the clustering property (4) to show that \(R\), \(S\) and \((R \cup S)''\) are factors. Let \(Q\) be an operator in the center of either \(R\), \(S\) or \((R \cup S)''\) and \(\|Q\| = 1\). Then \(Q\) must commute with all \((M_n \cup N_n)''\), \(n = 1, 2, 3, \ldots\) and hence it is in \(\{\bigcup_{n \geq N} (M_n \cup N_n)''\}\) for any \(N\). (Again use the fact that \(\{\bigcup_{n \leq N} (M_n \cup N_n)''\}\) is a finite type I factor and (2.4).) Since the unit ball of \(\bigcup_{n \geq N} (M_n \cup N_n)''\) is weakly dense in the unit ball of \(\bigcup_{n \leq N} (M_n \cup N_n)''\), we have

\[
| (\psi, Q\phi) - (\psi, \phi)(\phi, Q\phi) | \leq \sum_{n = N+1}^{\infty} \epsilon_n
\]

for any \(\psi \in D_{N+1}\), \(\phi \in D_{N+1}\). Since \(N\) is arbitrary, we obtain in the limit of \(N \to \infty\),

\[
(\psi, Q\phi) = (\psi, \phi)(\phi, Q\phi).
\]

The same equation for \(Q^*\), with \(\psi\) and \(\phi\) interchanged implies that

\[
(\psi, Q\psi) = (\phi, Q\phi)
\]

for \((\psi, \phi) \neq 0\). Since \(\psi, \phi\) run over a set of unit vectors \(\{\psi_n\}\) which is dense in the set of all unit vectors, (3.8) and (3.7) imply that \(Q = c1\). This proves that \(R\), \(S\) and \((R \cup S)''\) are factors.

Since the projection on \(H_0\) commutes with \(R\), \(S\) and \((R \cup S)''\), the factors \(R\), \(S\) and \((R \cup S)''\) are isomorphic to its restriction on \(H_0\).
In particular, \( R \sim R \otimes R \) and \( (R \cup S)'' \) is a type I factor.

The proof of the theorem can now be completed by

**Lemma 3.2.** Let \( H = H_1 \otimes H_2 \), \( \hat{R} \) be an infinite tensor product of type I, factors on \( H_2 \), \( R = 1 \otimes \hat{R} \), \( S = 1 \otimes \hat{R}' \). Let \( M \) be a factor on \( H \) such that \( M \supseteq R, M' \supseteq S \). Then \( M = M_1 \otimes R \) for some factor \( M_1 \) on \( H_1 \).

**Proof.** Let

\begin{align*}
H_2 &= \otimes (H_2, \Omega_\nu), \quad \hat{R} = \otimes \hat{R}_\nu, \\
D(n) &= H_1 \otimes \left( \otimes_{\nu=1}^n H_2^\nu \right) \otimes \left( \otimes_{\nu>n} \Omega_\nu \right), \\
D(n, n+k) &= H_1 \otimes \left( \otimes_{\nu=1}^n H_2^\nu \right) \otimes \left( \otimes_{\nu=n+1}^i \Omega_{\nu+i} \right) \otimes \left( \otimes_{\nu>n+k} H_2^\nu \right).
\end{align*}

Let \( u_n \) be a standard matrix unit of

\begin{align*}
R_\nu &= 1 \otimes \left( \hat{R}_\nu \otimes \left( \otimes_{\nu=1}^n 1_\nu \right) \right)
\end{align*}

relative to \( \Omega_\nu \), \( \text{Sp}(\Omega_\nu/\hat{R}_\nu) \) be \( (\nu, 1-\nu) \) and

\begin{align*}
\tau_\nu A &= \lambda_\nu \tau_{11}(R_\nu) A + (1-\lambda_\nu) \tau_{22}(R_\nu) A, \\
\tau_{n,n+k} &= \prod_{i=1}^k \tau_{n+i}.
\end{align*}

Further let \([A]_n\) be the unique operator in \( \mathcal{B}(H_1 \otimes \left( \otimes_{\nu=1}^n H_2^\nu \right) \otimes \left( \otimes_{\nu>n} 1_\nu \right))\) satisfying

\begin{align*}
\phi_i (\tau_{n,n+k} A) &= \phi_i (A) \phi_2
\end{align*}

for all \( \phi_1, \phi_2 \in D(n) \).

If \( A \in M \), then \( \tau_{n,n+k} A \in M \), \( \|\tau_{n,n+k} A\| \leq \|A\| \) and

\begin{align*}
\phi_i (\tau_{n,n+k} A) \phi_2 &= \phi_i (A) \phi_2
\end{align*}

for all \( \phi_1, \phi_2 \in D(n, n+k) \). Hence

\begin{align*}
\phi_i (\tau_{n,n+k} A) \phi_2 &= \phi_i ([A]_n \phi_2)
\end{align*}

for \( \phi_1, \phi_2 \in D(n+k) \). Since \( D(n) \) is an increasing sequence of sets with a dense union and \( \|\tau_{n,n+k} A\| \) is bounded uniformly in \( k \), \([A]_n\) is the weak limit of \( \tau_{n,n+k} A \) as \( k \to \infty \) and hence is in \( M \). By definition, \([A]_n\) is then in
Because (3.15) holds for \( \phi_i, \phi_i \in D_n, D_n \) is an increasing sequence of sets with a dense union and \( \|[A]_n\| \) is uniformly bounded by \( \|A\| \) (which immediately follows from (3.15)), \( A \) is the weak limit of \([A]_n\) as \( n \to \infty \). Hence

(3.19) \[
M = (\bigcup_n M^{(\circ)})''.
\]

Since \( \bigcup R_n \) is a finite type I factor, \( M^{(\circ)} \) is generated by \( \bigcup R_n \) and \( M^{(\circ)} = M \cap R' \). Hence

(3.20) \[
M = (M^{(\circ)} \cup R)'' = ((M \cap R') \cup R)''.
\]

Since \( M \cap R' \) commutes with \( R \) and \( S \), it is isomorphic to \( M \otimes 1 \) on \( H_1 \otimes H_2 \) for some \( M \) and \( M = M \otimes \hat{R} \).

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References
