Proper Stacks

By

Luca Prelli

Abstract

We generalize the notion of proper stack introduced by Kashiwara and Schapira to the case of a general site, and we prove that a proper stack is a stack.

Contents

Introduction

§1. Review on Grothendieck Topologies and Sheaves

§2. Review on Stacks

§3. Proper Stacks

References

Introduction

In [3] Kashiwara and Schapira defined the notion of proper stack on a locally compact topological space \( X \). A proper stack is a separated prestack \( S \) satisfying suitable hypothesis. They proved that a proper stack is a stack. In this paper, we generalize the notion of proper stack to the case of a site \( X \) associated to a small category \( C_X \) and we prove that a proper stack is a stack.
§1. Review on Grothendieck Topologies and Sheaves

Let \( C \) be a category\(^1\). As usual we denote by \( C^\wedge \) the category of functors from \( C^{\text{op}} \) to \( \text{Set} \) and we identify \( C \) with its image in \( C^\wedge \) via the Yoneda embedding. If \( A \in C^\wedge \), we will denote by \( C_A \) the category of arrows \( U \rightarrow A \) with \( U \in C \). When taking inductive and projective limits on a category \( I \) we will always assume that \( I \) is small.

We recall here some classical definitions (see [2]), following the presentation of [4].

Definition 1.1. A Grothendieck topology on a small category \( C_X \) is a collection of morphisms in \( C_X^\wedge \) called local epimorphisms, satisfying the following conditions:

LE1 For any \( U \in C_X \), \( \text{id}_U : U \rightarrow U \) is a local epimorphism.

LE2 Let \( A_1 \xrightarrow{u} A_2 \xrightarrow{v} A_3 \) be morphisms in \( C_X^\wedge \). If \( u \) and \( v \) are local epimorphisms, then \( v \circ u \) is a local epimorphism.

LE3 Let \( A_1 \xrightarrow{u} A_2 \xrightarrow{v} A_3 \) be morphisms in \( C_X^\wedge \). If \( v \circ u \) is a local epimorphism, then \( v \) is a local epimorphism.

LE4 A morphism \( u : A \rightarrow B \) in \( C_X^\wedge \) is a local epimorphism if and only if for any \( U \in C_X \) and any morphism \( U \rightarrow B \), the morphism \( A \times_B U \rightarrow U \) is a local epimorphism.

Definition 1.2. A morphism \( A \rightarrow B \) in \( C_X^\wedge \) is a local monomorphism if \( A \rightarrow A \times_B A \) is a local epimorphism.

A morphism \( A \rightarrow B \) in \( C_X^\wedge \) is a local isomorphism if it is both a local epimorphism and a local monomorphism.

Definition 1.3. A site \( X \) is a category \( C_X \) endowed with a Grothendieck topology.

Let \( \mathcal{A} \) be a category admitting small inductive and projective limits.

Definition 1.4. An \( \mathcal{A} \)-valued presheaf on \( X \) is a functor \( C_X^{\text{op}} \rightarrow \mathcal{A} \). A morphism of presheaves is a morphism of such functors. One denotes by \( \text{Psh}(X, \mathcal{A}) \) the category of \( \mathcal{A} \)-valued presheaves on \( X \).

\(^1\)We shall work in a given universe \( \mathcal{U} \), small means \( \mathcal{U} \)-small (i.e. a set is \( \mathcal{U} \)-small if it is isomorphic to a set belonging to \( \mathcal{U} \)) and a category \( C \) means a \( \mathcal{U} \)-category (i.e. \( \text{Hom}_C(X,Y) \) is \( \mathcal{U} \)-small for any \( X,Y \in C \)).
If \( F \in \text{Psh}(X, A) \), it extends naturally to \( C_X \) by setting
\[
F(A) = \lim_{(U \to A) \in C_A} F(U),
\]
where \( A \in C_X \) and \( U \in C_X \).

**Definition 1.5.** Let \( X \) be a site.

- One says that \( F \in \text{Psh}(X, A) \) is separated, if for any local isomorphism \( A \to U \) with \( U \in C_X \) and \( A \in C_X \), \( F(U) \to F(A) \) is a monomorphism.
- One says that \( F \in \text{Psh}(X, A) \) is a sheaf, if for any local isomorphism \( A \to U \) with \( U \in C_X \) and \( A \in C_X \), \( F(U) \to F(A) \) is an isomorphism.

§2. Review on Stacks

Let \( C_X \) be a small category. We suppose that a Grothendieck topology on \( C_X \) is defined and we denote by \( X \) the associated site. We recall some classical definitions (see [1]), following the presentation of [4].

**Definition 2.1.** A prestack \( S \) on \( X \) is the data of:

- for each \( U \in C_X \), a category \( S(U) \),
- for each \( V \to U \in C_U \), a functor \( j_{VU*} : S(U) \to S(V) \),
- given \( U, V, W \in C_X \) and \( W \to V \to U \), an isomorphism of functors \( \lambda_{WVVU} : j_{WV*} \circ j_{VU*} \to j_{WU*} \),

such that

- \( j_{UU*} = \text{id}_{S(U)} \),
- given \( \{U_i\}_{i \in I} \in C_X \), \( i = 1, 2, 3, 4 \) and \( U_1 \to U_2 \to U_3 \to U_4 \), the following diagram commutes:

\[
\begin{array}{ccc}
j_{12*} \circ j_{23*} \circ j_{34*} & \xrightarrow{\lambda_{234}} & j_{12*} \circ j_{24*} \\
\downarrow{\lambda_{123}} & & \downarrow{\lambda_{124}} \\
\downarrow{\lambda_{134}} & & \downarrow{\lambda_{14}} \\
\end{array}
\]

Let \( \lim_{U \in C_X} S(U) \) denote a category defined as follows. An object \( F \) of \( \lim_{U \in C_X} S(U) \) is a family \( \{(F_U)_U, (\psi_u)_u\} \) where...
• for any $U \in \mathcal{C}_X$, $F_U \in \text{Ob}(\mathcal{S}(U))$, 

• for any morphism $U_1 \to U_2$ in $\mathcal{C}_X$, $\psi_{12} : j_{12*} F_{U_2} \to F_{U_1}$ is an isomorphism, such that for any sequence $U_1 \to U_2 \to U_3$ the following diagram commutes

\[
\begin{array}{ccc}
j_{12*} j_{23*} F_{U_3} & \xrightarrow{\psi_{23}} & j_{12*} F_{U_2} \\
\downarrow{\lambda_{12}} & & \downarrow{\psi_{12}} \\
j_{13*} F_{U_3} & \xrightarrow{\psi_{13}} & F_{U_1}.
\end{array}
\]

Note that $\psi_{\text{id}U} = \text{id}_{F_U}$ for any $U \in \mathcal{C}_X$.

The morphisms are defined in natural way. Let $F, G \in \varprojlim_{U \in \mathcal{C}_X} \mathcal{S}(U)$. Then

\[
\text{Hom}_{\varprojlim_{U \in \mathcal{C}_X} \mathcal{S}(U)}(F, G) \simeq \lim_{U \in \mathcal{C}_X} \text{Hom}_{\mathcal{S}(U)}(F_U, G_U).
\]

For any $A \in \mathcal{C}_X^\wedge$, we set

\[
\mathcal{S}(A) = \varprojlim_{(U \to A) \in \mathcal{C}_A} \mathcal{S}(U).
\]

A morphism $\varphi : A \to B$ in $\mathcal{C}_X^\wedge$ defines a functor $j_{AB*} : \mathcal{S}(B) \to \mathcal{S}(A)$, therefore a prestack on $\mathcal{C}_X$ extends naturally to a prestack on $\mathcal{C}_X^\wedge$.

**Definition 2.2.** Let $X$ be a site.

• A prestack $\mathcal{S}$ on $X$ is called separated if for any $U \in \mathcal{C}_X$, and for any local isomorphism $A \to U$ in $\mathcal{C}_X^\wedge$, $j_{AU*} : \mathcal{S}(U) \to \mathcal{S}(A)$ is fully faithful.

• A prestack $\mathcal{S}$ on $X$ is called a stack if for any $U \in \mathcal{C}_X$, and for any local isomorphism $A \to U$ in $\mathcal{C}_X^\wedge$, $j_{AU*} : \mathcal{S}(U) \to \mathcal{S}(A)$ is an equivalence.

**Proposition 2.3.** Let $\mathcal{S}$ be a prestack on $X$. Then $\mathcal{S}$ is a stack if and only if $\mathcal{S}$ satisfies the following conditions:

(i) $\mathcal{S}$ is separated,

(ii) for any $U \in \mathcal{C}_X$ and for any local isomorphism $A \to U$ the restriction functor $j_{AU*} : \mathcal{S}(U) \to \mathcal{S}(A)$ admits a left adjoint $j_{AU}^{-1}$ satisfying $j_{AU*} \circ j_{AU}^{-1} \simeq \text{id}$ (or, equivalently, the functor $j_{AU}^{-1}$ is fully faithful).

**Proof.** The result follows from the fact that two categories are equivalent if and only if they admit a pair of fully faithful adjoint functors. \qed
§3. Proper Stacks

Let \( C_X \) be a small category. In this section we extend a result of \([3]\) to the case of a site \( X \) associated to a small category \( C_X \).

Let \( S \) be a prestack on \( X \) and assume the following hypothesis

\[
\begin{cases}
\text{(i)} & \text{for any } U, V \in C_X \text{ and any morphism } U \to V \in C_X, \text{ the functor } j_{UV*} : S(V) \to S(U) \text{ admits a left adjoint } j_{UV}^{-1} \text{ satisfying } \\
\quad \text{id}_{S(U)} \to j_{UV*} \circ j_{UV}^{-1} \text{ (or, equivalently, } j_{UV}^{-1} \text{ is fully faithful),} \\
\text{(ii)} & \text{for all } U \in C_X \text{ the category } S(U) \text{ admits small inductive limits.}
\end{cases}
\]

**Lemma 3.1.** Let \( S \) be a prestack and assume (1). Let \( A \in C_X^\Delta, V \in C_X \) and \( A \to V \). Then the functor \( j_{AV*} \) admits a left adjoint, denoted by \( j_{AV}^{-1} \).

**Proof.** Let \( F = \{ F_U \}_{(U \to A) \in C_A} \in S(A) \), and let \( j_{AV}^{-1} F := \lim_{(U \to A) \in C_A} j_{UV}^{-1} F_U \).

This defines a functor \( j_{AV}^{-1} : S(A) \to S(V) \). Let \( G \in S(V) \). We have the chain of isomorphisms

\[
\text{Hom}_{S(V)}(j_{AV}^{-1} F, G) = \text{Hom}_{S(V)}(\lim_{(U \to A) \in C_A} j_{UV}^{-1} F_U, G) \]

\[
\cong \lim_{(U \to A) \in C_A} \text{Hom}_{S(V)}(j_{UV}^{-1} F_U, G) \]

\[
\cong \lim_{(U \to A) \in C_A} \text{Hom}_{S(U)}(F_U, j_{UV*} G) \]

\[
\cong \text{Hom}_{S(A)}(F, j_{AV*} G). \]

\[ \square \]

**Lemma 3.2.** Let \( S \) be a prestack on \( X \) satisfying (1), let \( U', U, V \in C_X \) and \( U' \to U \to V \). Then

(i) there exists a canonical morphism \( j_{U'V}^{-1} \circ j_{UV*} \to j_{U'V}^{-1} \circ j_{UV*} \),

(ii) we have \( j_{U'V}^{-1} \circ j_{UV*} \simeq j_{U'V}^{-1} \circ j_{UV*} \circ j_{UV}^{-1} \circ j_{UV*} \).

**Proof.** (i) The adjunction morphism \( j_{U'V}^{-1} \circ j_{UV} \circ j_{UV}^{-1} \to \text{id}_{S(U)} \) defines

\[
\text{id}_{S(U)} \to j_{UV*} \circ j_{UV}^{-1} \circ j_{UV} \circ j_{UV*} \circ j_{UV}^{-1} \circ j_{UV*} \to j_{UV*} \circ j_{UV*}.
\]

(ii) We have \( j_{UV*} \simeq j_{UV*} \circ j_{UV} \cdot j_{UV}^{-1} \), and then

\[
j_{UV*} \circ j_{UV}^{-1} \simeq j_{UV*} \circ j_{UV} \cdot j_{UV}^{-1} \simeq j_{UV*}.
\]
Hence we have the chain of isomorphisms
\[ j_{UV}^{-1} \circ j_{UV*} \circ j_{UV}^{-1} \circ j_{UV*} \simeq j_{UV}^{-1} \circ j_{UV*} \circ j_{UV}^{-1} \circ j_{UV*} \simeq j_{UV}^{-1} \circ j_{UV*}. \]

**Lemma 3.3.** Let \( S \) be a prestack on \( X \) satisfying (1). Let \( U, V, W \in C_X \) and let \( U \to W, V \to W \) be morphisms. Consider the diagram

\[
\begin{array}{ccc}
U \times_W V & \longrightarrow & V \\
\downarrow & & \downarrow \\
U & \longrightarrow & W
\end{array}
\]

where \( U \times_W V \in \mathcal{C}_X \). Then there exists a canonical morphism

\[ j_{U \times_W V}^{-1} \circ j_{U \times_W V*} \to j_{UW}^{-1} \circ j_{UW*} \circ j_{VW}^{-1}. \]

**Proof.** Since \( U \times_W V \in \mathcal{C}_X \) for each \( F \in S(W) \) we have

\[ j_{U \times_W V*} F = \{ j_{W*} F \}_{(W' \to U \times_W V) \in C_{U \times_W V}} \in S(U \times_W V) \]

hence as in Lemma 3.1

\[ j_{U \times_W V*}^{-1} j_{U \times_W V*} \simeq \lim_{(W' \to U \times_W V) \in C_{U \times_W V}} j_{W*}^{-1} j_{W*}. \]

By Lemma 3.2 we have \( j_{W*}^{-1} j_{W*} \circ j_{W*} \simeq j_{W*}^{-1} j_{W*} \circ j_{W*} \) for each \( (W' \to U \times_W V) \in C_{U \times_V W} \). We have natural morphisms

\[ j_{U \times_W V*}^{-1} j_{U \times_W V*} \to j_{U \times_W V*}^{-1} j_{U \times_W V*} \to j_{U \times_W V*} \to j_{U \times_W V*} \to j_{U \times_W V*}. \]

**Definition 3.4.** A proper stack \( S \) on \( X \) is a prestack satisfying

PRS1 \( S \) is separated,
Proper Stacks

PRS2 for each \( U \in \mathcal{C}_X \), \( S(U) \) admits small inductive limits,

PRS3 for all \( U, V \in \mathcal{C}_X \) and \( U \rightarrow V \) the functor \( j_{UV*} : S(V) \rightarrow S(U) \) commutes with \( \operatorname{lim}_{\rightarrow} \),

PRS4 for all \( U, V \in \mathcal{C}_X \) and \( U \rightarrow V \) the functor \( j_{UV*} : S(V) \rightarrow S(U) \) admits a left adjoint \( j_{UV}^{-1} \), satisfying \( \text{id}_{S(U)} \sim j_{UV*} \circ j_{UV}^{-1} \) (or, equivalently, the functor \( j_{UV}^{-1} \) is fully faithful),

PRS5 for all \( V, U, W \in \mathcal{C}_X \), \( U \rightarrow W \) and \( V \rightarrow W \), the morphism

\[
j_{U \times_W V}^{-1} \circ j_{U \times_W V} \circ j_{U \times_W V}^{-1} \rightarrow j_{VW} \circ j_{UW}^{-1}
\]

is an isomorphism.

Remark 3.5. Here \( U \times_W V \in \mathcal{C}_X \), since we have not assumed that \( \mathcal{C}_X \) admits fiber products.

Lemma 3.6. Let us consider the following diagram

\[
\begin{array}{c}
A \times_V U \\
\downarrow \\
U \\
\downarrow \\
A \\
\downarrow \\
V
\end{array}
\]

where \( U, V \in \mathcal{C}_X \) and \( A \in \mathcal{C}^A_X \). Let \( S \) be a proper stack on \( X \). Then we have

\[
j_{UV*} \circ j_{AV}^{-1} \simeq j_{A \times_V U}^{-1} \circ j_{A \times_V U A*}.
\]

Proof. Let \( F = \{ F_W \}_{W \rightarrow A} \in \mathcal{C}_A \). We have the chain of isomorphisms

\[
j_{UV*} \circ j_{AV}^{-1} F \simeq j_{UV*} \lim_{(W \rightarrow A) \in \mathcal{C}_A} j_{WV}^{-1} F_W \\
\simeq \lim_{(W \rightarrow A) \in \mathcal{C}_A} j_{UV*} j_{WV}^{-1} F_W \\
\simeq \lim_{(W \rightarrow A) \in \mathcal{C}_A} j_{U \times_V WU}^{-1} j_{U \times_V W W*} F_W \\
\simeq \lim_{(W \rightarrow A) \in \mathcal{C}_A} \lim_{(W' \rightarrow W \times_V U) \in \mathcal{C}_{W \times_V U}} j_{W'U}^{-1} F_{W'} \\
\simeq \lim_{(W' \rightarrow A \times_V U) \in \mathcal{C}_{A \times_V U}} j_{W'U}^{-1} F_{W'},
\]

as required.
where the second and the third isomorphism follow from PRS3 and PRS5 respectively. The fourth isomorphism follows since $W \times_V U \in \mathcal{C}_X$ and we have

$$j_{U \times_V W \times_V F_W} \simeq \{ j_{W \times_V F_W} \}_{(W' \to W \times_V U) \in \mathcal{C}_{W \times_V U}} \simeq \{ F_{W'} \}_{(W' \to W \times_V U) \in \mathcal{C}_{W \times_V U}}.$$ 

On the other hand we have $j_{A \times_V U \times_A F} \simeq \{ F_{U'} \}_{(U'' \to A \times_V U') \in \mathcal{C}_{A \times_V U}}$, hence

$$\overline{\overline{j}}_{A \times_V U \times_A F} \simeq \varinjlim_{(U'' \to A \times_V U') \in \mathcal{C}_{A \times_V U}} \overline{\overline{j}}_{W'' \times_V F_{W''}}.$$ 

\hfill \Box

**Theorem 3.7.** Let $X$ be a site associated to a small category $\mathcal{C}_X$. Let $S$ be a proper stack on $X$. Then $S$ is a stack.

**Proof.** Let $A \to V$ be a local isomorphism. By Proposition 2.3 it is enough to show that $j_{AV} \circ j_{AV}^{-1} \simeq \text{id}$. Let $F = \{ F_{V_i} \}_{(V_i \to A) \in \mathcal{C}_A} \in S(A)$. It satisfies, for each $V_i \to V_j$

$$j_{V_i V_j} F_{V_j} \simeq F_{V_i}. \tag{3}$$

We have to show that $j_{V_i V_j} j_{AV}^{-1} F \simeq F_{V_i}$ for each $V_i \to A$. Let us consider $V_i \to A$. By PRS5 and (3), for each $V_k \to A$ we have the chain of isomorphisms

$$j_{V_k V_{V_k V_{V_k V_{V_k V_i V_j V_i V_j V_k}}} F_{V_k} \simeq j_{V_k V_{V_k V_{V_k V_{V_k V_i V_j V_i V_j V_k}}} F_{V_k} \simeq j_{V_k V_{V_i V_j V_i V_j V_k} F_{V_k} \simeq j_{V_k V_{V_i V_j V_i V_j V_k} F_{V_k} \simeq j_{V_k V_{V_i V_j V_i V_j V_k} F_{V_k}}. \tag{4}$$

Hence we obtain the isomorphism

$$j_{V_k V_{V_k V_{V_k V_{V_k V_i V_j V_i V_j V_k}}} F_{V_k} \simeq j_{V_k V_{V_k V_{V_k V_{V_k V_i V_j V_i V_j V_k}}} F_{V_k}} \simeq j_{A \times_V U} F_{V_k} \simeq F_{V_k}.$$ 

and $j_{A \times_V U} F_{V_k} \simeq F_{V_k}$ since $S$ is separated and $A \times_V U \to V_k$ is a local isomorphism. \hfill \Box

**Example 3.8.** Let $k$ be a field, and $X$ a topological space (or, more generally, let $X$ be a site associated to an ordered-set category). The prestack associating to an open set $U$ of $X$ the category of sheaves of $k$-vector spaces\footnote{More generally, one can consider sheaves with values in a category $\mathcal{A}$ admitting small inductive and projective limits, such that filtrant inductive limits are exact and satisfying the ICP property (see [4] for a detailed exposition).} on $U$ is a proper stack.
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References