Fixed Points
for Multi-Valued Mixed Increasing Operators
in Ordered Banach Spaces
with Applications to Integral Inclusions

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Abstract. Some new fixed point and coupled fixed point theorems for multi-valued mixed increasing operators in ordered Banach spaces are presented in this paper. As applications, we prove the existences of solutions for a class of integral inclusions.

Keywords: Ordered Banach spaces, multi-valued operators, mixed increasing operators, fixed points, coupled fixed points, integral inclusions

AMS subject classification: 47H10, 47H07, 47H04, 54H25

1. Introduction and Preliminaries

Fixed point theorems for single-valued increasing operators and coupled fixed point theorems for single-valued mixed increasing operators in ordered Banach spaces are widely investigated and have found various applications to nonlinear integral equations and differential equations. For details, we can refer to [1, 3 - 12, 14, 16, 17, 20] and the references therein. However, only few results on fixed points of multi-valued increasing operators in ordered Banach spaces are obtained. In 1984, Nishnianidze [16] introduced monotone multi-valued operators and proved some fixed point theorems for such operators. Recently, Huy and Khash [11] gave some new fixed point theorems for multi-valued increasing operators in ordered Banach spaces by means of the concept of Nishnianidze [16].

Motivated and inspired by [11], in this paper, we introduce a class of multi-valued mixed increasing operators in Banach spaces and prove some new fixed point and coupled fixed point theorems. As applications, we discuss the existences of solutions for a class of integral inclusions.

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Let $X$ be a real Banach space and $K$ be a non-empty pointed closed convex cone in $X$. Recall that $K \subset X$ is called a pointed closed convex cone if $K$ is closed and the following conditions hold:

(i) $K + K \subset K$
(ii) $tK \subset K$ for all $t \geq 0$
(iii) $K \cap (-K) = \{0\}$.

A partial ordering $\leq$ can be induced by $K$ by

\[ x \leq y \text{ if and only if } y - x \in K. \]

We always say that $(X, \leq)$ is an ordered Banach space induced by $K$. A pointed convex cone $K$ is said to be normal if there exists some constant $N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq N \|y\|$.

**Definition 1.1.** Let $(X, \leq)$ be an ordered Banach space induced by $K$ and let $A, B$ be two non-empty subsets of $X$. Then we denote

(i) $A \leq_{1} B$ if for each $a \in A$ there exists $b \in B$ such that $a \leq b$
(ii) $A \leq_{2} B$ if for each $b \in B$ there exists $a \in A$ such that $a \leq b$
(iii) $A \leq B$ if $A \leq_{1} B$ and $A \leq_{2} B$
(iv) $A \ll B$ if $a \leq b$ for any $a \in A$ and $b \in B$.

**Definition 1.2.** Let $(X, \leq)$ be an ordered Banach space induced by $K$ and let $L : X \rightarrow X$ be a mapping. We say that $L$ is positive if $Lx \in K$ whenever $x \in K$.

**Definition 1.3.** Let $M$ be a non-empty subset of an ordered Banach space $(X, \leq)$ and let $f : M \times M \rightarrow 2^{X}$ be a multi-valued operator. We say that $f$ is mixed increasing if, for any $x_1, x_2, y_1, y_2 \in M$, $x_1 \leq x_2$ and $y_2 \leq y_1$ imply $f(x_1, y_1) \leq f(x_2, y_2)$.

**Definition 1.4.** Let $M$ be a non-empty subset of an ordered Banach space $(X, \leq)$ and let $f : M \times M \rightarrow 2^{X}$ be a multi-valued operator. We say that $(x^*, y^*) \in M \times M$ is a coupled fixed point of $f$ if $x^* \in f(x^*, y^*)$ and $y^* \in f(y^*, x^*)$. We say that $x^* \in M$ is a fixed point of $f$ if $x^* \in f(x^*, x^*)$.

**2. Main Results**

In this section, we prove some new fixed point and coupled fixed point theorems for multi-valued mixed increasing operators in ordered Banach spaces.

**Theorem 2.1.** Let $(X, \leq)$ be an ordered Banach space induced by a pointed closed convex cone $K$ and $M$ be a non-empty closed subset of $X$. Suppose that $f : M \times M \rightarrow 2^{M}$ is a multi-valued operator satisfies the following conditions:

(i) $f(x, y)$ is closed for any $x, y \in M$.
(ii) There exist $x_0, y_0 \in M$ such that $\{x_0\} \leq_{1} f(x_0, y_0)$ and $f(y_0, x_0) \leq_{2} \{y_0\}$.
(iii) For any \( u_1, u_2, v_1, v_2 \in M \), \( u_1 \leq u_2 \) and \( v_2 \leq v_1 \) imply

\[
\begin{align*}
f(u_1, v_1) & \subset f(u_2, v_2) - K \cap B(r \|u_1 - u_2\| + s \|v_1 - v_2\|) \\
f(v_1, u_1) & \subset f(v_2, u_2) + K \cap B(r \|v_1 - v_2\| + s \|u_1 - u_2\|)
\end{align*}
\]

where \( r, s \) are two non-negative constants with \( r + s < 1 \) and \( B(l) \) denotes the closed ball with radius \( l \) and center at origin.

Then \( f \) admits a coupled fixed point \((x^*, y^*) \in M \times M\).

**Proof.** From condition (ii), there exist \( x_1 \in f(x_0, y_0) \) and \( y_1 \in f(y_0, x_0) \) such that \( x_0 \leq x_1 \) and \( y_1 \leq y_0 \). By condition (iii), we can choose \( x_2 \in f(x_1, y_1) \) and \( y_2 \in f(y_1, x_1) \) such that

\[
\begin{align*}
x_1 & \leq x_2, \quad \|x_2 - x_1\| \leq r \|x_1 - x_0\| + s \|y_1 - y_0\| \\
y_2 & \leq y_1, \quad \|y_2 - y_1\| \leq r \|y_1 - y_0\| + s \|x_1 - x_0\|.
\end{align*}
\]

Repeating the arguments above for \( x_1, x_2, y_1, y_2 \) in place \( x_0, x_1, y_0, y_1 \) and so on, we can construct two sequences

\[
\begin{align*}
\{x_n\}, \quad x_n & \in f(x_{n-1}, y_{n-1}) \\
\{y_n\}, \quad y_n & \in f(y_{n-1}, x_{n-1})
\end{align*}
\]

such that

\[
\begin{align*}
x_{n-1} & \leq x_n, \quad \|x_n - x_{n-1}\| \leq r \|x_{n-1} - x_{n-2}\| + s \|y_{n-1} - y_{n-2}\| \\
y_{n-1} & \leq y_n, \quad \|y_n - y_{n-1}\| \leq r \|y_{n-1} - y_{n-2}\| + s \|x_{n-1} - x_{n-2}\|.
\end{align*}
\]

(2.1)

We claim that

\[
\begin{align*}
\|x_{n+1} - x_n\| & \leq (r + s)^n (\|x_1 - x_0\| + \|y_1 - y_0\|) \\
\|y_{n+1} - y_n\| & \leq (r + s)^n (\|x_1 - x_0\| + \|y_1 - y_0\|)
\end{align*}
\]

(2.2)

for all \( n \geq 1 \). In fact, for \( n = 1 \) it follows from (2.1) that

\[
\begin{align*}
\|x_2 - x_1\| & \leq r \|x_1 - x_0\| + s \|y_1 - y_0\| \leq (r + s)(\|x_1 - x_0\| + \|y_1 - y_0\|) \\
\|y_2 - y_1\| & \leq r \|y_1 - y_0\| + s \|x_1 - x_0\| \leq (r + s)(\|x_1 - x_0\| + \|y_1 - y_0\|).
\end{align*}
\]

Suppose that (2.2) holds for \( n = k \) (\( \geq 1 \)). For \( n = k + 1 \) it follows from (2.1) that

\[
\begin{align*}
\|x_{k+2} - x_{k+1}\| & \leq r \|x_{k+1} - x_k\| + s \|y_{k+1} - y_k\| \\
& \leq r(r + s)^k (\|x_1 - x_0\| + \|y_1 - y_0\|) + s(r + s)^k (\|x_1 - x_0\| + \|y_1 - y_0\|) \\
& = (r + s)^{k+1} (\|x_1 - x_0\| + \|y_1 - y_0\|) \\
\|y_{k+2} - y_{k+1}\| & \leq r \|y_{k+1} - y_k\| + s \|x_{k+1} - x_k\| \\
& \leq r(r + s)^k (\|x_1 - x_0\| + \|y_1 - y_0\|) + s(r + s)^k (\|x_1 - x_0\| + \|y_1 - y_0\|) \\
& = (r + s)^{k+1} (\|x_1 - x_0\| + \|y_1 - y_0\|).
\end{align*}
\]
By induction, we can conclude that (2.2) holds for all \( n \geq 1 \). Since \( 0 \leq r + s < 1 \), from (2.2) we know that \( \{x_n\}, \{y_n\} \) are two Cauchy sequences. Let \( x_n \to x^* \) and \( y_n \to y^* \). Obviously, \((x^*, y^*) \in M \times M\) since \( M \) is closed. Further, \( x_n \leq x^* \) and \( y_n \leq y^* \) for all \( n \) since \( K \) is closed. Again from condition (iii), we can choose \( x^*_{n+1} \in f(x^*, y^*) \) and \( y^*_{n+1} \in f(y^*, x^*) \) such that

\[
\begin{align*}
\|x^*_{n+1} - x_{n+1}\| &\leq r\|x_n - x^*\| + s\|y_n - y^*\|, \\
\|y^*_{n+1} - y_{n+1}\| &\leq r\|y_n - y^*\| + s\|x_n - x^*\|.
\end{align*}
\]

(2.3)

Since \( x_n \to x^* \) and \( y_n \to y^* \), it follows from (2.3) that \( x^*_{n+1} \to x^* \) and \( y^*_{n+1} \to y^* \). By condition (i), we know that \( x^* \in f(x^*, y^*) \) and \( y^* \in f(y^*, x^*) \). The proof is complete

**Theorem 2.2.** Let \((X, \leq)\) be an ordered Banach space induced by a pointed closed convex cone \( K \) and \( M \) be a non-empty closed subset of \( X \). Suppose that \( f : M \times M \to C(M) \) (the family of all non-empty compact subsets of \( M \)) is a multi-valued operator satisfying the following conditions:

1. There exist \( x_0, y_0 \in M \) such that \( \{x_0\} \leq 1 f(x_0, y_0) \) and \( f(y_0, x_0) \leq 2 \{y_0\} \).
2. For any \( u_1, u_2, v_1, v_2 \in M \), \( u_1 \leq u_2 \) and \( v_2 \leq v_1 \) imply \( f(u_1, v_1) \ll f(u_2, v_2) \).
3. For any fixed \( u \in M \), \( x \leq y \) implies

\[
\begin{align*}
H(f(x, u), f(y, u)) &\leq r\|x - y\|, \\
H(f(u, x), f(u, y)) &\leq s\|x - y\|
\end{align*}
\]

where \( r, s \) are two non-negative constants with \( r + s < 1 \) and \( H(\cdot, \cdot) \) is the Hausdorff metric on \( C(M) \).

Then \( f \) admits a coupled fixed point \((x^*, y^*) \in M \times M\).

**Proof.** From condition (i), there exist \( x_1 \in f(x_0, y_0) \) and \( y_1 \in f(y_0, x_0) \) such that \( x_0 \leq x_1 \) and \( y_1 \leq y_0 \). By Nadler [15], we can choose \( x'_1 \in f(x_1, y_0) \) and \( y'_1 \in f(y_1, x_0) \) such that

\[
\begin{align*}
\|x_1 - x'_1\| &\leq H(f(x_0, y_0), f(x_1, y_0)) \\
\|y_1 - y'_1\| &\leq H(f(y_0, x_0), f(y_1, x_0)).
\end{align*}
\]

(2.4)

Since \( x'_1 \in f(x_1, y_0) \) and \( y'_1 \in f(y_1, x_0) \), again from Nadler [15] we can choose \( x_2 \in f(x_1, y_1) \) and \( y_2 \in f(y_1, x_0) \) such that

\[
\begin{align*}
\|x_2 - x'_1\| &\leq H(f(x_1, y_1), f(x_1, y_0)) \\
\|y_2 - y'_1\| &\leq H(f(y_1, x_1), f(y_1, x_0)).
\end{align*}
\]

(2.5)

It follows from (2.4) - (2.5) and condition (iii) that

\[
\begin{align*}
\|x_2 - x_1\| &\leq \|x_2 - x'_1\| + \|x'_1 - x_1\| \\
&\leq H(f(x_1, y_1), f(x_1, y_0)) + H(f(x_0, y_0), f(x_1, y_0)) \\
&\leq r\|x_1 - x_0\| + s\|y_1 - y_0\| \\
\|y_2 - y_1\| &\leq \|y_2 - y'_1\| + \|y'_1 - y_1\| \\
&\leq H(f(y_1, x_1), f(y_1, x_0)) + H(f(y_0, x_0), f(y_1, x_0)) \\
&\leq s\|x_1 - x_0\| + r\|y_1 - y_0\|.
\end{align*}
\]
Furthermore, by condition (ii) we know that \( x_1 \leq x_2 \) and \( y_2 \leq y_1 \). Repeating the arguments above for \( x_1, x_2, y_1, y_2 \) in place \( x_0, x_1, y_0, y_1 \) and so on, we can construct two sequences

\[
\{x_n\}, \quad x_n \in f(x_{n-1}, y_{n-1}) \\
\{y_n\}, \quad y_n \in f(y_{n-1}, x_{n-1})
\]

such that

\[
x_{n-1} \leq x_n, \quad \|x_n - x_{n-1}\| \leq r\|x_{n-1} - x_{n-2}\| + s\|y_{n-1} - y_{n-2}\| \leq 1 \\
y_{n-1} \leq y_n, \quad \|y_n - y_{n-1}\| \leq r\|y_{n-1} - y_{n-2}\| + s\|x_{n-1} - x_{n-2}\|.
\]

The rest of proof now follows as in Theorem 2.1 and is therefore omitted.

**Theorem 2.3.** Let \((X, \leq)\) be an ordered Banach space induced by a pointed closed convex normal cone \(K\) with normal constant \(N > 0\) and \(M\) be a non-empty closed subset of \(X\). Suppose that \(f : M \times M \to 2^M\) be a multi-valued operator satisfying the following conditions:

(i) \(f(x, y)\) is closed for any \(x, y \in M\).

(ii) There exist \(x_0, y_0 \in M\) such that \(\{x_0\} \subseteq f(x_0, y_0)\) and \(f(y_0, x_0) \subseteq \{y_0\}\).

(iii) There exist two positive linear operators \(L, S : X \to X\) with \(r(S + L) < 1\) such that, for any \(u_1, u_2, v_1, v_2 \in M\), \(u_1 \leq u_2\) and \(v_2 \leq v_1\) imply:

(a) for any \(x_1 \in f(u_1, v_1)\), there exists \(x_2 \in f(u_2, v_2)\) satisfying \(0 \leq x_2 - x_1 \leq L(u_2 - u_1) + S(v_1 - v_2)\)

(b) for any \(y_1 \in f(v_1, u_1)\), there exists \(y_2 \in f(v_2, u_2)\) satisfying \(0 \leq y_1 - y_2 \leq L(v_2 - v_1) + S(u_2 - u_1)\)

where \(r(S + L)\) denotes the spectral radius of \(S + L\).

Then \(f\) admits a coupled fixed point \((x^*, y^*) \in M \times M\).

**Proof.** From condition (ii), there exist \(x_1 \in f(x_0, y_0)\) and \(y_1 \in f(y_0, x_0)\) such that \(x_0 \leq x_1\) and \(y_1 \leq y_0\). By condition (iii), we can choose \(x_2 \in f(x_1, y_1)\) and \(y_2 \in f(y_1, x_1)\) such that

\[
0 \leq x_2 - x_1 \leq L(x_1 - x_0) + S(y_0 - y_1) \\
0 \leq y_2 - y_1 \leq L(y_0 - y_1) + S(x_1 - x_0).
\]

Repeating the arguments above for \(x_1, x_2, y_1, y_2\) in place \(x_0, x_1, y_0, y_1\) and so on, we can construct two sequences

\[
\{x_n\}, \quad x_n \in f(x_{n-1}, y_{n-1}) \\
\{y_n\}, \quad y_n \in f(y_{n-1}, x_{n-1})
\]

such that

\[
0 \leq x_{n+1} - x_n \leq L(x_n - x_{n-1}) + S(y_{n-1} - y_n) \\
0 \leq y_{n+1} - y_n \leq L(y_n - y_{n-1}) + S(x_{n-1} - x_n).
\]

We claim that

\[
0 \leq x_{n+1} - x_n \leq (L + S)^n(x_1 - x_0 + y_0 - y_1) \\
0 \leq y_{n+1} - y_n \leq (L + S)^n(x_1 - x_0 + y_0 - y_1).
\]
for all $n \geq 1$. In fact, for $n = 1$ it follows from (2.6) that

$$0 \leq x_2 - x_1 \leq L(x_1 - x_0) + S(y_0 - y_1) \leq (L + S)(x_1 - x_0 + y_0 - y_1)$$

$$0 \leq y_1 - y_2 \leq L(y_0 - y_1) + S(x_1 - x_0) \leq (L + S)(x_1 - x_0 + y_0 - y_1).$$

Suppose that (2.7) holds for $n = k$ $(k \geq 1)$. For $n = k + 1$ it follows from (2.6) that

$$0 \leq x_{k+2} - x_{k+1}$$

$$\leq L(x_{k+1} - x_k) + S(y_k - y_{k+1})$$

$$\leq L[(L + S)^k(x_1 - x_0 + y_0 - y_1)] + S[(L + S)^k(x_1 - x_0 + y_0 - y_1)]$$

$$= (L + S)^{k+1}(x_1 - x_0 + y_0 - y_1).$$

By induction, we can conclude that (2.7) holds for all $n \geq 1$. Since $K$ is normal, it follows from (2.7) that

$$\|x_{n+1} - x_n\| \leq N\|(L + S)^n\|\|x_1 - x_0 + y_0 - y_1\|$$

$$\|y_{n+1} - y_n\| \leq N\|(L + S)^n\|\|x_1 - x_0 + y_0 - y_1\|. \quad (2.8)$$

Since $\lim_{n \to \infty} \|(L + S)^n\| = r(S + L) < 1$, we have

$$\|(L + S)^n\| \leq q^n \quad (2.9)$$

for some $q \in (0, 1)$ and $n$ large enough. It follows from (2.8) and (2.9) that

$$\|x_{n+1} - x_n\| \leq Nq^n\|x_1 - x_0 + y_0 - y_1\|$$

$$\|y_{n+1} - y_n\| \leq Nq^n\|x_1 - x_0 + y_0 - y_1\|$$

which implies that $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences. Let $x_n \to x^*$ and $y_n \to y^*$. It is easy to see that $x_n \leq x^*$ and $y^* \leq y_n$ for all $n$ since $K$ is closed. Since $x_{n+1} \in f(x_n, y_n)$ and $y_{n+1} \in f(y_n, x_n)$, by condition (iii) we can choose $x_{n+1}^* \in f(x^*, y^*)$ and $y_{n+1}^* \in f(y^*, x^*)$ such that

$$0 \leq x^* - x_{n+1}^* \leq L(x^* - x_n) + S(y_n - y^*)$$

$$0 \leq y_{n+1}^* - y^* \leq L(y_n - y^*) + S(x^* - x_n).$$

Since $K$ is normal, it follows that

$$\|x_{n+1}^* - x^*\| \leq \|L\|\|x_n - x^*\| + \|S\|\|y_n - y^*\|$$

$$\|y_{n+1}^* - y^*\| \leq \|L\|\|y_n - y^*\| + \|S\|\|x_n - x^*\|.$$
Theorem 2.4. Let \((X, \leq)\) be an ordered Banach space induced by a pointed closed convex normal cone \(K\) with normal constant \(N > 0\), and let \(x_0, y_0 \in X\) with \(x_0 \leq y_0\). Denote
\[ D = [x_0, y_0] = \{ x \in X : x_0 \leq x \leq y_0 \} \]
and let \(f : D \times D \to 2^X\) be a multi-valued mixed increasing operator satisfying the following conditions:

(i) \(f(x, y)\) is closed for any \(x, y \in M\).

(ii) \(\{x_0\} \leq_1 f(x_0, y_0)\) and \(f(y_0, x_0) \leq_2 \{y_0\}\).

(iii) There exists a positive linear operator \(L : X \to X\) with \(r(L) < 1\) such that, for any \(x, y \in D\), \(x \leq y\) implies \(0 \leq v - u \leq L(y - x)\) for any \(u \in f(x, y)\) and any \(v \in f(y, x)\).

Then there exists \(x^* \in D\) such that \(\{x^*\} = f(x^*, x^*)\).

Proof. First we show that \(f(x, x)\) is single-valued for each \(x \in D\). Indeed, since \(K\) is normal, it follows from condition (iii) that
\[ \|u - v\| \leq N\|L\| \|x - x\| = 0 \quad (u, v \in f(x, x)) \]
which implies that \(f(x, x)\) is single-valued for every \(x \in D\). From condition (ii), there exist \(x_1 \in f(x_0, y_0)\) and \(y_1 \in f(y_0, x_0)\) such that \(x_0 \leq x_1\) and \(y_1 \leq y_0\). Further, since \(x_0 \leq y_0\), it follows from condition (iii) that \(x_0 \leq x_1 \leq y_1 \leq y_0\). Since \(f\) is mixed increasing, we can choose \(x_2 \in f(x_1, y_1)\) and \(y_2 \in f(y_1, x_1)\) such that \(x_1 \leq x_2\) and \(y_2 \leq y_1\). Again from condition (iii), we know that \(x_1 \leq x_2 \leq y_2 \leq y_1\). Repeating the arguments above for \(x_1, x_2, y_1, y_2\) in place \(x_0, x_1, y_0, y_1\) and so on, we can construct two sequences
\[ \{x_n\}, \quad x_{n+1} \in f(x_n, y_n), \quad x_n \leq x_{n+1} \leq y_{n+1} \leq y_n. \]
\[ \{y_n\}, \quad y_{n+1} \in f(y_n, x_n), \quad y_n \leq y_{n+1} \leq x_{n+1} \leq x_n. \]
From here and condition (iii) we have
\[ 0 \leq x_{n+1} - x_n \leq y_n - x_n \leq L(y_{n-1} - x_{n-1}) \leq L^n (y_0 - x_0) \]
\[ 0 \leq y_n - y_{n+1} \leq y_n - x_n \leq L(y_{n-1} - x_{n-1}) \leq L^n (y_0 - x_0). \]
Since \(K\) is normal, we now have
\[ \|x_{n+1} - x_n\| \leq N\|L\|^{n-1}\|y_0 - x_0\|. \]
\[ \|y_{n+1} - y_n\| \leq N\|L\|^n\|y_0 - x_0\|. \]
\[ \|x_n - y_n\| \leq N\|L\|^n\|y_0 - x_0\|. \]
Since \(r(L) < 1\), it follows that both \(\{x_n\}\) and \(\{y_n\}\) are Cauchy sequences with the same limit. Let \(\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = x^*\). It is easy to see that \(x_n \leq x^* \leq y_n\) for all \(n \geq 0\). Since \(f\) is mixed increasing, we can choose \(x^*_{n+1} \in f(x^*, x^*)\) and \(y^*_{n+1} \in f(y_n, x_n)\) such that \(x_{n+1} \leq x^*_{n+1} \leq y^*_{n+1} \leq y_{n+1}\), i.e.
\[ 0 \leq x_{n+1} - x_{n+1} \leq y^*_{n+1} - x_{n+1}. \]
This and condition (iii) imply
\[ \|x^*_{n+1} - x_{n+1}\| \leq N\|y^*_{n+1} - x_{n+1}\| \leq N\|L\|\|y_n - x_n\|. \]
Since \(\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = x^*\), then \(x^*_n \to x^*\). It follows from condition (i) that \(\{x^*\} = f(x^*, x^*)\). The proof is complete. \(\square\)
3. Applications

Let \((E, \leq)\) be a real ordered separable Banach space induced by a pointed closed convex normal cone \(P\) with normal constant \(N > 0\), let \(C([0,1], E) = \{u : [0,1] \rightarrow E | u \text{ is continuous}\}\) and \(P_c = \{u \in C([0,1], E) : u(t) \geq 0 \ (t \in [0,1])\}\). For each \(u \in C([0,1], E)\) we define \(\|u\|_c = \max_{t \in [0,1]} \|u(t)\|\). Then \(C([0,1], E)\) is a real Banach space with norm \(\|\cdot\|_c\) and \(P_c\) is a pointed closed convex normal cone with normal constant \(N\). In this section, we also denote by \(\leq\) the order induced by \(P_c\).

Let \((\Omega, \Sigma)\) be a measurable space and \(X\) be a non-empty subset of \(E\). We will use the notations

\[
P_f(X) = \{A \subseteq X : A \text{ non-empty, closed}\}
\]

\[
P_{kc}(X) = \{A \subseteq X : A \text{ non-empty, compact, convex}\}.
\]

A multi-valued mapping \(F : \Omega \rightarrow P_f(X)\) is said to be measurable if, for every \(x \in X\),

\[
\omega \rightarrow d(x, F(\omega)) = \inf_{z \in F(\omega)} \|x - z\|
\]

is measurable.

In the following, we always suppose that \(x \in C([0,1], E)\), \(k : [0,1] \times [0,1] \rightarrow (-\infty, +\infty)\) is a non-negative continuous function, and \(f : [0,1] \times E \times E \rightarrow 2^E\) is a multi-valued operator.

**Theorem 3.1.** Assume that the following conditions hold:

(C1) \(f : [0,1] \times E \times E \rightarrow 2^E\) is a multi-valued operator such that

(a) \(f(\cdot, \cdot, \cdot)\) has values in \(P_{kc}(E)\)

(b) for each \(u, v \in C(I, E)\), \(t \mapsto f(t, u(t), v(t))\) is measurable

(c) for each \(t \in I\) and \(u, v \in C(I, E)\), \(\sup_{x \in f(t, u(\cdot), v(\cdot))} \|x\| \in L^1_{\Delta} E\).

(C2) There exist \(u_0, v_0 \in C([0,1], E)\) such that

\[
\{u_0(t)\} \leq_1 x(t) + \int_0^t k(t, s)f(s, u_0(s), v_0(s))ds
\]

\[
\{v_0(t)\} \geq_2 x(t) + \int_0^t k(t, s)f(s, v_0(s), u_0(s))ds
\]

(C3) There exist two non-negative constants \(L', S'\) such that, for any \(u_1, u_2, v_1, v_2 \in C([0,1], E)\), \(u_1 \leq u_2\) and \(v_2 \leq v_1\) imply

(a) for any \(x_1(t) \in \int_0^t k(t, s)f(s, u_1(s), v_1(s))ds\), there exists \(x_2(t) \in \int_0^t k(t, s)x_1(t) \times f(s, u_2(s), v_2(s))ds\) such that

\[
0 \leq x_2(t) - x_1(t)
\]

\[
\leq \int_0^t L'k(t, s)(u_2(s) - u_1(s))ds + \int_0^t S'k(t, s)(v_1(s) - v_2(s))ds
\]
(b) for any \( y_1(t) \in \int_0^t k(t, s)f(s, v_1(s), u_1(s))ds \), there exists \( y_2(t) \in \int_0^t k(t, s) \times f(s, v_2(s), u_2(s))ds \) such that
\[
0 \leq y_1(t) - y_2(t) \\
\leq \int_0^t L'k(t, s)(v_1(s) - v_2(s))ds + \int_0^t S'k(t, s)(u_2(s) - u_1(s))ds.
\]

(C4) There exists a constant \( K \geq 0 \) such that \( K(L' + S') < 1 \) and \( \int_0^t k(t, s)ds \leq K \). Then there exist \( u^*, v^* \in C(I, E) \) such that
\[
u^*(t) \in x(t) + \int_0^t k(t, s)f(s, u^*(s), v^*(s))ds \\
v^*(t) \in x(t) + \int_0^t k(t, s)f(s, v^*(s), u^*(s))ds.
\]

Proof. Define \( F : C([0, 1], E) \times C([0, 1], E) \rightarrow 2^{C([0, 1], E)} \) as
\[
F(u, v)(t) = x(t) + \int_0^t k(t, s)f(s, u(s), v(s))ds \quad (u, v \in C([0, 1], E)). \tag{3.1}
\]

From condition (C1) we know that \( F \) has non-empty values. Because of the Rådström embedding theorem (see Klein and Thompson [13]), it is easy to see that
\[
\int_0^t k(t, s)f(s, u(s), v(s))ds \in P_{kc}(E) \quad (t \in [0, 1]).
\]

So a straightforward application of the Arzela and Ascoli theorem tells us that \( F \) has values in \( P_{kc}(C([0, 1], E)) \). It follows from condition (C2) and (3.1) that \( \{u_0\} \leq_1 F(u_0, v_0) \)

and \( F(v_0, u_0) \leq_2 \{v_0\} \). We now define \( L, S : C([0, 1], E) \rightarrow C([0, 1], E) \) by
\[
Lu(t) = \int_0^t L'k(t, s)u(s)ds \\
Su(t) = \int_0^t S'k(t, s)u(s)ds.
\]

From here and conditions (C3) - (C4), it is easy to see that condition (iii) of Theorem 2.3 holds for \( F \). Thus, by Theorem 2.3, there exist \( u^*, v^* \in C([0, 1], E) \) such that
\[
u^*(t) \in x(t) + \int_0^t k(t, s)f(s, u^*(s), v^*(s))ds \\
v^*(t) \in x(t) + \int_0^t k(t, s)f(s, v^*(s), u^*(s))ds.
\]

The proof is complete
Remark 3.1. If \( \dim E < \infty \), then condition (C4) of Theorem 3.1 can be relaxed by requiring only \( KS' < 1 \). In fact, \( L \) in the proof of Theorem 3.1 is a compact Volterra operator, and so the operator \( S + T \) has the same spectrum as \( S \) by [2: Theorem 2.3]; in particular, \( r(S + T) = r(S) \). Using this fact and \( KS' < 1 \), we know that \( r(S + T) < 1 \).

Remark 3.2. If \( f : [0, 1] \times E \times E \to P_{kc}(E) \) is a multi-valued operator such that, for all \( u, v \in C([0, 1], E) \), \( t \mapsto f(t, u(t), v(t)) \) is integrably bounded (see, for example, [12] or [19]), then condition (C1) of Theorem 3.1 holds. If \( f : [0, 1] \times E \times E \to E \) is a single-valued operator satisfying the Carathéodory condition, then condition (C1) of Theorem 3.1 can be satisfied.

Theorem 3.2. Let \( u_0, v_0 \in C([0, 1], E) \) with \( u_0 \leq v_0 \), let \( D = [u_0, v_0] = \{ u \in C([0, 1], E) : u_0 \leq u \leq v_0 \} \) and let \( f : [0, 1] \times E \times E \to 2^E \) be a mixed increasing operator satisfying the following conditions:

(C1) (a) \( f(\cdot, \cdot, \cdot) \) has values in \( P_{kc}(E) \)

(b) for each \( u, v \in D \), \( t \mapsto f(t, u(t), v(t)) \) is measurable

(c) for each \( t \in [0, 1] \) and \( u, v \in D \), \( \sup_{x \in f(\cdot, u(\cdot), v(\cdot))} \| x \| \in L_+^1 \).

(C2) \( u_0 \) and \( v_0 \) are such that

\[
\{ u_0(t) \} \leq_1 x(t) + \int_0^t k(t, s)f(s, u_0(s), v_0(s)) \, ds \\
\{ v_0(t) \} \geq_2 x(t) + \int_0^t k(t, s)f(s, v_0(s), u_0(s)) \, ds.
\]

(C3) There exists a non-negative constant \( L' \) such that for any \( \mu, \nu \in D \), \( \mu \leq \nu \) implies

\[
0 \leq v(t) - u(t) \leq \int_0^t L'k(t, s)(\nu(s) - \mu(s)) \, ds
\]

for any \( v(t) \in \int_0^t k(t, s)f(s, \nu(s), \mu(s)) \, ds \) and \( u(t) \in \int_0^t f(s, \mu(s), \nu(s)) \, ds \).

(C4) There exists a constant \( K \geq 0 \) such that \( KL' < 1 \) and \( \int_0^t k(t, s) \, ds \leq K \). Then there exists \( u^* \in D \) such that

\[
\{ u^*(t) \} = x(t) + \int_0^t k(t, s)f(s, u^*(s), u^*(s)) \, ds.
\]

Proof. By using Theorem 2.4 and the similar arguments in Theorem 3.1, the conclusion can be proved but we omit the details.

Example 3.1. Let \( u_0, v_0 \in C([0, 1], E) \) with \( u_0 \leq v_0 \). Let

\( D = [u_0, v_0] = \{ u \in C([0, 1], E) : u_0 \leq u \leq v_0 \} \)

and let \( f : [0, 1] \times E \times E \to E \) be a single-valued mixed increasing operator satisfying the following conditions:

(C1) For each \( u, v \in D \), \( t \mapsto f(t, u(t), v(t)) \) is measurable.
(C2) $u_0$ and $v_0$ are such that
\[ u_0(t) \leq x(t) + \int_0^t k(t, s)f(s, u_0(s), v_0(s)) \, ds \]
\[ v_0(t) \geq x(t) + \int_0^t k(t, s)f(s, v_0(s), u_0(s)) \, ds. \]

(C3) There exists a non-negative constant $L'$ such that, for any $\mu, \nu \in D$, $\mu \leq \nu$ implies
\[ 0 \leq f(t, \nu(t), \mu(t)) - f(t, \mu(t), \nu(t)) \leq L'(\nu(t) - \mu(t)). \]

(C4) There exists a constant $K \geq 0$ such that $KL' < 1$ and $\int_0^t k(t, s) \, ds \leq K$. Then by using Theorem 3.2, there exists $u^* \in D$ such that
\[ u^*(t) = x(t) + \int_0^t k(t, s)f(s, u^*(s), u^*(s)) \, ds. \]

However, the standard technique used in [18] is invalid since $f$ is not continuous.

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References


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