Regularity for a Variational Inequality with a Pseudodifferential Operator of Negative Order

R. Schumann

Abstract. We prove that the solution of a variational inequality on a submanifold in \( \mathbb{R}^n \) involving a pseudodifferential operator of order -1 is bounded.

Keywords: Variational inequalities, regularity of solutions, pseudodifferential operators

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1. Introduction

Consider the variational inequality to find \( u \in K \) such that \((v - u, Au) \geq (v - u, b)\) for all \( v \in K \), where \( b \in W^{\frac{1}{2},2}(S) \) is given, \( K \) denotes the positive cone of the Hilbert space \( W^{-\frac{1}{2},2}(S) \) and \( A \) is an elliptic pseudodifferential operator of the negative order -1 on a closed manifold \( S \subset \mathbb{R}^n \).

Variational inequalities are nonlinear problems even if the operator \( A \) is linear because \( K \) fails to be a linear subspace of \( W^{-\frac{1}{2},2}(S) \). The usual setting is that \( A \) maps a Banach or Hilbert space \( X \) into its dual \( X^* \). In many applications \( X \) is a Sobolev space and \( A \) denotes a linear elliptic differential operator of order \( m \). By energetic considerations, for example, it is often easy to prove the (weak) solvability of the variational inequality. Concerning the regularity of weak solutions we find two different situations: For elliptic equations \( Au = b \) the inclusion \( b \in W^{k,2} \) implies, in general, the inclusion \( u \in W^{k+m,2} \). In contrast to this case, problems for variational inequalities have limited regularity, i.e. even if \( b \) is smooth, their solutions \( u \) cannot overcome a certain threshold of smoothness. For instance, Shamir [14] gave an example where \( u \notin W^{3,2}(\Omega) \cup W^{2,4}(\Omega) \) for \( A = -\Delta + I, b \in W^{1,p} \) for all \( p > 1 \) and \( K = \{ u \in W^{1,2}(\Omega) : u \geq 0 \text{ on } \Gamma \subset \partial \Omega \} \) (cf. Lions [9: Section 8.2] and Rodrigues [12: p. 279]). For variational inequalities with elliptic differential operators the regularity of solutions was investigated, e.g., by Kinderlehrer [6], Kinderlehrer and Stampacchia [8], and Uralzeva [2, 17]. The case of systems of variational inequalities with one-sided obstacles was treated in the papers of Kinderlehrer [7] (systems in \( \mathbb{R}^2 \)) and Schumann [13] (Lamé's system of elasticity in \( \mathbb{R}^N \) \((N \geq 2)\).

It seems however that problems concerning regularity of solutions of variational inequalities have not been considered if the operator \( A \) is a pseudodifferential operator.

R. Schumann: Universität Leipzig, Institut für Mathematik, Augustuspl. 10, D - 04109 Leipzig
of negative order. This case can also be motivated by a physical example (see [10]). A-priori the solution $u$ of the variational inequality only belongs to the Sobolev space $W^{-\frac{1}{2}}(S)$ of negative order $-\frac{1}{2}$. Thus we are interested to prove more regularity for the solution. In Section 5 we shall prove the following result.

**Theorem.** Suppose $b \in W^{\gamma,p}(S)$ for some $\gamma \in (1,2)$ and $r > \frac{N}{\gamma - 1}$. Then the solution $u \in K$ of the variational inequality (1) below is essentially bounded, i.e. $u \in L_\infty(S)$.

We use the following notation. The norm in the Lebesgue space $L_p(U)$ where $U \subset \mathbb{R}^n$ denotes an open set is

$$||u||_p = ||u||_{p,U} = \left( \int_U |u(x)|^p dx \right)^{1/p},$$

and

$$||u||_{\gamma,p} = \left( ||u||_p^p + |u|_{\gamma,p}^p \right)^{1/p}$$

denotes the norm in the Sobolev space $W^{\gamma,p}(U)$ with $\gamma \in (0,1)$ where the seminorm $|u|_{\gamma,p}$ is defined by

$$|u|_{\gamma,p} = \left( \int_{U \times U} |x-y|^{-N-\gamma p} |u(x) - u(y)|^p dx dy \right)^{1/p}.$$ 

The set of pseudodifferential operators of order $m$ acting on $U$ is denoted by $\mathcal{P}^m(U)$.

### 2. Problem and approximation (I)

We suppose that $S$ is a smooth compact $N$-dimensional manifold ($N \geq 2$) without boundary ($\partial S = \emptyset$). Consider the following variational inequality:

Find $u \in K$ such that

$$(v - u, Au) \geq (v - u, b) \quad \text{for all } v \in K \quad (1)$$

where $b \in W^{\frac{1}{2},2}(S)$ is given and $K$ is the positive cone of the Hilbert space $W^{-\frac{1}{2},2}(S)$, i.e.

$$K = \left\{ v \in W^{-\frac{1}{2},2}(S) : (v, \varphi) \geq 0 \text{ for all } \varphi \in D(S) \text{ such that } \varphi \geq 0 \text{ on } S \right\} \quad (2)$$

Clearly $K$ is a closed cone of the Sobolev space $X = W^{-\frac{1}{2},2}(S)$. We denote the norm in $X$ by $|| \cdot ||_{-\frac{1}{2},2}$ and make the following hypotheses on the linear continuous operator $A : W^{-\frac{1}{2},2}(S) \to W^{\frac{1}{2},2}(S)$:

*(H1)* There exists a constant $c > 0$ such that $(v, Av) \geq c ||v||_{-\frac{1}{2},2}^2$ for all $v \in X$. 

For sake of technical simplicity, we assume that a part \( \Gamma' \) of \( S \) lies in the hyperplane \( \mathbb{R}^N \subset \mathbb{R}^n \) \((n = N + 1)\). Furthermore we suppose that the principal symbol of the pseudodifferential operator \( A \in \Psi^{-1}(S) \) on \( \Gamma \) is given by

\[
\sigma_{-1}(A)(x',\xi') = |\xi'|^{-1} \quad \text{for } (x',0) \in \Gamma
\]  

where \( x' = (x_1, \ldots, x_N) \) and \( \xi' = (\xi_1, \ldots, \xi_N) \) (the general case can be handled after a coordinate transform).

It follows from hypothesis (H1) that the variational inequality (1) has a unique solution \( u \in K \) (for a proof cf. Lions [9: Chapter 2.8.2/Theorem 8.1]). Hypothesis (H2) will be used in Sections 4 and 5 to prove regularity of the solution.

To prove regularity we first approximate the solution \( u \) of variational inequality (1) by solutions \( u^\delta \) \((\delta > 0)\) of the following family of variational inequalities:

\[
\text{Find } u^\delta \in K_1 \text{ such that }
\delta (v - u^\delta | u^\delta) + (v - u^\delta, A u^\delta) \geq (v - u^\delta, b^\delta) \quad \text{for all } v \in K_1
\]  

where

\[
K_1 = K \cap L^2(S) = \left\{ v \in L^2(S) : v(x) \geq 0 \ a.e. \ on \ S \right\}.
\]

\( b^\delta \in W^{1,2} \) and \((\cdot | \cdot)\) denotes the inner product in \( L^2(S) \).

We will show that the family \((u^\delta)_{\delta > 0}\) of solutions of variational inequalities (4) approximates the solution \( u \) of variational inequality (1).

**Proposition 1.** Let \( b, b^\delta \in W^{1,2}(S) \). Then the following assertions are true.

1. For any \( \delta > 0 \), there exists a unique solution \( u^\delta \in K_1 \) of inequality (4).

2. If \( \sup_{\delta} \|b^\delta\|_{1,2} < +\infty \), then \( \sup_{\delta} \|u^\delta\|_{-\frac{1}{2},2} < +\infty \).

3. If \( b^\delta \rightharpoonup b \) in \( W^{1,2}(S) \) as \( \delta \to +0 \), then \( u^\delta \rightharpoonup u \) in \( X = W^{-\frac{1}{2},2}(S) \) where \( u \) is the unique solution of inequality (1).

**Proof.** Assertion 1: \( K_1 \) is a closed, convex cone of \( L^2(S) \). The linear continuous operator \( A \) defined by

\[
(v, Au) = \delta (v | u) + (v, Au) \quad \text{for all } u, v \in X
\]

is strongly coercive on \( L^2(S) \) since \( (u, Au) \geq \delta\|u\|_2^2 + c\|u\|_{\frac{1}{2},2}^2 \) for all \( u \in L^2(S) \) (cf. (2)). Thus existence and uniqueness of the solution \( u^\delta \) of variational inequality (4) follow immediately.

Assertion 2: We set \( v = 0 \) in (4) and get

\[
\delta\|u^\delta\|_2^2 + (u^\delta, Au^\delta) \leq \|b^\delta\|_{1,2}\|u^\delta\|_{-\frac{1}{2},2}.
\]
Thus, by (2) and Young's inequality
\[ \delta\|u^\delta\|_2^2 + \frac{c}{2}\|u^\delta\|_{-\frac{1}{2},2}^2 \leq c_1\|b^\delta\|_{\frac{1}{2},2}^2. \]

This means that there exists a constant \( C > 0 \) such that
\[ \sup_\delta \|u^\delta\|_{-\frac{1}{2},2} \leq C \quad \text{and} \quad \sup_\delta \sqrt{\delta}\|u^\delta\|_2 \leq C. \] (6)

Assertion 3: Now, we suppose that \( b^\delta \to b \) in \( W^{\frac{1}{2},2}(S) \) and that \( (\delta_n) \) is a sequence converging to zero. For simplicity we write only \( \delta \) instead of \( \delta_n \) in what follows. Then we may conclude that, at least for a subsequence, \( u^\delta \to u_1 \in K \) in \( X \) and \( \sqrt{\delta}u^\delta \to w \) in \( L_2(S) \). By compact embedding, \( \sqrt{\delta}u^\delta \to w \) in \( X \). Since \( (u^\delta) \) is bounded in \( X \) it follows that \( \sqrt{\delta}u^\delta \to 0 \) in \( X \) as \( \delta \to +0 \). Therefore \( w = 0 \) and \( \sqrt{\delta}u^\delta \to 0 \) in \( L_2(S) \).

(a) To prove \( u = u_1 \) we want to show that \( u_1 \) satisfies the inequality
\[ (v - u_1, Au_1) \geq (v - u_1, b) \quad \text{for all} \quad v \in K_1. \] (7)

Then a density argument proves that \( u_1 \) is a solution of inequality (1) and the uniqueness of the solution gives \( u = u_1 \). Indeed, from (4) we get
\[ \delta(u^\delta \mid u^\delta) + (u^\delta, Au^\delta) \leq (u^\delta - v, b^\delta) + \delta(v \mid u^\delta) + (v, Au^\delta). \] (8)

Since the positive bilinear form \( v \mapsto (Av, v) \) is weakly sequentially lower semicontinuous (cf. Zeidler [19: Vol. 3, p. 156]) it follows from \( \delta \to +0 \) that
\[ (u_1, Au_1) \leq \liminf(u^\delta, Au^\delta) \leq \liminf ((u^\delta, Au^\delta) + \delta\|u^\delta\|^2) \leq (u_1 - v, b) + (v, Au_1) \] (9)

for all \( v \in K_1 \). Thus (7) is proved and we have \( u = u_1 \). A well-known argument concerning subsequences (cf. Zeidler [19: Vol. 1, p. 480]) shows that the whole sequence \( (u^\delta_n) \) is weakly convergent to \( u \).

(b) We prove the strong convergence \( u^\delta \to u \) in \( X \). Let us use (8) with \( v = u \) to get
\[ (u, Au) \leq \liminf(u^\delta, Au^\delta) \leq \limsup(u^\delta, Au^\delta) \leq \limsup ((u^\delta, Au^\delta) + \delta\|u^\delta\|^2) \leq \limsup ((u^\delta - u, b^\delta) + \delta(u \mid u^\delta) + (u, Au^\delta)) = (u, Au) \]

and therefore \( (u^\delta, Au^\delta) \to (u, Au) \) as \( \delta \to +0 \). Then (2) implies
\[ c\|u^\delta - u\|^2_{-\frac{1}{2},2} \leq (u^\delta - u, Au^\delta - Au) \to 0 \]

and Assertion 3 is proved. \( \blacksquare \)
3. Approximation (II)

In Section 2 we replaced the variational inequality (1) acting in \( X = W^{-\frac{1}{2},2}(S) \) by a family of approximate variational inequalities depending on \( \delta > 0 \) with cone \( K_1 \subseteq L_2(S) \) (see (4)). Now we suppose that \( \delta > 0 \) is fixed and introduce a penalization of the negative part of the functions of \( L_2(S) \). The aim is to get a variational inequality over the whole of \( L_2(S) \). This variational inequality has a unique solution \( u_\varepsilon = u_\varepsilon^\delta \) where \( \varepsilon > 0 \) is the penalization parameter. (Since \( \delta \) is fixed in this section we shall omit the supercript \( \delta \)).

Later, in Sections 4 and 5 we are going to derive bounds on the solutions depending neither on \( \varepsilon \) nor on \( \delta \) in order to get regularity results for the solution \( u \) of variational inequality (1).

Suppose \( \varepsilon > 0 \). We construct the following approximation of the variational inequality (4):

Find \( u_\varepsilon \in L_2(S) \) such that

\[
\delta(v - u_\varepsilon | u_\varepsilon) + (v - u_\varepsilon, Au_\varepsilon) + F_\varepsilon(v) - F_\varepsilon(u_\varepsilon) \geq (v - u_\varepsilon, b_\varepsilon) \tag{10}
\]

for all \( v \in L_2(S) \), where \( b_\varepsilon \in W^{\frac{1}{2},2}(S) \) and the penalization functional \( F_\varepsilon \) is defined by

\[
F_\varepsilon(v) = \frac{1}{2\varepsilon} \int_S |v|^{-1} dS
\]

for \( v \in L_2(S) \), denoting for any real function \( \varphi \) by \( \varphi^\pm \) the positive and negative parts of \( \varphi \), respectively, i.e. \( \varphi = \varphi^+ + \varphi^- \).

Parallel with (10) we consider the following variational inequality:

Find \( u^\delta \in L_2(S) \) such that

\[
\delta(v - u^\delta | u^\delta) + (v - u^\delta, Au^\delta) + F(v) - F(u^\delta) \geq (v - u^\delta, b^\delta) \tag{11}
\]

for all \( v \in L_2(S) \), where \( F \) is the indicatrix of the convex set \( K_1 \), i.e. for \( v \in L_2(S) \) we have \( F(v) = 0 \) if \( v \in K_1 \) and \( F(v) = +\infty \) otherwise.

We get now the following statement.

**Proposition 2.** Let \( \delta > 0 \) be fixed and \( b^\delta, b^\varepsilon \in W^{\frac{1}{2},2}(S) \). Then the following assertions are true.

1. For any \( \varepsilon > 0 \) the variational inequality (10) has exactly one solution \( u_\varepsilon \in L_2(S) \).
2. The variational inequality (11) has exactly one solution \( u^\delta \in L_2(S) \).
3. If \( M_0 = \sup_\varepsilon \|b_\varepsilon\|_{\frac{1}{2},2} < +\infty \), then there exists a constant \( M > 0 \) independent of \( \delta \) such that \( M = \sup_\varepsilon (\|u_\varepsilon\|_{\frac{1}{2},2} + \delta \|u_\varepsilon\|_2^2 + F_\varepsilon(u_\varepsilon)) < +\infty \).
4. \( b_\varepsilon \to b^\delta \) in \( W^{\frac{1}{2},2}(S) \) as \( \varepsilon \to +0 \) implies \( u_\varepsilon \to u^\delta \) in \( L_2(S) \) and in \( W^{-\frac{1}{2},2}(S) \).
Proof. Assertion 1 follows from the coercivity of the operator $A$ defined by (5) and the fact that $F_\varepsilon(v) \geq 0$ for all $v \in L_2(S)$ (cf. Lions [9: Chapter 2.8.5/Theorem 8.5]). Since (11) and (4) are equivalent Assertion 2 is obvious. To prove Assertion 3 we set $v = 0$ in (10). As $F_\varepsilon(0) = 0$ we get

$$\delta \|u_\varepsilon\|_2^2 + (u_\varepsilon, Au_\varepsilon) + F_\varepsilon(u_\varepsilon) \leq \|b_\varepsilon\|_{\frac{1}{2},2} \|u_\varepsilon\|_{\frac{1}{2},2}.$$ 

Therefore

$$\delta \|u_\varepsilon\|_2^2 + \frac{c}{2} \|u_\varepsilon\|_{\frac{1}{2},2}^2 + F_\varepsilon(u_\varepsilon) \leq c_1 \|b_\varepsilon\|_{\frac{1}{2},2}$$ (12)

which gives Assertion 3.

To prove Assertion 4 suppose $\varepsilon = \varepsilon_n \to +0$. If $\|b_\varepsilon - b^\delta\|_{\frac{1}{2},2} \to 0$ we get from estimate (12) that at least for a subsequence $u_\varepsilon \to u_1$ in $L_2(S)$. Thus $u_\varepsilon \to u_1$ in $X$. We need to prove that $u_1 = u^\delta$. From the variational inequality (10) it follows that

$$\delta \|u_\varepsilon\|_2^2 + (u_\varepsilon, Au_\varepsilon) \leq \delta(v \mid u_\varepsilon) + (v, Au_\varepsilon) + F_\varepsilon(v) - F_\varepsilon(u_\varepsilon) + (u_\varepsilon - v, b_\varepsilon)$$ (13)

for all $v \in L_2(S)$. By virtue of Barbu and Precupanu [3: Theorem 2.3/p. 107] we have

$$F_\varepsilon(\varphi) = \frac{1}{2\varepsilon} \|\varphi - J_\varepsilon(\varphi)\|_2^2 + F(J_\varepsilon(\varphi))$$ (14)

where $J_\varepsilon = (I + \varepsilon \partial F)^{-1}$ denotes the resolvent of $\partial F$. Then $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$ implies $\|u_\varepsilon - J_\varepsilon u_\varepsilon\| \to 0$ if $\varepsilon \to +0$. Therefore we have $J_\varepsilon u_\varepsilon \to u_1$ in $L_2(S)$ and, since the convex function $F$ is weakly sequentially lower semincontinuous (see [3: p. 102]),

$$F(u_1) \leq \lim inf F(J_\varepsilon u_\varepsilon)$$

$$\leq \lim inf \left( -\frac{1}{2\varepsilon} \|u_\varepsilon - J_\varepsilon u_\varepsilon\|_2^2 + F_\varepsilon(u_\varepsilon) \right)$$ (15)

$$\leq \lim inf F_\varepsilon(u_\varepsilon).$$

Since $u_\varepsilon \to u_1$ in $L_2(S)$ and $u_\varepsilon \to u_1$ in $X$ we get from (13)

$$\delta \|u_1\|_2^2 + (u_1, Au_1)$$

$$\leq \lim inf (\delta \|u_\varepsilon\|_2^2 + (u_\varepsilon, Au_\varepsilon))$$

$$\leq \lim sup (\delta \|u_\varepsilon\|_2^2 + (u_\varepsilon, Au_\varepsilon))$$

$$\leq \lim sup \{\delta(v \mid u_\varepsilon) + (v, Au_\varepsilon) + F_\varepsilon(v) - F_\varepsilon(u_\varepsilon) + (u_\varepsilon - v, b_\varepsilon)\}$$

$$\leq F(v) - \lim inf F_\varepsilon(u_\varepsilon) + \delta(v \mid u_1) + (v, Au_1) + (u_1 - v, b)$$

$$\leq F(v) - F(u_1) + \delta(v \mid u_1) + (v, Au_1) + (u_1 - v, b)$$ (16)

for all $v \in L_2(S)$, i.e. $u_1$ is a solution of variational inequality (11). Observe that $F_\varepsilon(v) \to F(v)$ for all $v \in L_2(S)$ (see Barbu and Precupanu [3: p. 107]). Uniqueness implies $u_1 = u^\delta$.
4. Regularity

In this section we derive $L_p$-bounds for the solution $u_\varepsilon = u_\varepsilon^\delta$ of the variational inequality (10) that are independent of $\varepsilon$ and $\delta$. (Here again, we shall omit the superscript $\delta$.) We are going to consider $u_\varepsilon$ on the hyperplane part $\Gamma$ of $S$ defined in hypothesis (H2). The solution $u_\varepsilon \in L_2(S)$ satisfies the inequality

$$\delta(v - u_\varepsilon | u_\varepsilon) + (v - u_\varepsilon, Au_\varepsilon) + F_\varepsilon(v) - F_\varepsilon(u_\varepsilon) \geq (v - u_\varepsilon, b_\varepsilon)$$  \hspace{1cm} (17)

for all $v \in L_2(S)$. We multiply inequality (17) by the test function $v = u_\varepsilon + t\eta$, where $0 \neq t \in \mathbb{R}$ and $\eta \in C_0^\infty(S)$ satisfies the condition $\text{supp}\eta \subset \subset \Gamma$. Thus

$$\delta(\eta | u_\varepsilon) + (\eta, Au_\varepsilon) + \frac{1}{t}(F_\varepsilon(u_\varepsilon + t\eta) - F_\varepsilon(u_\varepsilon)) \left\{ \begin{array}{ll} > & t > 0 \\ = & t = 0 \\ < & t < 0 \end{array} \right. (\eta, b_\varepsilon) \text{ for } t \neq 0.$$

From

$$\lim_{t \to 0} \frac{1}{t}(F_\varepsilon(u_\varepsilon + t\eta) - F_\varepsilon(u_\varepsilon)) = \varepsilon^{-1} \int_{\Gamma} \eta u_\varepsilon^- \, dS$$

it follows that

$$\delta \int_{\Gamma} \eta u_\varepsilon^- \, dS + \int_{\Gamma} \eta Au_\varepsilon \, dS + \varepsilon^{-1} \int_{\Gamma} \eta u_\varepsilon^- \, dS = \int_{\Gamma} \eta b_\varepsilon \, dS$$  \hspace{1cm} (18)

for all $\eta \in C_0^\infty(S)$ and by approximation for all $\eta \in L_2(S)$ with $\text{supp}\eta \subset \subset \Gamma$. Since $\eta$ can be chosen arbitrarily we get

$$\delta u_\varepsilon + Au_\varepsilon + \varepsilon^{-1} u_\varepsilon^- = b_\varepsilon \quad \text{in } L^1_2(\Gamma).$$  \hspace{1cm} (19)

4.1 (Localization and preliminary regularity). In the following we are going to use local properties of pseudodifferential operators. We choose an open subset $U \subset \subset \Gamma$ and an arbitrary but fixed test function $\varphi \in C_0^\infty(U)$ with $\varphi \geq 0$. Setting $g_\varepsilon = \varphi u_\varepsilon$, relation (19) gives

$$\delta g_\varepsilon + \varphi Au_\varepsilon + \varepsilon^{-1} g_\varepsilon^- = \varphi b_\varepsilon =: \tilde{b}_\varepsilon.$$  \hspace{1cm} (20)

Remark that $\text{supp}\tilde{b}_\varepsilon \subset U$. Furthermore we choose a function $\mu \in C_0^\infty(U)$ such that $\mu \equiv 1$ on an open set $W \subset \subset U$ with $K_\varphi = \text{supp}\varphi \subset W$. Then relation (20) may be written in the form

$$\delta g_\varepsilon + (\varphi A\mu) u_\varepsilon + \varepsilon^{-1} g_\varepsilon^- = \tilde{b}_\varepsilon - \varphi A(1 - \mu) u_\varepsilon = \tilde{b}_\varepsilon + R_1 u_\varepsilon = \tilde{b}_\varepsilon + \mu R_1 u_\varepsilon$$  \hspace{1cm} (21)

where $R_1 = -\varphi A(1 - \mu)$ is a so-called regularizing $\psi$do: $R_1 \in \Psi^{-\infty}(S)$ (see Dieudonné [4: Vol. 7, Prop. 23.26.11/p. 212]). Therefore $R_1 : W^{-\frac{1}{2},2}(S) \longrightarrow W^{m,2}(U) \subset W^{m,2}(S)$ is a continuous operator for all $m \in \mathbb{N}$.

Next we make use of the principal symbol $\sigma_{-1}(A)$ defined in hypothesis (H2). Let us agree to write $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^N$ in the following instead of $x'$ and $\xi'$, respectively. Since the principal symbols of both $\varphi A\mu$ and $\mu A\varphi$ are the same: $\sigma_{-1}(\varphi A\mu)(x, \xi) = \sigma_{-1}(\mu A\varphi)(x, \xi) = \varphi(x)|\xi|^{-1}$, we only get a perturbation of order $-2$ exchanging $\varphi$ and
μ in the term \((φAμ)\) of (21): \(φAμ = μAφ + P_{-2}\) where \(P_{-2} \in Ψ^{-2}(U)\) is a proper \(ψ\)do of order \(-2\). Thus
\[
δg_ε + (μAφ)u_ε + ε^{-1}g_ε^- = b_ε + μR_1u_ε + P_{-2}g_ε =: f_ε.
\] (22)

By the mapping properties of proper \(ψ\)do's, we see that \(P_{-2} : W^{-\frac{1}{2}}(U) → W^{\frac{3}{2}}(U)\) is a continuous linear mapping. Introducing a third cut-off function \(μ_1\) such that \(μ_1 \equiv 1\) on \(supp μ\) we can re-write (22) as
\[
δg_ε + (μAμ_1)g_ε + ε^{-1}g_ε^- = f_ε.
\] (23)

The principal symbol of \(μAμ_1\) on \(r\) is \(σ_{-1}(μAμ_1) = μ(x)|χ|^{-1}\).

Let us fix \(ε > 0\) and study the individual function \(g_ε\) for a moment.

**Lemma 1.** Let us assume \(b_ε \in W^{1,p}_{loc}(Γ)\) for all \(p < +∞\). Then \(g_ε = φu_ε^δ \in W^{1,p}(U)\) for all \(ε, δ > 0\) and \(p < +∞\).

**Proof.** The solution \(u_ε\) of inequality (17) belongs to \(L_2(S)\). Therefore \(f_ε \in W^{1,2}(U)\). From Treves [15: Theorem 2.1/p. 16] we get \((μAμ_1)g_ε \in W^{1,2}(U)\) and relation (23) gives the inclusion
\[
δg_ε + ε^{-1}g_ε^- \in W^{1,2}(U)
\] (24)

Therefore \(δg_ε^+\) and \((δ + ε^{-1})g_ε^-\) both belong to \(W^{1,2}(U)\), and \(g_ε \in W^{1,2}(U)\) for each fixed pair \(δ, ε > 0\). From the embedding theorem it follows that \(g_ε \in L_{p_1}(U)\) with \(p_1 = \frac{2N}{N-2}\) for \(N ≥ 3\) and \(p_1 < +∞\) arbitrary for \(N = 2\). From the same argument we derive the inclusion \(f_ε, (μAμ_1)g_ε \in W^{1,p_1}(U)\) and finally \(g_ε \in W^{1,p_1}(S) \subset L_{p_2}(U)\) with \(p_2 = \frac{2N}{N-4}\) for \(N ≥ 5\) and \(p_2 < +∞\) arbitrary for \(N ≤ 4\). Repeating the argument we conclude that for each \(ε, δ > 0\)
\[
g_ε = φu_ε^δ \in W^{1,p}(U) \quad \text{for all } p < +∞.
\] (25)

Then it follows from the embedding theorem that \(g_ε \in C^β(U)\) for all \(β \in (0, 1)\). \(\square\)

4.2 (\(L^p\)-regularity). We intend first to apply a \(ψ\)do \(P\) with principal symbol \(|χ|\) to equality (23). Then we multiply it by the test function \((g_ε)^{p-1} = |g_ε|^{p-2}g_ε\). In order to avoid additional regularizing terms containing \(ε^{-1}g_ε^-\) we need some preparation. For this define
\[
(Pv)(x) = \int_{\mathbb{R}^N} e^{iξx}χ(ξ)|ξ|^β\hat{v}(ξ)\frac{dξ}{(2π)^N}
\]
for \(v \in C^∞_0(\mathbb{R}^N)\), where \(χ \in C^∞(\mathbb{R}^N)\) is a cut-off function characterized, e.g., by
\[
χ(ξ) = \begin{cases} 
0 & \text{if } |ξ| < 1 \\
1 & \text{if } |ξ| ≥ 2.
\end{cases}
\]

Now we put \(∫ Pv \cdot w dx\) into a form adapted for considerations of the positive and negative part of the functions involved. Taking real functions \(v, w ∈ C^∞_0(\mathbb{R}^N)\) the
theorem of Fubini gives

\[
(P_v, w) = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} e^{i \xi \cdot x} \chi(\xi) |\xi| \hat{v}(\xi) \frac{d\xi}{2\pi N} \right) w(x) \, dx
\]

\[
= \int_{\mathbb{R}^N} \chi(\xi) |\xi| \hat{v}(\xi) \frac{d\xi}{2\pi N} \int_{\mathbb{R}^N} \frac{d\xi}{2\pi N}
\]

\[
= \int_{\mathbb{R}^N} |\xi| \hat{v}(\xi) \frac{d\xi}{2\pi N} + \int_{\mathbb{R}^N} (\chi(\xi) - 1) |\xi| \hat{v}(\xi) \frac{d\xi}{2\pi N}
\]

\[=: I_1 + I_2.
\]

The operator $R_2$ defined by

\[
(R_2 v)(x) = \int_{\mathbb{R}^N} e^{i \xi \cdot x} (\chi(\xi) - 1) |\xi| \hat{v}(\xi) \frac{d\xi}{2\pi N}
\]

for $v \in C_0^\infty(\mathbb{R}^N)$ is regularizing: $R_2 \in \mathcal{P}^{-\infty}(\mathbb{R}^N)$, since the amplitude $\chi(\xi) - 1$ vanishes outside the ball $B_2(0)$ (cf. Dieudonné [4: Remark 23.19.5(iii)/p.149]). Applying Parseval’s equality to $I_1$ we get

\[
I_1 = a \int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y|^{-N-1} (v(x) - v(y)) (w(x) - w(y)) \, dx \, dy
\]

where $a = a(N) > 0$ is a constant (see Wloka [18: p. 97] and Hörmander [5: Vol. 1/p. 241]). We stress that both integrals $I_1$ and $I_2$ depend on $v$ and $w$. We have $(R_2 v, w) = I_2$ and define an operator $J_1$ by

\[
(J_1 v, w) = I_1 = a \int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y|^{-N-1} (v(x) - v(y)) (w(x) - w(y)) \, dx \, dy
\]

for all $v, w \in C_0^\infty(\mathbb{R}^N)$ to get

\[
(J_1 v, w) = (P_v, w) - (R_2 v, w).
\]

We now prove $L_p$-regularity of the solution $u$ of the variational inequality (1).

**Theorem 1.** Let $2 \leq p < +\infty$ and $b \in W^{1,2}(S) \cap W^{1,p}_{loc}(\Gamma)$. Then $u \in L^p_{loc}(\Gamma)$.

**Remark 1.** For $2 \leq p < +\infty$, the inclusion $b \in W^{1,p}(S)$ implies the inclusion $u \in L^p(S)$ if after a coordinate transform the operator $A$ has the principal symbol (3) in each coordinate patch of a partition of unity on $S$.

**Proof of Theorem 1.** To prove the theorem we consider the approximate problems and derive uniform bounds for the solutions $u_\varepsilon = u_\varepsilon^\delta$ of inequality (10) and $u_\delta$ of inequality (4).

(a) For simplicity we set $b_\delta = b \in W^{1,2}(S) \cap W^{1,p}_{loc}(\Gamma)$. By approximation, we may assume that the family $(b_\varepsilon)$ belongs to $W^{1,2}(S) \cap W^{1,q}_{loc}(\Gamma)$ for all $q < +\infty$ and,
furthermore, \( b_\varepsilon \to b^\delta \) in \( W^{1,2} \)}(S) in \( W^{1,p}_{\text{loc}}(\Gamma) \) as \( \varepsilon \to +0. \) In particular, for any open set \( O \subset \subset \Gamma \)

\[
\sup_{\varepsilon} \| b_\varepsilon \|_{1,p,0} \leq M = M(O) < +\infty. \tag{29}
\]

It follows from Lemma 1 that \( g_\varepsilon \in W^{1,q}(U) \) for all \( q < +\infty. \) Therefore also \( g_\varepsilon^- \in W^{1,q}(U) \) for all \( q < +\infty. \) Suppose \( q > N, \) arbitrary. Then \( W^{1,q}(U) \) is a Banach algebra (see Adams [1: p. 115]) and it follows that \( (g_\varepsilon)^{p-1} = |g_\varepsilon|^{p-2}g_\varepsilon \in W^{1,q}(U) \) for each \( q \geq 2. \) It is our goal to show that (29) implies

\[
\sup_{\varepsilon} \| g_\varepsilon \|_p \leq M_1 < +\infty \tag{30}
\]

where the constant \( M_1 \) is independent of \( \delta. \) This gives the local boundedness of \( u_\varepsilon \in L_p(\Gamma). \) In fact, we may choose \( \varphi \) such that \( \varphi \equiv 1 \) on any open set \( V \subset \subset U \) and estimation (30) implies

\[
\sup_{\varepsilon} \| u_\varepsilon \|_{p,V} \leq M_1 < +\infty. \tag{31}
\]

(b) We apply operator \( J_1 \) to equality (23) and multiply it by \( h_\varepsilon = (g_\varepsilon)^{p-1} \) to get

\[
\delta(J_1 g_\varepsilon, h_\varepsilon) + (J_1 (\mu A \mu_1) g_\varepsilon, h_\varepsilon) + \epsilon^{-1}(J_1 g_\varepsilon^-, h_\varepsilon) = (J_1 f_\varepsilon, h_\varepsilon),
\]

that is

\[
L_1 + L_2 + L_3 := \delta a \int \int |x-y|^{-N-1} (g_\varepsilon(x) - g_\varepsilon(y)) (h_\varepsilon(x) - h_\varepsilon(y)) \, dxdy
\]

\[
+ (P(\mu A \mu_1) g_\varepsilon, h_\varepsilon)
\]

\[
+ \epsilon^{-1} a \int \int |x-y|^{-N-1} (g_\varepsilon^-(x) - g_\varepsilon^-(y)) (h_\varepsilon(x) - h_\varepsilon(y)) \, dxdy
\]

\[
= ((P - R_2)f_\varepsilon, h_\varepsilon) + (R_2(\mu A \mu_1) g_\varepsilon, h_\varepsilon).
\]

Now we have to consider the terms \( L_1, L_2 \) and \( L_3 \) of (32) separately. The function \( t \mapsto |t|^{p-2} t \) is uniformly monotone for \( p \geq 2: \)

\[
(|s|^{p-2} s - |t|^{p-2} t)(s - t) \geq c|s - t|^p \quad \text{for all } s, t \in \mathbb{R} \tag{33}
\]

where \( c > 0 \) is a constant (cf. Zeidler [19: Vol. 2/p. 503]). Then

\[
L_1 = \delta a \int \int |x-y|^{-N-1} (g_\varepsilon(x) - g_\varepsilon(y)) (|g_\varepsilon(x)|^{p-2} g_\varepsilon(x) - |g_\varepsilon(y)|^{p-2} g_\varepsilon(y)) \, dxdy
\]

\[
\geq \delta ca \int \int |x-y|^{-N-1} |g_\varepsilon(x) - g_\varepsilon(y)|^p dxdy
\]

\[
= \delta ca |g_\varepsilon|^p_{p,p}.
\]

The third term \( L_3 \) in (32) is the penalization term. Observing that

\[
(|s|^{p-2} s - |t|^{p-2} t)(s^- - t^-) \geq (|s^-|^{p-2} s^- - |t^-|^{p-2} t^-)(s^- - t^-)
\]
it follows from (33) that

\[
L_3 = \varepsilon^{-1} a \int \int |x - y|^{-N-1} (g^-_\varepsilon(x) - g^-_\varepsilon(y)) (h^-_\varepsilon(x) - h^-_\varepsilon(y)) \, dx \, dy
\]

\[
\geq \varepsilon^{-1} c a \int \int |x - y|^{-N-1} |g^-_\varepsilon(x) - g^-_\varepsilon(y)|^p \, dx \, dy
\]

\[
= \varepsilon^{-1} c a |g^-_\varepsilon|_{p,p}^p.
\]

The second term of \(L_2 = (P(\mu A_1)g_\varepsilon, h_\varepsilon)\) of (32) contains the composition of \(P \in \Psi^1(U)\) and the proper \(\psi \) do \(\mu A_1 \in \Psi^{-1}(U)\). The principal symbol of \(P(\mu A_1) \in \Psi^0(U)\) is \(\sigma_0(P(\mu A_1))(x, \xi) = \chi(\xi)\mu(x)\). Thus there exists a \(\psi \) do \(P^{-1} \in \Psi^{-1}(U)\) such that

\[
\int (P(\mu A_1)(v)) \cdot w \, dx
\]

\[
= \int \left\{ \int e^{i(x-y)\xi} \chi(\xi)\mu(y)v(y) \, dy \cdot \frac{d\xi}{(2\pi)^N} \right\} w(x) \, dx + \int P_{-1} v \cdot w \, dx
\]

\[
= \int \left( \int e^{ix\xi} \chi(\xi)\bar{v}(\xi) \frac{d\xi}{(2\pi)^N} \right) w(x) \, dx + (P_{-1} v, w)
\]

\[\leq (1 + |P_{-1} v, w|) (1 + |P_{-1} v, w|)
\]

for all \(v, w \in C^0_c(W)\) where \(\int \int\) denotes an oscillatory integral and \(R_3\) is regularizing by the argument already used for \(R_2\). Then, by approximation,

\[
L_2 = \int g_\varepsilon^p \, dx + (R_3 g_\varepsilon, h_\varepsilon) + (P_{-1} g_\varepsilon, h_\varepsilon).
\]

By Hölder's inequality, equations (32) and (34) together give

\[
\delta c a |g_\varepsilon|_{p,p}^p + \|g_\varepsilon\|_{p,p}^p + \varepsilon^{-1} c a |g^-_\varepsilon|_{p,p}^p
\]

\[
\leq \left( \|P - R_2\|_p + \|R_2(\mu A_1)g_\varepsilon\|_p + \|R_3 g_\varepsilon\|_p + \|P_{-1} g_\varepsilon\|_p \right) \|g_\varepsilon\|_{p-1}^p
\]

\[
\leq C \left( \|\varphi b\|_{1,p,W} + \|P - R_2\|_p u \|_p, w + \|P_{-2} g_\varepsilon\|_{1,p,W}
\]

\[
+ \|R_2(\mu A_1)g_\varepsilon\|_p, w + \|R_3 g_\varepsilon\|_p, w + \|P_{-1} g_\varepsilon\|_p, w \right) \|g_\varepsilon\|_{p-1}^p
\]

since \(K_\varphi = \text{supp} \varphi \subset W \subset U\). Young's inequality and Proposition 2 imply

\[
\delta |g_\varepsilon|_{1,p,p}^p + \|g_\varepsilon\|_{p,p}^p + \varepsilon^{-1} |g^-_\varepsilon|_{p,p}^p
\]

\[
\leq C \left( \|b\|_{1,p,W} + \|u\|_{p,2,s} + \|P_{-2} g_\varepsilon\|_{1,p,W} + \|g_\varepsilon\|_{\frac{1}{2},2} + \|P_{-1} g_\varepsilon\|_{p,w} \right)
\]

\[
\leq C \left( 1 + \|P_{-2} g_\varepsilon\|_{1,p,W} + \|P_{-1} g_\varepsilon\|_{p,w} \right)
\]
since $R_1$ and $R_2$ are regularizing.

(c) We are going to apply a bootstrap argument. Using the embedding theorem and the fact that $P_{-1} : W^{rac{1}{2},2}_{comp}(U) \to W^{rac{1}{2},2}_{loc}(U)$ and $P_{-2} : W^{rac{1}{2},2}_{comp}(U) \to W^{rac{3}{2},2}_{loc}(U)$ are continuous linear mappings we get

\[
\|P_{-1}g\|_{q_1,w} \leq c_1\|P_{-1}g\|_{\frac{1}{2},2,w} \leq c_2\|g\|_{-\frac{1}{2},2} \tag{36}
\]
\[
\|P_{-2}g\|_{1,q_1,w} \leq c_1\|P_{-2}g\|_{\frac{3}{2},2,w} \leq c_2\|g\|_{-\frac{1}{2},2} \tag{37}
\]

for some constants $c_1 > 0$ and $c_2 > 0$, where $q_1 = \frac{2N}{N-1}$. We stress that these constants depend upon $W$ and $K_y$, but neither on $\epsilon$ nor on $\delta$. It follows from (35) with $p = q_1$ that

\[
\sup_{\epsilon} \left( \epsilon|g_\epsilon|_{q_1, q_1, w}^q + \|g_\epsilon\|_{q_1, q_1, w} + \epsilon^{-1}|g_\epsilon^{-1}|_{q_1, q_1, w} \right) < +\infty. \tag{38}
\]

This implies $\sup_{\epsilon} \|g_\epsilon\|_{q_1, w} < +\infty$. As in the first step we get

\[
\sup_{\epsilon} \{\|P_{-2}g\|_{2,q_1,w} + \|P_{-1}g\|_{1,q_1,w}\} < +\infty. \tag{39}
\]

With $q_2 = \frac{2N}{N-3}$ the embedding theorem implies

\[
\|P_{-2}g\|_{1,q_2,w} \leq c_3\|P_{-2}g\|_{2,q_1,u} \quad \text{and} \quad \|P_{-1}g\|_{q_2,w} \leq c_3\|P_{-1}g\|_{1,q_1,w}
\]

and we get from (35) with $p = q_2$

\[
\sup_{\epsilon, \delta} \left( \delta|g_\epsilon|_{q_2, q_2}^q + \|g_\epsilon\|_{q_2, q_2} + \epsilon^{-1}|g_\epsilon^{-1}|_{q_2, q_2} \right) < +\infty.
\]

We can repeat this procedure as far as $q_j \leq p$. In the last step we get

\[
\sup_{\epsilon} \left( \delta|g_\epsilon|^p_{p,p} + \|g_\epsilon\|_{p,p} + \epsilon^{-1}|g_\epsilon^{-1}|_{p,p} \right) \leq M_1 < +\infty \tag{39}
\]

where the estimates used above show that the constant $M_1$ is independent of $\delta > 0$. This proves estimations (30) and (31).

(d) Let $\epsilon_n \to +0$ for fixed $\delta > 0$. Since $\sup \|g_\epsilon\|_p \leq M_1$ we can extract a subsequence with $\varphi u_\epsilon \to g^\delta$ in $L_p(U)$. As $u_\epsilon \to u^\delta$ in $L^2(S)$ (Proposition 2) we conclude that $g^\delta = \varphi u^\delta \in L^p(S)$, i.e. $u^\delta \in L^p_{loc}(\Gamma)$. Let $\varphi \equiv 1$ on $V$. The weak sequential lower semicontinuity of the norm gives $\|u^\delta\|_{p, V} \leq \|\varphi u^\delta\|_p \leq M_1$ for $V \subset U$.

(e) If $\delta_n \to +0$, there exists a subsequence such that $\varphi u^\delta \to u_0$ in $L_p(S)$ and $\varphi u^\delta \to u_0$ in $W^{-\frac{1}{2},2}(U)$. Proposition 1 gives $\varphi u^\delta \to \varphi u$ in $W^{-\frac{1}{2},2}(S)$. Consequently $u_0 = \varphi u \in L_p(U)$, and it follows that $u \in L^p_{loc}(\Gamma)$ with $\|u\|_{p, V} \leq \|\varphi u\|_p \leq M_1$ for $V \subset U$.
5. $L^\infty$-regularity

5.1. To prove $L^\infty$-regularity for the solutions $u_\varepsilon$ of equation (19) we apply a method from the classical theory of differential equations due to Stampacchia. It depends on estimates for the size of level sets. As in Subsection 4.2 we begin with a kind of differentiation of equation (23). Here we are going to use the operator

\[ (P^\gamma v)(x) := \int_{\mathbb{R}^N} e^{i x \xi} \chi(\xi) |\xi|^\gamma \hat{\nu}(\xi) \frac{d\xi}{(2\pi)^N} \]  

(40)

for $v \in C^\infty_0(\mathbb{R}^N)$ where $1 < \gamma < 2$ and $\chi \in C^\infty(\mathbb{R}^N)$ is the same function as in Subsection 4.2. For $g_\varepsilon = \varphi u_\varepsilon$ we have the following estimate.

**Lemma 2.** Suppose $b_\varepsilon \in W^{1,2}(U)$ for some $\gamma \in (1, 2)$. Then there exist appropriate pseudo's $Q_\gamma$ and $Q_{\gamma-2}$ from $\Psi^\gamma(U)$ and $\Psi^{\gamma-2}(U)$, respectively, such that

\[ \delta a \left[ |g_\varepsilon(x) - k|^2 + a |g_\varepsilon(x) - k|^2 \right]_{2,2} \leq \int_U \left( |Q_{\gamma} g_\varepsilon| + |Q_{\gamma-2} g_\varepsilon| \right) |g_\varepsilon(x) - k|^2 dx. \]  

(41)

**Proof.** (a) For $v, w \in C^\infty_0(\mathbb{R}^N)$ we get

\[ (P^\gamma v, w) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{i x \xi} \chi(\xi) |\xi|^\gamma \hat{\nu}(\xi) \frac{d\xi}{(2\pi)^N} \ w(x) \ dx \]

\[ = \int_{\mathbb{R}^N} |\xi|^\gamma \hat{\nu}(\xi) \overline{w(\xi)} \frac{d\xi}{(2\pi)^N} + \int_{\mathbb{R}^N} (\chi(\xi) - 1) |\xi|^\gamma \hat{\nu}(\xi) \overline{w(\xi)} \frac{d\xi}{(2\pi)^N} \]  

(42)

Concerning the integral $I_2^\gamma$ we observe that the operator $R_2^\gamma$ defined by

\[ (R_2^\gamma v)(x) = \int_{\mathbb{R}^N} e^{i x \xi} (\chi(\xi) - 1) |\xi|^\gamma \hat{\nu}(\xi) \frac{d\xi}{(2\pi)^N} \]  

for $v \in C^\infty_0(\mathbb{R}^N)$ is regularizing, whereas Parseval's inequality implies

\[ I_1^\gamma = a \int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y|^{-N-\gamma} (v(x) - v(y)) (w(x) - w(y)) \ dx \ dy \]  

(43)

with $a = a(\gamma, N) > 0$. Defining

\[ (J^\gamma v, w) = I_1^\gamma \]

\[ = a \int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y|^{-N-\gamma} (v(x) - v(y)) (w(x) - w(y)) \ dx \ dy \]

for all $v, w \in C^\infty_0(\mathbb{R}^N)$ we get

\[ (J^\gamma v, w) = (P^\gamma v, w) - (R_2^\gamma v, w). \]  

(44)
(b) The application of the operator \( J \) to equality (23) and scalar multiplication by a test function \( h_\varepsilon \) gives

\[
L_1 + L_2 + L_3 := \delta a \int \int |x - y|^{-N-\gamma}(g_\varepsilon(x) - g_\varepsilon(y))(h_\varepsilon(x) - h_\varepsilon(y))\,dxdy \\
+ (P^\gamma(\mu A\mu_1)g_\varepsilon, h_\varepsilon) \\
+ \varepsilon^{-1} a \int \int |x - y|^{-N-\gamma}(g_\varepsilon^-(x) - g_\varepsilon^-(y))(h_\varepsilon(x) - h_\varepsilon(y))\,dxdy
\]

\[
= \left( (P^\gamma - R_2^\gamma)f_\varepsilon, h_\varepsilon \right) + (R_2^\gamma(\mu A\mu_1)g_\varepsilon, h_\varepsilon).
\]

For \( k \geq 0 \), choose \( h_\varepsilon = |g_\varepsilon - k|^+ \in W^{\frac{3}{2}, 2}(U) \) in (45). It follows that \( \text{supp} [g_\varepsilon(x) - k]^+ \subseteq \text{supp} \varphi \) for \( k \geq 0 \). We first get

\[
L_1 = \delta a \int \int |x - y|^{-N-\gamma}([g_\varepsilon(x) - k] - [g_\varepsilon(y) - k]) \\
\times ([g_\varepsilon(x) - k]^+ - [g_\varepsilon(y) - k]^+)\,dxdy \\
\geq \delta a \int \int |x - y|^{-N-\gamma}||g_\varepsilon(x) - k|^+ - [g_\varepsilon(y) - k]^+|^2\,dxdy \\
= \delta a [g_\varepsilon(k)^+]^2|^{2}_{\frac{3}{2}, 2}.
\]

Observing that

\[
(s^- - t^-)(|s - k|^+ - |t - k|^+) \geq 0 \quad \text{for all } s, t \in \mathbb{R}
\]

we see that

\[
L_3 = \varepsilon^{-1} a \int \int |x - y|^{-N-\gamma}(g_\varepsilon^-(x) - g_\varepsilon^-(y)) \\
\times ([g_\varepsilon(x) - k]^+ - [g_\varepsilon(y) - k]^+)\,dxdy \\
\geq 0.
\]

In the second term \( L_2 \) of (45), the principal symbol of the composition \( P^\gamma(\mu A\mu_1) \) is \( \sigma_{\gamma - 1}(P^\gamma(\mu A\mu_1))(x, \xi) = \mu(x)|\xi|^{\gamma - 1}\chi(\xi) \). It follows that there exists a \( \psi \)-do \( P_{\gamma - 2} \in \Psi^{\gamma - 2}(U) \) such that \( P^\gamma(\mu A\mu_1) = P^{\gamma - 1}\mu + P_{\gamma - 2} \) where \( P^{\gamma - 1} \in \Psi^{\gamma - 1}(U) \) is defined by (40) with \( \gamma \) replaced by \( \gamma - 1 \). Thus (44) with \( \gamma - 1 \) instead of \( \gamma \) gives

\[
L_2 = (P^{\gamma - 1}g_\varepsilon, h_\varepsilon) + (P_{\gamma - 2}g_\varepsilon, h_\varepsilon) \\
= a \int \int |x - y|^{-N-\gamma + 1}([g_\varepsilon(x) - k] - [g_\varepsilon(y) - k])([g_\varepsilon(x) - k]^+ - [g_\varepsilon(y) - k]^+) \\
+ (R_3 g_\varepsilon, h_\varepsilon) + (P_{\gamma - 2} g_\varepsilon, h_\varepsilon) \\
\geq a [g_\varepsilon(k)^+]^{2}_{\frac{3}{2}, 2} + (R_3 g_\varepsilon, h_\varepsilon) + (P_{\gamma - 2} g_\varepsilon, h_\varepsilon).
\]

The regularizing operator \( R_3 = R_2^{\gamma - 1} \) arises from (44). Observe that \( \mu \equiv 1 \) on \( K_\varphi = \text{supp} \varphi \). Summarizing we get

\[
\delta a [g_\varepsilon(k)^+]^{2}_{\frac{3}{2}, 2} + a [g_\varepsilon(k)^+]^{2}_{\frac{3}{2}, 2} \\
\leq \int_U \left\{(P^\gamma - R_2^\gamma)f_\varepsilon + (R_2^\gamma(\mu A\mu_1)g_\varepsilon - P_{\gamma - 2} g_\varepsilon - R_3 g_\varepsilon)\right\}[g_\varepsilon(x) - k]^+\,dx
\]

\[
= \int_U (Q_\gamma f_\varepsilon + Q_{\gamma - 2} g_\varepsilon)[g_\varepsilon(x) - k]^+\,dx.
\]
where we have introduced $Q_{\gamma} = P^\gamma - R_{2}^\gamma$ and $Q_{\gamma - 2} = R_{2}^\gamma (\mu A \mu_1) - P_{\gamma - 2} - R_{3}$ to keep the notation short. This proves the lemma.

5.2 We prove an embedding theorem which is needed later in this section.

Lemma 3. Suppose $\Omega \subset \mathbb{R}^N$ is a domain and $s \in (0, 1)$ is given. We set $\frac{1}{q} = \frac{1}{2} - \frac{s}{N}$, i.e. $q = \frac{2N}{N-2s} > 2$. Then the following assertions are true.

1. We have the continuous embedding $W^{s,2}(\Omega) \subset L_q(\Omega)$, such that
   \[ \|u\|_q \leq c \|u\|_{s,2} \quad \text{for all } u \in W^{s,2}(\Omega). \]

2. If $\Omega_1 \subset \subset \Omega$ is an open set, then there exists a constant $C = C(\Omega, \Omega_1) > 0$ such that
   \[ \|u\|_q \leq C \|u\|_{s,2} \quad \text{for all } u \in W^{s,2}(\Omega) \text{ with } \text{supp } u \subseteq \Omega_1. \] (47)

Proof. For Assertion 1 cf. Triebel [16: p. 196]. For Assertion 2 we prove that

\[ u \mapsto \|u\|_a = \left\{ |u|_{s,2}^2 + \int_{\Omega \setminus \Omega_1} |u|^2 \right\}^{1/2} \] (48)

is an equivalent norm on $W^{s,2}(\Omega)$, i.e. there exist constants $c_1, c_2 > 0$ such that

\[ c_1 \left\{ |u|_{s,2}^2 + \int_{\Omega \setminus \Omega_1} |u|^2 \right\} \leq \left\{ |u|_{s,2}^2 + \int_{\Omega} |u|^2 \right\} \leq c_2 \left\{ |u|_{s,2}^2 + \int_{\Omega \setminus \Omega_1} |u|^2 \right\} \] (49)

for all $u \in W^{s,2}(\Omega)$. The first inequality in (49) is obvious. To prove the second one we suppose the contrary. Then there exists an sequence $(u_n)_{n \in \mathbb{N}}$ such that $\|u_n\|_{s,2} \geq n \|u_n\|_a$ ($n \in \mathbb{N}$). We define $v_n = u / \|u_n\|_{s,2}$. Thus $\|v_n\|_{s,2} = 1$ and $\|v_n\|_a \to 0$, and we can select a subsequence, again denoted by $(v_n)$ such that $v_n \to v$ in $W^{s,2}(\Omega)$, $v_n \to v$ in $L_2(\Omega)$ and $v_n(x) \to v(x)$ a.e. in $\Omega$. From $\|v_n\|_a \to 0$ it follows that

\[ v_n(x) \to 0 \quad \text{a.e. in } \Omega \setminus \Omega_1 \] (50)

and

\[ |v_n|_{s,2}^2 = \int_{\Omega \times \Omega} |x-y|^{-N-2s} |v_n(x) - v_n(y)|^2 \, dx \, dy \to 0. \]

Therefore $|v_n(x) - v_n(y)| \to 0$ a.e. in $\Omega \times \Omega$ and (50) implies $v_n(x) \to 0$ a.e. in $\Omega$. This gives $v_n \to 0$ in $L_2(\Omega)$ and because of $|v_n|_{s,2} \to 0$ we see that $\|v_n\|_{s,2} \to 0$, which contradicts $\|v_n\|_{s,2} = 1$. Thus (49) is proved, and (47) follows immediately.

5.3 Now we define sets $A_\epsilon(k)$ where $g_\epsilon = \varphi u_\epsilon^\delta$ superceeds a level $k$:

\[ A_\epsilon(k) = \{ x \in \Gamma : g_\epsilon \geq k \}. \]

We age going to estimate the size of $A_\epsilon(k)$. Remember that $1 < \gamma < 2$. 

Lemma 4. We suppose \( b \in W^{\gamma, r}_{\text{loc}}(\Gamma) \) for some \( \gamma \in (1, 2) \) and \( r > \frac{N}{\gamma - 1} \). Set \( b^\delta := b \). Then there exist constants \( C > 0 \) and \( \beta > 1 \), independent from \( \varepsilon \) and \( \delta \), such that
\[
|A_\varepsilon(h)| \leq \frac{C}{(h - k)^\beta} |A_\varepsilon(k)|^\beta \quad \text{for all } h > k \geq 0
\]
where \( q = \frac{2N}{N+1-\gamma} \).

Proof. Set \( s = \frac{2 - 1}{2} \), \( q = \frac{2N}{N-2s} = \frac{2N}{N+1-\gamma} > 2 \) and \( \frac{1}{q} + \frac{1}{q'} = 1 \). It follows from Lemma 2, Lemma 3 and the inclusion \( \text{supp} \{ g_\varepsilon(x) - k \}^+ \subseteq \text{supp} \varphi \subseteq U \) that
\[
\left\{ \int_{A_\varepsilon(k)} |g_\varepsilon(x) - k|^q dx \right\}^{2/q} \leq c \left\{ \int_{A_\varepsilon(k)} (|Q_\gamma f_\varepsilon| + |Q_{\gamma - 2}g_\varepsilon|)^{q'} dx \right\}^{1/q'} \left\{ \int_{A_\varepsilon(k)} |g_\varepsilon(x) - k|^{|q|} dx \right\}^{1/q} \quad (52)
\]
for \( k \geq 0 \). Young's inequality gives
\[
\left\{ \int_{A_\varepsilon(k)} |g_\varepsilon(x) - k|^q dx \right\}^{2/q} \leq c \left\{ \int_{A_\varepsilon(k)} (|Q_\gamma f_\varepsilon| + |Q_{\gamma - 2}g_\varepsilon|)^{q'} dx \right\}^{2/q'}
\]
Therefore, for \( h > k \geq 0 \),
\[
|A_\varepsilon(h)|(h - k)^q \leq c \left\{ \int_{A_\varepsilon(k)} (|Q_\gamma f_\varepsilon| + |Q_{\gamma - 2}g_\varepsilon|)^{q'} dx \right\}^{q/q'}
\]
and, by Hölder's inequality with \( r > \frac{q}{q-2} = \frac{N}{\gamma - 1} \) and \( r > q' \),
\[
|A_\varepsilon(h)|(h - k)^q \leq c \left( \|Q_\gamma f_\varepsilon\|_{r,U} + \|Q_{\gamma - 2}g_\varepsilon\|_{r,U} \right)^q |A_\varepsilon(k)|^{q - 1 - \frac{q}{r}} \quad (53)
\]
We see that \( \beta = q - 1 - \frac{q}{r} > 1 \). It follows from (22) and (30) in the proof of Theorem 1 that \( \sup (\|Q_\gamma f_\varepsilon\|_{r,U} + \|Q_{\gamma - 2}g_\varepsilon\|_{r,U}) < +\infty \). This gives (51).

Now we are in the position to prove the uniform boundedness of the family \( (u_\varepsilon) = (u^\delta_\varepsilon) \). We are going to use the following result of Stampacchia.

Lemma 5 (see Kinderlehrer and Stampacchia [8: p. 63]). Let \( \phi : [k_0, +\infty) \to \mathbb{R} \) be a non-negative and non-increasing function such that
\[
\phi(h) \leq \frac{C}{(h - k)^\alpha} |\phi(k)|^\beta \quad \text{for } h > k > k_0
\]
where \( C, \alpha \) and \( \beta \) are positive constants with \( \beta > 1 \). Then
\[
\phi(k_0 + M) = 0
\]
where

\[ M = 2^{\frac{2}{\gamma + 1}} C \frac{1}{\gamma} \left| \phi(k_0) \right|^{\frac{\gamma - 1}{\gamma}}. \]  

(55)

**Theorem 2.** Suppose \( b \in W^{1,\gamma}(U) \) for some \( \gamma \in (1,2) \) and \( r > \frac{N}{\gamma - 1} \). Then the solution \( u \) of the variational inequality (1) is locally bounded on \( \Gamma : u \in L^{\infty}(\Gamma) \), i.e. for all \( V \subset \subset \Gamma \) there exists a constant \( M > 0 \) such that \( 0 \leq u(x) \leq M \) a.e. on \( V \).

**Remark 2.** Under the hypotheses of Remark 1 one may prove the inclusion \( u \in L_{\infty}(S) \).

**Proof of Theorem 2.** We shall prove the theorem in three steps.

(a) First we define \( b_\varepsilon^\delta = b_\varepsilon \) for all \( \varepsilon, \delta > 0 \). We are going to apply Lemma 5 and suppress the superindices \( \varepsilon, \delta \) again. Set \( \phi_\varepsilon(k) = |A_\varepsilon(k)| \) and \( k_0 = 0 \). Then \( \phi_\varepsilon(k_0) = |\{x \in \Gamma : g_\varepsilon \geq 0\}| \leq |U| \) and it follows from (51) that there exists a bound \( M > 0 \) independent of \( \varepsilon \) and \( \delta \) such that

\[ \varphi(x) u_\varepsilon^\delta(x) = g_\varepsilon(x) \leq M := \sup_\varepsilon 2^{\delta - \varepsilon} C \frac{1}{\gamma} \left| \phi_\varepsilon(0) \right|^{\frac{\gamma - 1}{\gamma}} \leq c_1 |U|^{\frac{\gamma - 1}{\gamma}} \]  

(56)

a.e. on \( U \).

(b) Next, we keep \( \delta > 0 \) fixed and let \( \varepsilon := \varepsilon_\delta \to +0 \). For simplicity, we omit the subscript \( n \). From Proposition 2 we know that \( u_\varepsilon \to u^\delta \) in \( L_2(S) \), \( g_\varepsilon \to g^\delta = \varphi u^\delta \) in \( L_2(U) \) and along a subsequence \( g_\varepsilon(x) \to g^\delta(x) \) a.e. in \( U \). Since \( u^\delta \in K_1 \) (56) gives

\[ 0 \leq \varphi(x) u^\delta(x) = g^\delta(x) \leq M \]

a.e. in \( U \).

(c) Finally, let \( \delta := \delta_\varepsilon \to +0 \). As in the proof of Theorem 1 we have \( \varphi u^\delta \to \varphi u \) in \( L_2(U) \), and \( \varphi u^\delta \to \varphi u \) in \( W^{-\frac{1}{2},2}(S) \). Along a subsequence, a theorem of Banach and Saks (see Riesz and Sz.-Nagy [11: p.72]) implies the strong \( L_2 \)-convergence of the sequence of arithmetic means, i.e. \( v_n = \frac{1}{n}(\varphi u^\delta_1 + \varphi u^\delta_2 + \ldots + \varphi u^\delta_n) \to \varphi u \) in \( L_2(U) \). Again, passing to a subsequence if necessary, \( v_n(x) \to \varphi(x) u(x) \) a.e. in \( U \). Since for the means \( 0 \leq v_n(x) \leq M \) we have also \( 0 \leq \varphi(x) u(x) \leq M \) a.e. in \( U \). As we may choose \( \varphi \) in Subsection 4.1 such that \( \varphi \equiv 1 \) on an arbitrary open set \( V \subset \subset U \) the assertion follows □

**References**


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