On the Existence of Conjugate Points for Sturm-Liouville Differential Equations

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There are various results in the literature on estimating the distance between consecutive zeros of solutions of second order differential equations (cf. [10], for instance). The following investigation is devoted to this problem. We consider the Sturm-Liouville differential equation on a bounded interval,

\[-(p(x) u')' + q(x) u = 0, \quad (-a \leq x \leq a < \infty; \ p, q \in C[-a, a]) \quad (1)\]

and suppose that \( p \) is positive and piecewise continuously differentiable on \([-a, a]\). The points \( x_1, x_2, -a \leq x_1 < x_2 \leq a \), are said to be conjugate with respect to the equation (1) if there exists a nontrivial solution \( u \) to (1) with \( u(x_1) = u(x_2) = 0 \). Solutions to (1) are always real-valued functions belonging to \( C^1[-a, a] \) (cf. [10, p. 25]). Set, for \( 0 \leq s < a \),

\[Q(s) = \sup_{s < h < a} \frac{1}{2h} \int_{-h}^{h} q \ dx, \quad \bar{p}(s) = \frac{1}{2(a - s)} \left( \int_{-s}^{a} p \ dx + \int_{s}^{a} p \ dx \right).\]

**Theorem 1:** If there exists a number \( s, 0 \leq s < a \), such that

\[\frac{3}{(a - s) (a + 2s)} \bar{p}(s) + Q(s) \leq 0, \quad (2) \]

then there exists a pair of conjugate points on \((-a, a)\) with respect to (1) and the constant \( 3(a - s)^{-1} (a + 2s)^{-1} \) in (2) is the best possible one.

**Proof:** Consider the sesquilinear form

\[t(f, g) = \int_{-a}^{a} (p(x)' \bar{g} + q(x) g) \ dx \quad (f, g \in D(t)) \quad (3)\]
to (1). The domain $D(t)$ of this form is identical with the Sobolev space $W^2_2(-a, a)$. In the following the form is estimated by means of the test function

$$v(x) = \begin{cases} a - |x|, & s \leq |x| \leq a, \\ a - s, & |x| \leq s, \end{cases}$$

which belongs to $D(t)$. By Fubini's theorem we obtain

$$[[v, v]] = \int_{-a}^a (p(v')^2 + qv^2) \, dx = 2(a - s) \overline{p}(s) + \int_{-a}^a q(x) \, dy \, dx$$

$$= 2(a - s) \overline{p}(s) + 2 \int_{-a}^a \left( \frac{1}{2h(y)} \int_{-h(y)}^{h(y)} q(x) \, dx \right) h(y) \, dy$$

$$\leq 2(a - s) \overline{p}(s) + 2Q(s) \int_{-a}^a h(y) \, dy,$$

where $h(y) = a - \sqrt{y}$, $0 \leq y \leq (a - s)^2$. Hence, in view of (2), there follows that

$$[[v, v]] \leq 2(a - s) \overline{p}(s) + \frac{2}{3} Q(s) (a' - s)^2 (a + 2s) \leq 0.\] Consequently, we have

$$\inf \{[[v, f]] : f \in D(t), \|f\| = 1\} \leq 0,$$

where $\|\|$ denotes the norm in the Hilbert space $L_2(-a, a)$. If this infimum is less than zero, then there exists a nontrivial solution $v$ to (1) having at least two zeros on $(-a, a)$ (cf. [8]). If the infimum is equal to zero, the (normalized) test function $v$ is realizing the infimum and, consequently, it is a solution to (1) (cf. [9]). This, however, is impossible, because a solution to (1) belongs to the Sobolev space $W^2_2(-a, a)$ (cf. [2]). The function $v$, however, does not belong to $W^2_2(-a, a)$.

We prove now that the constant $\sigma = 3(a - s)^{-1} (a + 2s)^{-1}$ is the best possible one. Let us discuss the case $0 < s < a$. We prove that for any $\varepsilon > 0$ there exist functions $p_\varepsilon$ and $q_\varepsilon$ with

$$(\sigma - \varepsilon) \overline{p}_\varepsilon(s) + Q_\varepsilon(s) \leq 0,$$

where

$$\overline{p}_\varepsilon(s) = \frac{1}{2(a - s)} \left( \int_{-s}^a p_\varepsilon \, dx + \int_{s}^a p_\varepsilon \, dx \right), \quad Q_\varepsilon(s) = \sup_{s < h < a} \frac{1}{2h} \int_{-h}^h q_\varepsilon \, dx,$$

such that there does not exist a pair of conjugate points on $(-a, a)$ with respect to the differential equation

$$- (p_\varepsilon(x) u')' + q_\varepsilon(x) u = 0, \quad (-a \leq x \leq a).$$

Obviously, it suffices to assume that $\varepsilon < \sigma$ (if $\varepsilon \geq \sigma$, choose $p_\varepsilon \equiv 1$ and $q_\varepsilon \equiv 0$). Choose $q_\varepsilon = \varepsilon - \sigma$ and

$$p_\varepsilon(x) = (\varepsilon - \sigma) \begin{cases} (x^2/2 - (a + 2t^2(a + s)^{-1}) x) + A(t), & |x| \geq s \\ (t^2 - x \sinh^{-1} (x/t)) B(t), & |x| \leq s, \end{cases}$$

where, for $0 < t < \infty$,

$$A(t) = \frac{s^2}{2} + t^2 - st \coth \frac{s}{t}, \quad B(t) = \left( a - s + \frac{2t^2}{a + s} \right) \sinh \frac{s}{t} + t \cosh \frac{s}{t}.\]
Here \( t \) is a parameter which will be fixed later. The value \( p_0(0) \) is defined by the limit \( (e^{-t \sigma}) \left( t^2 - tB(t) \right) \) of \( p_0(x) \) for \( x \to 0 \). Clearly, \( p_0 \) is an even function. Further, it can easily be verified that \( p_0 \) is continuous and piecewise continuously differentiable on \([-a, a]\). We have

\[
p_o'(x) = (\sigma - \epsilon) \begin{cases} (a + 2t^2(a + s)^{-1} - x), & s < x \leq a, \\ \sinh^{-1} \left( \frac{x}{t} \right) B(t) \left( 1 - \frac{x}{t} \right) \coth \left( \frac{x}{t} \right), & 0 < x < s, \end{cases}
\]

\( p_o'(0) = 0 \), and \( p_o'(x) > 0 \) \( (0 < x \leq a) \), \( p_o'(x) < 0 \) \( (0 < x < s) \). Hence

\[
\min_{0 \leq x < s} p_o(x) = p_o(s) = (\epsilon - \sigma) \left( s^2/2 - (a + 2t^2(a + s)^{-1}) s + A(t) \right)
\]

\[
> (\epsilon - \sigma) \left( t^2 - s \coth \left( \frac{s}{t} \right) - (a + 2t^2(a + s)^{-1}) s \right)
\]

Thus, it follows that \( p_o(x) > 0 \), \( -a \leq x \leq a \), whenever \( 0 < t < s \). Next we prove that (6) holds if \( t \) is chosen sufficiently small. The inequality (6) is equivalent to

\[
(a - s)^{-1} \int_s^a p_o dx \leq 1.
\]

An easy calculation shows that

\[
(a - s)^{-1} \int_s^a p_o dx = 1 - e\sigma^{-1} + (\sigma - \epsilon) s \coth \left( \frac{s}{t} \right).
\]

A value \( t = t_o, 0 < t_o < s \), can be chosen so small that \( t_o \coth \left( \frac{s}{t_o} \right) \leq \epsilon/\sigma(\sigma - \epsilon) s \). By such choice the inequalities (8), and, consequently, (6) are fulfilled. It is easily seen that the function

\[
u(x) = \begin{cases} a + 2t_o^2(a + s)^{-1} - |x|, & |x| \geq s, \\ \sinh^{-1} \left( \frac{x}{t_o} \right) (B(t_o) - t_o \cosh \left( \frac{x}{t_o} \right)), & |x| \leq s, \end{cases}
\]

belongs to \( C^1[-a; a] \) and is positive. The function \( p_o u' \) belongs also to \( C^1[-a; a] \) and by calculation one can prove that \( (p_o u')' = q u \). The function \( u \) is a positive solution to (7). Finally, assume that there exists a nontrivial solution \( u_0 \) to (7) possessing at least two zeros \( x_1 \) and \( x_2 \) on \((-a, a)\). Then, by Sturm's comparison theorem, each solution to (7) has a zero between \( x_1 \) and \( x_2 \) or is a constant multiple of \( u_0 \). The solution (9), however, contradicts this conclusion. Hence, there cannot exist a pair of conjugate points on \((-a, a)\) with respect to (7). This proves the theorem in the case \( 0 < s < a \). Similarly, one can prove that the constant \( 3a^{-2} \) is best possible in the case \( s = 0 \).

The mean value \((1/2h) \int_h^0 q dx, s \leq h \leq a \)) is an increasing or a decreasing function of \( h \) if its derivative is non-negative or non-positive, i.e. if

\[
\frac{1}{2h} \int_{-h}^h q dx \leq \frac{1}{2} \left( g(-h) + g(h) \right) \quad \text{or} \quad \frac{1}{2h} \int_{-h}^h q dx \geq \frac{1}{2} \left( g(-h) + g(h) \right),
\]

1) Set \( q(0) \) for the mean value when \( s = h = 0 \).
respectively. Then, the corresponding supremae \( Q(s) \) are
\[
Q(s) = \frac{1}{2a} \int_{-s}^{a} q \, dx \quad \text{and} \quad Q(s) = \frac{1}{2s} \int_{-s}^{s} q \, dx,
\]
respectively. Thus we obtain the following corollary.

**Corollary 1:** If there exists a number \( s, 0 \leq s < a \), such that
\[
\frac{1}{2a} \int_{-s}^{a} q \, dx \left( \frac{3}{2} \frac{1}{2} \right) \left( q(-s) + q(s) \right) \quad \text{if} \quad (s \leq h \leq a),
\]
then there exists a pair of conjugate points on \((-a, a)\) with respect to (1) if
\[
\frac{3}{(a - s)(a + 2s)} p(s) + \frac{1}{2a} \int_{-a}^{a} q \, dx \leq 0 \left( \frac{3}{(a - s)(a + 2s)} p(s) + \frac{1}{2s} \int_{-s}^{s} q \, dx \leq 0 \right).
\]
In each case the factor \( 3(a - s)^{-1}(a + 2s)^{-1} \) is best possible.

If \( q \) can be written as a sum \( q = q_1 + q_2 \) where \( q_1 \) is an odd function and \( q_2 \) is monotone decreasing on \([-a, -s]\) and monotone increasing on \([s, a]\), then one can easily prove that \( \frac{1}{2s} \int_{-s}^{s} q \, dx \leq \frac{1}{2} \left( q(-s) + q(s) \right) \) implies (10) written with the sign \( \leq \). If \( q_2 \) is monotone increasing on \([-a, -s]\) and monotone decreasing on \([s, a]\), then
\[
\frac{1}{2s} \int_{-s}^{s} q \, dx \geq \frac{1}{2} \left( q(-s) + q(s) \right) \implies (10) \text{written with the sign} \geq.
\]
Hence, we obtain the following corollary from Corollary 1.

**Corollary 2:** Suppose that there exists a point \( s, 0 \leq s < a \), such that \( q \) can be written as a sum \( q = q_1 + q_2 \) in \( C[-a, a] \) with \( q_1 \) odd and \( q_2(x_1) \leq q_2(x_2) \) \((x_1 < x_2 \leq -s)\), \( q_2(x_1) \leq q_2(x_2) \) \((s \leq x_1 < x_2)\), and suppose further that
\[
\frac{1}{2s} \int_{-s}^{s} q(x) \, dx \left( \frac{3}{2} \frac{1}{2} \right) \left( q(-s) + q(s) \right).
\]
Then there exists a pair of conjugate points on \((-a, a)\) with respect to (1) if
\[
\frac{3}{(a - s)(a + 2s)} p(s) + \frac{1}{2a} \int_{-a}^{a} q \, dx \leq 0 \left( \frac{3}{(a - s)(a + 2s)} p(s) + \frac{1}{2s} \int_{-s}^{s} q \, dx \leq 0 \right).
\]
In the special case \( s = 0 \) the hypotheses (11) are always satisfied and the conditions (12) call
\[
\frac{3}{a^2} \int_{-a}^{a} p \, dx + \int_{-a}^{a} q \, dx \leq 0 \left( \frac{3}{2a^2} \int_{-a}^{a} p \, dx + q(0) \leq 0 \right).
\]
Corollary 3: Suppose that \( q \) belongs to \( C^2(-a, a) \) and is convex (concave) on \([-a, a]\).
If the hypotheses (13) are satisfied, respectively, then there exists a pair of conjugate points on \((-a, a)\) with respect to (1). The constants \( 3/\alpha^2 \) (\( 3/2\alpha^3 \)) are best possible.

Proof: Set \( q(x) = q(0) + q'(0)x + r(x), -a \leq x \leq a, \) and note that \( r \) has the properties \( r'(0) = 0 \) and \( r'' \geq 0 \) (\( r'' \leq 0 \)). Hence, we have \( r'(x) \leq 0 \) (\( r'(x) \geq 0 \)) on \([-a, 0] \) and \( r'(x) \geq 0 \) (\( r'(x) \leq 0 \)) on \([0, a]\). Further, \( q_1(x) = q'(0)x \) is an odd function and \( q_2(x) = q(0) + r(x) \) has the concerned properties formulated in Corollary 2 (with \( s = 0 \)). The corollary now follows from Corollary 2.

Corollary 4: Let \( p \) be positive and piecewise continuously differentiable on \([0, X)\), \( X \leq \infty, \) and assume that \( q(\in C^2(0, X)] \) is convex (concave). Let \( u \) be a nontrivial solution to (1) considered on \([0, X)\) with \( u(0) = 0 \). The first conjugate point to \( x = 0 \) is smaller than \( b \), where \( b \) is the smallest positive root of the equation

\[
12 \int_0^b p \, dx + b^2 \int_0^b q \, dx = 0 \quad \left( 12 \int_0^b p \, dx + b^2 q(b/2) = 0 \right),
\]

when \( b \) exists.

Proof: Use Corollary 3 and Sturm's comparison theorem.

Set \( s = 0 \) in the estimate (5) for \( \ell[v, v] \) and suppose that \( q \) is monotone increasing on \([-a, a]\). Then we obtain

\[
\ell[v, v] = \int_{-a}^a p(x) \, dx + \int_{-a}^a \left( \frac{1}{h(y)} \int_0^y q(x) \, dx \right) h(y) \, dy + \int_{-a}^a \left( \frac{1}{h(y)} \int_0^y q(x) \, dx \right) h(y) \, dy
\]

\[
\leq \int_{-a}^a p(x) \, dx + q(0) \int_0^a h(y) \, dy + \int_{-a}^a q(x) \, dx \int_0^a h(y) \, dy
\]

\[
= \int_{-a}^a p(x) \, dx + \frac{1}{3} a^3 q(0) + \frac{1}{3} a^2 \int_{-a}^a q(x) \, dx.
\]

Analogously, if \( q \) is monotone decreasing on \([-a, a]\) we have

\[
\ell[v, v] \leq \int_{-a}^a p(x) \, dx + \frac{1}{3} a^3 q(0) + \frac{1}{3} a^2 \int_{-a}^a q(x) \, dx.
\]

These estimates yield analogous corollaries to the Corollaries 3 and 4. The following corollary corresponds to Corollary 4.

Corollary 5: Let \( p \) be positive and piecewise continuously differentiable on \([0, X)\), \( X \leq \infty, \) and assume that \( q \) is monotone increasing (decreasing) on \([0, X] \) with \( u(0) = 0 \). The first conjugate point to \( x = 0 \) is smaller than \( b \) where \( b \) is the smallest positive root of the equation

\[
12 \int_0^b p \, dx + \frac{b^3}{2} q \left( \frac{b}{2} \right) + b^2 \int_{b/2}^b q \, dx = 0 \quad \left( 12 \int_0^b p \, dx + \frac{b^3}{2} q \left( \frac{b}{2} \right) + b^2 \int_0^{b/2} q \, dx = 0 \right),
\]

when \( b \) exists.

\(^2\) The hypothesis \( q \in C^2 \) can be weakened.
Corollary 6: If there exists a nontrivial solution $u$ to (1) with $u(-a) = 0 = u(a)$ and $u(x) \neq 0$, $-a < x < a$, then
\[ \int_{-a}^{a} p(x) \, dx + \int_{-a}^{a} (a + x)^2 q(x) \, dx + \int_{-a}^{a} (a - x)^2 q(x) \, dx > 0. \] (14)

Proof: By assuming the contrariety of (14) there follows from (5) that $l[v, v] \leq 0$ where $v$ is defined by (4). This implies that there exists a pair of conjugate points on $(-a, a)$ with respect to (1). This, however, is impossible because by Sturm's comparison theorem the solution $u$ would have a zero in $(-a, a)$.

In the special case $p \equiv 1$ and $q \leq 0$ the condition (14) is due to Lovelady [7].

By using the test function $v(x) = \cos(\pi x / 2a)$, $-a \leq x \leq a$, in place of (4) and setting
\[ P = \sup_{0 < h < a} \frac{1}{2(a - h)} \left( \int_{-h}^{h} p \, dx + \int_{-h}^{h} p \, dx \right), \quad Q = \sup_{0 < h < a} \frac{1}{2h} \int_{-h}^{h} q \, dx \]
we obtain the following theorem.

Theorem 2: If
\[ (\pi^2 / 4a^2) P + Q \leq 0, \] (15)
then there exists a pair of conjugate points on $[-a, a]$ with respect to (1). The constant $\pi^2 / 4a^2$ is the best possible one.

Proof: The proof is analogous to that of Theorem 1. By means of Fubini's theorem and setting
\[ \sigma(x) = \sin^2 (\pi x / 2a), \quad \gamma(x) = \cos^2 (\pi x / 2a), \quad -a \leq x \leq a, \]
\[ h(y) = (2a / \pi) \arcsin \sqrt{y}, \quad k(y) = (2a / \pi) \arccos \sqrt{y}, \quad 0 \leq y \leq 1, \]
we obtain
\[ l[v, v] = \int_{-a}^{a} p(x) (v'(x))^2 \, dx + \int_{-a}^{a} q(x) v^2(x) \, dx \]
\[ = \frac{\pi^2}{4a^2} \int_{-a}^{a} p(x) \sin^2 \left( \frac{\pi}{2a} x \right) \, dx + \int_{-a}^{a} q(x) \cos^2 \left( \frac{\pi}{2a} x \right) \, dx \]
\[ = \frac{\pi^2}{4a^2} \int_{-a}^{a} \int_{0}^{\sigma(x)} p(x) \, dy \, dx + \int_{-a}^{a} \int_{0}^{\gamma(x)} q(x) \, dy \, dx \]
\[ = \frac{\pi^2}{2a^2} \int_{0}^{1} \frac{1}{2(a - h(y))} \left( \int_{-h(y)}^{h(y)} p(x) \, dx + \int_{-h(y)}^{h(y)} p(x) \, dx \right) (a - h(y)) \, dy \]
\[ + 2 \int_{0}^{1} \frac{1}{2k(y)} \left( \int_{k(y)}^{k(y)} q(x) \, dx \right) k(y) \, dy \]
\[ \leq \frac{\pi^2}{2a^2} P \int_{0}^{1} (a - h(y)) \, dy + 2Q \int_{0}^{1} k(y) \, dy = \frac{\pi^2}{4a} P + aQ. \] (16)
It follows from (15) that \( (v, v) \leq 0 \). Hence, there exists a pair of conjugate points on \([-a, a]\). To prove that the constant \( \pi^2/4a^2 \) is best possible observe that \( \cos (\pi x/2a), \alpha > a \), is a positive solution to the differential equation
\[
-u'' - (\pi^2/4a^2) u = 0 \quad (-a \leq x \leq a).
\]
(17)
The coefficients \( p \equiv 1 \) and \( q \equiv -\pi^2/4a^2 \) satisfy \( (\pi^2/4a^2) P + Q = 0 \), and, on the other hand, there does not exist a pair of conjugate points on \([-a, a]\) with respect to (17). Therefore, the constant \( \pi^2/4a^2 \) in (15) cannot be chosen smaller.

In the special case \( p \equiv 1 \) the inequality (15) calls
\[
\sup_{0 < h < a} \frac{1}{2h} \int_{-h}^{h} q(x) dx \leq - \frac{\pi^2}{4a^2}.
\]
This condition for the existence of conjugate points with respect to the equation \(-u'' + q(x) u = 0\) \((-a \leq x \leq a)\) is due to LEIGHTON [5, 6].

Assume now that the coefficients \( p \) and \( q \) can be written as \( p = p_1 + p_2, q = q_1 + q_2 \), on \([-a, a]\), where the following possibilities are to be discussed:

1. \( p_1, q_1 \) are odd functions;
2. \( p_2(x_1) \geq p_2(x_2) \) \((x_1 < x_2 \leq 0), p_2(x_1) \leq p_2(x_2) \) \((0 \leq x_1 < x_2)\);
3. \( q_2(x_1) \geq q_2(x_2) \) \((x_1 < x_2 \leq 0), q_2(x_1) \leq q_2(x_2) \) \((0 \leq x_1 < x_2)\);
4. \( q_2(x_1) \leq q_2(x_2) \) \((x_1 < x_2 \leq 0), q_2(x_1) \geq q_2(x_2) \) \((0 \leq x_1 < x_2)\).

Corollary 7: There exists a pair of conjugate points on \([-a, a]\) with respect to (1) in each of the following cases.

1. (1), (2a), (3a) are fulfilled (this is so, for instance, if the functions \( p, q \) are convex and \( \pi^2/4a \) \((p(-a) + p(a)) + \int_a^0 q dx \leq 0\).

2. (1), (2a), (3b) are fulfilled (this is so, for instance, if \( p \) is convex and \( q \) is concave and \( \pi^2/8a^2 \) \((p(-a) + p(a)) + q(0) \leq 0\).

3. (1), (2b), (3a) are fulfilled (this is so, for instance, if \( p \) is concave and \( q \) is convex and \( \pi^2/4a^2 \) \(\int_a^0 p dx + \int_a^0 q dx \leq 0\).

4. (1), (2b), (3b) are fulfilled (this is so, for instance, if \( p, q \) are concave and \( \pi^2/8a^3 \) \(\int_a^0 p dx + q(0) \leq 0\).

The constants \( \pi^2/4a \) and so on are all best possible.

The following corollary follows directly from Corollary 7.

Corollary 8: An upper bound \( b \) for the first conjugate point of \( x = 0 \) with respect to (1) considered on \([0, X], X \leq \infty\), is given by the first roots of the following equations:

1. \( (\pi^2/2b) \left( p(0) + p(b) \right) + \int_0^b q dx = 0 \) when \( p, q \) are convex;
2. \( (\pi^2/2b^2) \left( p(0) + p(b) \right) + q(b/2) = 0 \) when \( p \) is convex and \( q \) is concave.
(iii) \( (\pi^2/\beta) \int_0^b p \, dx + \int_0^b q \, dx = 0 \) when \( p \) is concave and \( q \) is convex;

(iv) \( (\pi^2/\beta) \int_0^b p \, dx + q(b/2) = 0 \) when \( p, q \) are concave.

In the special case that \( p \equiv 1, q \leq 0, q \) is monotone and convex the conditions (i) and (iii) are due to Leighton [6]. These conditions also improve a condition by Fink [3, Th. 3]. In the special case that \( p \equiv 1, q \leq 0, q \) is monotone and concave the conditions (ii) and (iv) are due to Leighton and Kian Ke [4, 6].

Finally, let \( p \) and \( q \) be monotone functions. If both are monotone increasing on \([-a, a]\) the estimate (16) can be modified as follows:

\[
\{v, v\} = \frac{\pi^2}{4a^2} \int_0^1 \left( \frac{1}{a - h(y)} \int_0^a p(x) \, dx \right) \left( a - h(y) \right) dy
\]

\[
+ \frac{\pi^2}{4a^2} \int_0^1 \left( \frac{1}{a - h(y)} \int_0^a p(x) \, dx \right) \left( a - h(y) \right) dy
\]

\[
+ \int_0^1 \left( \frac{1}{k(y)} \int_0^a q(x) \, dx \right) k(y) dy + \int_0^1 \left( \frac{1}{k(y)} \int_0^a q(x) \, dx \right) k(y) dy
\]

\[
= \frac{\pi^2}{8a^2} \left( ap(a) + \int_{-a}^0 p(x) \, dx \right) + \frac{1}{2} \left( aq(0) + \int_0^a q(x) \, dx \right).
\]

This estimate proves the assertion of the following corollary under the hypothesis (i). The other assertions can analogously be proved.

**Corollary 9:** An upper bound \( b \) for the first conjugate point of \( x = 0 \) with respect to the equation (1) considered on \([0, X], X \leq \infty\), is given by the smallest roots of the following equations:

(i) \( \frac{\pi^2}{b^2} \left( \frac{b}{2} p(b) + \int_0^{b/2} p \, dx \right) + \frac{b}{2} q \left( \frac{b}{2} \right) + \int_{b/2}^b q \, dx = 0 \) when \( p, q \) are monotone increasing;
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(ii) \( \frac{\pi^2}{b^2} \left( \frac{b}{2} p(b) + \int_0^{b/2} p \, dx \right) + \frac{b}{2} q \left( \frac{b}{2} \right) + \int_0^{b/2} q \, dx = 0 \) when \( p \) is monotone increasing and \( q \) is monotone decreasing;

(iii) \( \frac{\pi^2}{b^2} \left( \frac{b}{2} p(0) + \int_0^{b/2} p \, dx \right) + \frac{b}{2} q \left( \frac{b}{2} \right) + \int_0^{b/2} q \, dx = 0 \) when \( p \) is monotone decreasing and \( q \) is monotone increasing;

(iv) \( \frac{\pi^2}{b^2} \left( \frac{b}{2} p(0) + \int_0^{b/2} p \, dx \right) + \frac{b}{2} q \left( \frac{b}{2} \right) + \int_0^{b/2} q \, dx = 0 \) when \( p, q \) are monotone decreasing.

Examples: a) Consider the differential equation

\[-u'' + (x^2 - 7) u = 0 \quad (0 < x < \infty).\]

The interesting solution defined by the boundary condition \( u(0) = 0 \) is the Hermite function \( H_3(x) = c e^{x^2/2}(e^{-x^2})^{(3)} \). The first conjugate point of \( x = 0 \) is \( x_1 = 1.2247 \ldots \). By means of Corollary 8/(i) we obtain the upper bound \( b = 1.2328 \ldots \) for \( x_1 \).

b) Consider the equation

\[-[(1 - x^2) u']' + \left( \frac{1}{1 - x^2} - 12 \right) u = 0 \quad (-a \leq x \leq a < 1).\]  

(18)

\( p \) is concave and \( q \) is convex. By applying Corollary 7/(iii), it is seen that there exists a pair of conjugate points on \([-a, a]\) with \( a = 0.4584 \ldots \) (18) is a Legendre differential equation the interesting solution of which is equal to

\[ P_3^1(x) = c_1 \sqrt{1 - x^2}((1 - x^2)^2)^{(4)} = c_2 \sqrt{1 - x^2} (5x^2 - 1)\]

(cf. [1, pp. 94, 96, 344]) with zeros at \( x_{1,2} = \pm 5^{-1/2} = \pm 0.4472 \ldots \)

Both examples show that the calculated upper bounds are good approximate values for the exact numbers.

REFERENCES


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