Mackey Topologies on Vector-Valued Function Spaces

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Abstract. Let $E$ be an ideal of $L^0$ over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$, and let $(X, \| \cdot \|_X)$ be a real Banach space. Let $E(X)$ be a subspace of the space $L^0(X)$ of $\mu$-equivalence classes of all strongly $\Sigma$-measurable functions $f : \Omega \to X$ and consisting of all those $f \in L^0(X)$ for which the scalar function $\| f(\cdot) \|_X$ belongs to $E$. Let $E(X)_n^\sim$ stand for the order continuous dual of $E(X)$. We examine the Mackey topology $\tau(E(X), E(X)_n^\sim)$ in case when it is locally solid. It is shown that $\tau(E(X), E(X)_n^\sim)$ is the finest Hausdorff locally convex-solid topology on $E(X)$ with the Lebesgue property. We obtain that the space $(E(X), \tau(E(X), E(X)_n^\sim))$ is complete and sequentially barreled whenever $E$ is perfect. As an application, we obtain the Hahn-Vitali-Saks type theorem for sequences in $E(X)_n^\sim$. In particular, we consider the Mackey topology $\tau(L^\Phi(X), L^\Phi(X)_n^\sim)$ on Orlicz-Bochner spaces $L^\Phi(X)$. We show that the space $(L^\Phi(X), \tau(L^\Phi(X), L^\Phi(X)_n^\sim))$ is complete iff $L^\Phi$ is perfect. Moreover, it is shown that the Mackey topology $\tau(L^\infty(X), L^\infty(X)_n^\sim)$ is a mixed topology.

Keywords: Vector-valued function spaces, Orlicz-Bochner spaces, locally solid topologies, Lebesgue topologies, Mackey topologies, mixed topologies, sequential barreledness

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1. Introduction and preliminaries

Given a topological vector space $(L, \xi)$ by $(L, \xi)^*$ we will denote its topological dual. We denote by $\sigma(L, K)$ and $\tau(L, K)$ the weak topology and the Mackey topology on $L$ with respect to a dual pair $(L, K)$. In the theory of topological function spaces the Mackey topology $\tau(E, E_n^\sim)$ on a function space $E$ is of importance (see [8, 7, 14]). It is well known that $\tau(E, E_n^\sim)$ is the finest Hausdorff locally convex-solid topology on $E$ with the Lebesgue property.
In this paper we consider the Mackey topology \( \tau(E(X), E(X)_n^\sim) \) on a vector-valued function space \( E(X) \) whenever \( E \) is an ideal of \( L^0 \) (over a \( \sigma \)-finite measure space), \( X \) is a Banach space and \( E(X)_n^\sim \) stand for the order continuous dual of \( E(X) \). In Section 2 we examine some properties of solid sets in the order continuous dual \( E(X)_n^\sim \) of \( E(X) \). We examine the properties of \( \tau(E(X), E(X)_n^\sim) \) in case it is locally solid. In Section 3 we show that \( \tau(E(X), E(X)_n^\sim) \) is the finest Hausdorff locally convex-solid topology on \( E(X) \) with the Lebesgue property (see Theorem 3.2). We obtain that the space \( (E(X), \tau(E(X), E(X)_n^\sim)) \) is complete and sequentially barreled whenever \( E \) is perfect (see Theorem 3.3 and Theorem 3.5). As an application, we obtain that \( E(X)_n^\sim \) is \( \sigma(E(X)_n^\sim, E(X)) \)-sequentially complete (see Theorem 3.6). In Section 4 we consider the Mackey topology \( \tau(L^\Phi(X), L^\Phi(X)_n^\sim) \) on Orlicz-Bochner spaces \( L^\Phi(X) \) (\( \Phi \) is not necessarily convex). It is shown that the space \( (L^\Phi(X), \tau(L^\Phi(X), L^\Phi(X)_n^\sim)) \) is complete if and only if \( L^\Phi \) is perfect (see Theorem 4.4). In particular, we obtain that \( \tau(L^\infty(X), L^\infty(X)_n^\sim) \) is a mixed topology (see Theorem 4.5).

First we establish terminology concerning function spaces (see \([2, 10, 27]\)). Let \((\Omega, \Sigma, \mu)\) be a complete \( \sigma \)-finite measure space. Let \( L^0 \) denote the space of \( \mu \)-equivalence classes of all \( \Sigma \)-measurable real valued functions defined and finite a.e. on \( \Omega \). For a subset \( M \) of \( L^0 \) by supp \( M \) we denote the support of \( M \), i.e., the smallest set in \( \Sigma \) containing (a.e.) the supports of all \( u \in M \) (see \([10, \text{Chapter 1.6}]\)). Let \( \chi_A \) stand for the characteristic function of a set \( A \), and let \( \mathbb{N} \) and \( \mathbb{R} \) denote the sets of all natural and real numbers.

Let \( E \) be an ideal of \( L^0 \) with supp \( E = \Omega \), and let \( E' \) stand for the Köthe dual of \( E \), i.e., \( E' = \{ v \in L^0 : \int_\Omega |u(\omega)v(\omega)| \, d\mu < \infty \text{ for all } u \in E \} \). Throughout the paper we assume that supp \( E' = \Omega \). Let \( E^\sim_n \), \( E^\sim_n^\sim \) and \( E^\sim_s \) stand for the order dual, the order continuous dual and the singular dual of \( E \), respectively. Then \( E^\sim_n \) separates points of \( E \) and it can be identified with \( E' \) through the mapping: \( E' \ni v \mapsto \varphi_v \in E^\sim_n \), where \( \varphi_v(u) = \int_\Omega u(\omega)v(\omega) \, d\mu \) for all \( u \in E \). \( E \) is said to be perfect whenever the natural embedding from \( E \) into \( (E^\sim_n^\sim)_n^\sim \) is onto, i.e., \( E'' = E \).

Now we collect notation along with some basic facts concerning vector-valued function spaces \( E(X) \) and locally solid topologies on \( E(X) \) as set out in \([3 - 5], [9] \) and \([19 - 21]\).

Let \((X, \| \cdot \|_X)\) be a real Banach space, and let \( S_X \) and \( B_X \) denote the unit sphere and the unit ball in \( X \). Let \( X^* \) stand for the Banach dual of \( X \). By \( L^0 \) we will denote the set of \( \mu \)-equivalence classes of strongly \( \Sigma \)-measurable functions \( f : \Omega \to X \). For \( f \in L^0(X) \) let \( \tilde{f}(\omega) = \|f(\omega)\|_X \) for \( \omega \in \Omega \). Let

\[
E(X) = \{ f \in L^0(X) : \tilde{f} \in E \}.
\]

A subset \( H \) of \( E(X) \) is said to be solid whenever \( \tilde{f}_1 \leq \tilde{f}_2 \) and \( f_1 \in E(X) \), \( f_2 \in H \) imply \( f_1 \in H \). A linear topology \( \tau \) on \( E(X) \) is said to be locally solid
if it has a local base at 0 consisting of solid sets. A linear topology on \( E(X) \) that is at the same time locally solid and locally convex will be called a locally convex-solid topology on \( E(X) \). A pseudonorm \( \varrho \) on \( E(X) \) is called solid if \( \varrho(f_1) \leq \varrho(f_2) \) whenever \( f_1, f_2 \in E(X) \) and \( \tilde{f}_1 \leq \tilde{f}_2 \). It is known that a linear topology \( \tau \) on \( E(X) \) is locally solid (resp. locally convex-solid) if and only if it is generated by some family of solid pseudonorms (resp. solid seminorms) defined on \( E(X) \) (see [9, Theorems 2.2 and 2.4]).

Recall that a locally solid topology \( \tau \) on \( E(X) \) is said to be a Lebesgue topology whenever for a net \( (f_\alpha) \) in \( E(X) \), \( \tilde{f}_\alpha \to 0 \) in \( E \) implies \( f_\alpha \to 0 \) (see [9, Definition 2.2]).

In the case when \( E \) is provided with a locally solid topology (resp. locally convex-solid topology) \( \xi \) one can topologize \( E(X) \) as follows. Let \( \{p_t : t \in T\} \) be a family of Riesz pseudonorms (resp. Riesz seminorms) on \( E \) that generates \( \xi \). By putting

\[
\overline{p}_i(f) := p_i(\tilde{f}) \quad \text{for } f \in E(X) \quad (t \in T)
\]

we obtain a family \( \{\overline{p}_t : t \in T\} \) of solid pseudonorms (resp. solid seminorms) on \( E(X) \) that defines a locally solid (resp. locally convex-solid) topology \( \tilde{\xi} \) on \( E(X) \) (called the topology associated with \( \xi \)).

Now we recall “vector valued analogues” of \( E^\sim, E_n^\sim \) and \( E_s^\sim \) as set out in [5, 20].

For a linear functional \( F \) on \( E(X) \) let us set

\[
|F|(f) = \sup \{ |F(h)| : h \in E(X), \tilde{h} \leq \tilde{f} \} \quad \text{for all } f \in E(X).
\]

Then the set

\[
E(X)^\sim = \{ F \in E(X)^\# : |F|(f) < \infty \quad \text{for all } f \in E(X) \}\]

will be called the order dual of \( E(X) \) (here \( E(X)^\# \) denotes the algebraic dual of \( E(X) \)) (see [5, §3, 18]).

It is well known that the Mackey topology \( \tau(E, E^\sim) \) is locally solid (see [1]). Moreover, one can show that the Mackey topology \( \tau(E(X), E(X)^\sim) \) is locally solid and \( \tau(E(X), E(X)^\sim) = \tau(E, E^\sim) \) (see [21, Theorem 3.3]).

Making use of the concept of \( |F| \) we can define in a natural way a positive linear functional \( \varphi_F \) on \( E \). Let \( F \in E(X)^\sim \) and \( x_0 \in S_X \) be fixed. For \( u \in E^+ \) let us set

\[
\varphi_F(u) := |F|(u \otimes x_0) = \sup \{ |F(h)| : h \in E(X), \tilde{h} \leq u \},
\]

where \( (u \otimes x_0)(\omega) := u(\omega)x_0 \) for \( \omega \in \Omega \). Clearly \( |F|(f) = \varphi_F(\tilde{f}) \) for all \( f \in E(X) \). Then \( \varphi_F : E^+ \to \mathbb{R}^+ \) is an additive mapping and \( \varphi_F \) has a unique positive extension to a linear mapping from \( E \to \mathbb{R} \) (denoted by \( \varphi_F \) again) and given by

\[
\varphi_F(u) := \varphi_F(u^+) - \varphi_F(u^-) \quad \text{for all } u \in E
\]
It is known (see [19, Corollary 2.5]) that for the rable functions $g$ (see [19]).

A subset $A$ of $E(X)^{\sim}$ is said to be solid whenever $|F_1| \leq |F_2|$ with $F_1 \in E(X)^{\sim}$ and $F_2 \in A$ imply $F_1 \in A$. A linear subspace $I$ of $E(X)^{\sim}$ will be called an ideal of $E(X)^{\sim}$ whenever $\tau$ is solid. It is known that $(E(X), \tau)^*$ is an ideal of $E(X)^{\sim}$ whenever $\tau$ is locally solid topology on $E(X)$ (see [19, Theorem 3.2]).

Every subset $A$ of $E(X)^{\sim}$ is contained in the smallest (with respect to inclusion) solid set called the solid hull of $A$ and denoted by $S(A)$. One can note that $S(A) = \{ F \in E(X)^{\sim} : |F| \leq |G|$ for some $G \in A \}$.

Recall that a functional $F \in E(X)^{\sim}$ is said to be order continuous whenever for a net $(f_\alpha)$ in $E(X)$, $\hat{f}_\alpha \overset{\sim}{\rightarrow} 0$ in $E$ implies $F(f_\alpha) \rightarrow 0$. The set $E(X)^{\sim}_{oc}$ consisting of all order continuous linear functionals on $E(X)$ is called the order continuous dual of $E(X)$. $E(X)^{\sim}_{oc}$ is an ideal of $E(X)^{\sim}$ (see [19]).

A functional $F \in E(X)^{\sim}$ is said to be singular if there is an ideal $B$ of $E$ with $\text{supp} B = \Omega$ and such that $F(f) = 0$ for all $f \in E(X)$ with $\hat{f} \in B$. The set consisting of all singular functionals on $E(X)$ will be denoted by $E(X)^{\sim}_{s}$ and called the singular dual of $E(X)$ (see [6, 18]). $E(X)^{\sim}_{s}$ is an ideal of $E(X)^{\sim}$ (see [19]).

Let $L^0(X^*, X)$ be the set of weak*-equivalence classes of all weak*-measurable functions $g : \Omega \rightarrow X^*$. One can define the so called abstract norm $\vartheta : L^0(X^*, X) \rightarrow L^0$ by $\vartheta(g) = \sup \{|g_\omega| : x \in B_X\}$, where $g_\omega(x) = g(\omega)(x)$ for $\omega \in \Omega$ and $x \in X$. One can show that $\vartheta(\lambda g) = |\lambda|\vartheta(g)$ and $\vartheta(g_1 + g_2) \leq \vartheta(g_1) + \vartheta(g_2)$ for $g, g_1, g_2 \in L^0(X^*, X)$ and $\lambda \in \mathbb{R}$. Then for $f \in L^0(X)$ and $g \in L^0(X^*, X)$ the function $(f, g) : \Omega \rightarrow \mathbb{R}$ defined by $(f, g)(\omega) := \langle f(\omega), g(\omega) \rangle$ is measurable, and $|\langle f, g \rangle| \leq \hat{f} \vartheta(g)$. Moreover, $\vartheta(g) = \hat{g}$ for $g \in L^0(X^*)$.

Let

$$E'(X^*, X) = \{ g \in L^0(X^*, X) : \vartheta(g) \in E' \}.$$ 

Due to A. V. Bukhvalov (see [4, Theorem 4.1]) $E(X)^{\sim}_{oc}$ can be identified with $E'(X^*, X)$ through the mapping $E'(X^*, X) \ni g \mapsto F_g \in E(X)^{\sim}_{oc}$, where

$$F_g(f) = \int_{\Omega} \langle f(\omega), g(\omega) \rangle \, d\mu \quad \text{for all } f \in E(X)$$

and moreover,

$$|F_g(f)| = \int_{\Omega} \hat{f}(\omega) \vartheta(g)(\omega) \, d\mu \quad \text{for all } f \in E(X).$$

It is known (see [19, Corollary 2.5]) that for $g_1, g_2 \in E'(X^*, X)$

$$|F_{g_1}| \leq |F_{g_2}| \quad \text{if and only if } \vartheta(g_1) \leq \vartheta(g_2).$$
Due to A. V. Bukhvalov and G.Y. Lozanowskii (see [5, §3, Theorem 2]) the following Yosida-Hewitt type decomposition holds

\[ E(X)^\sim = E(X)^\sim_n \oplus E(X)^\sim_s \]  

(1.3)

and moreover, if \( F = F_g + F_s \), where \( g \in E'(X^*, X) \) and \( F_s \in E(X)^\sim_s \), then \( \varphi_F = \varphi_{F_g} + \varphi_{F_s} \), where \( \varphi_{F_g}(u) = \int_{\Omega} u(\omega) \vartheta(g)(\omega) \, d\mu \) for \( u \in E \) and \( \varphi_{F_s} \in E^*_{\sim} \).

**Proposition 1.1.** Let \( E \) be an ideal of \( L^0 \). Then the following statements are equivalent:

(i) \( E(X)^\sim = E(X)^\sim_n \)

(ii) \( E(X)^\sim_s = \{0\} \)

(iii) \( E_s^\sim = \{0\} \)

(iv) \( E^\sim = E_n^\sim \).

**Proof.** (i) \( \iff \) (ii): It follows from (1.3).

(iii) \( \iff \) (iv): This is obvious, because \( E^\sim = E_n^\sim \oplus E_s^\sim \).

(ii) \( \iff \) (i): Assume that \( E(X)^\sim_s = \{0\} \) and let \( \varphi \in E^\sim_s \). Then there is an ideal \( B \) of \( E \) with \( \text{supp} \, B = \Omega \) and such that \( \varphi(u) = 0 \) for all \( u \in B \). Let \( x_o \in S_X \) and let \( x_o^* \in S_{X^*} \) be such that \( x_o^*(x_o) = 1 \). Define a linear functional \( F_\varphi \) on \( E(X) \) by setting \( F_\varphi(f) = \varphi(x_o^* \circ f) \) for \( f \in E(X) \). To show that \( F_\varphi \in E(X)^\sim \), let \( u \in E^+ \). Then for \( f \in E(X) \) with \( \tilde{f} \leq u \) we have \( |x_o^* \circ f| \leq \tilde{f} \), so

\[
\sup \{|F_\varphi(f)| : f \in E(X), \tilde{f} \leq u\} = \sup \{||\varphi(x_o^* \circ f)| : f \in E(X), \tilde{f} \leq u\} \\
\leq \sup \{|\varphi(w)| : w \in E, |w| \leq u\} < \infty.
\]

It is seen that \( F_\varphi(f) = 0 \) for \( f \in E(X) \) with \( \tilde{f} \in B \), because \( x_o^* \circ f \in B \). Hence \( F_\varphi \in E(X)^\sim = \{0\} \), so \( F_\varphi = 0 \). Then for \( u \in E \), we get \( \varphi(u) = \varphi(x_o^*(u \circ x_o)) = F_\varphi(u \circ x_o) = 0 \). Hence \( \varphi = 0 \), as desired.

(iii) \( \implies \) (ii): Assume that \( E_s^\sim = \{0\} \) and let \( F \in E(X)^\sim_s \). Then \( \varphi_F \in E_s^\sim = \{0\} \) (see 1.3), so \( F = 0 \).

2. Solid sets in the order continuous dual

In this section we shall show that the convex hull \( (\text{conv} \, A) \) of a solid subset \( A \) of \( E(X)^\sim_n \) is also solid in \( E(X)^\sim_n \). For this purpose we will need the following two lemmas.

**Lemma 2.1.** Let \( g \in L^0(X^*, X) \) and \( g_i \in L^0(X^*, X) \) for \( n = 1, 2, \ldots, n \), and assume that \( \vartheta(g) \leq \vartheta(\sum_{i=1}^n g_i) \). Then there exist \( g'_i \in L^0(X^*, X) \) for \( i = 1, 2, \ldots, n \) such that \( g = \sum_{i=1}^n g'_i \) and \( \vartheta(g'_i) \leq \vartheta(g_i) \) for \( i = 1, 2, \ldots, n \).
Proof. By using induction it is enough to establish this result for \( n = 2 \). For \( i = 1, 2 \) let us put
\[
\vartheta(g') = \begin{cases} 
\vartheta(g_1)(\omega) & \text{if } \vartheta(g_1)(\omega) + \vartheta(g_2)(\omega) > 0, \\
\vartheta(g_1)(\omega) + \vartheta(g_2)(\omega) & \text{if } \vartheta(g_1)(\omega) + \vartheta(g_2)(\omega) = 0.
\end{cases}
\]
It is seen that \( u_i \) are \( \mu \)-measurable, and let \( g'_i = u_i g \) for \( i = 1, 2 \). Then \( g'_i + g'_2 = u_i g + u_2 g = g \) and since \( \vartheta(g_i + g_2) \leq \vartheta(g_i) + \vartheta(g_2) \) for \( i = 1, 2 \) we have
\[
\vartheta(g') = \sup \{ |(u_i g)x| : x \in B_X \} \\
= \sup \{ u_i |g_x| : x \in B_X \} \\
\leq u_i \sup \{ |g_x| : x \in B_X \} = u_i \vartheta(g) \\
\leq u_i (\vartheta(g_1) + \vartheta(g_2)) = \vartheta(g).
\]
Thus the proof is complete. \( \square \)

Lemma 2.2. Let \( F \in E(X) \) and \( F_i \in E(X) \) for \( i = 1, 2, \ldots, n \), and assume that \( |F| \leq |\sum_{i=1}^n F_i| \). Then there exist \( F'_i \in E(X) \) for \( i = 1, 2, \ldots, n \) such that \( F = \sum_{i=1}^n F'_i \) and \( |F'_i| \leq |F_i| \) for \( i = 1, 2, \ldots, n \).

Proof. In view of (1.1) there exist \( g \in E(X^*, X) \) and \( g_i \in E'(X^*, X) \) for \( i = 1, 2, \ldots, n \) such that \( F = F_g \) and \( F_i = F_{g_i} \) for \( i = 1, 2, \ldots, n \). Then \( |F_g| \leq |\sum_{i=1}^n F_{g_i}| = |F\sum_{i=1}^n g_i| \), so \( \vartheta(g) \leq \vartheta(\sum_{i=1}^n g_i) \) by (1.2). Then in view of Lemma 2.1 there exist \( g'_i \in L^0(X^*, X) \) for \( i = 1, 2, \ldots, n \) such that \( g = \sum_{i=1}^n g'_i \) and \( \vartheta(g'_i) \leq \vartheta(g_i) \). Then \( g'_i \in E'(X^*, X) \) for \( i = 1, 2, \ldots, n \) and let \( F'_i = F_{g'_i} \) for \( i = 1, 2, \ldots, n \). Then \( F = F_g = F\sum_{i=1}^n g'_i = \sum_{i=1}^n F_{g'_i} = \sum_{i=1}^n F'_i \) and \( |F'_i| = |F_{g'_i}| \leq |F_{g_i}| = |F_i| \) for \( i = 1, 2, \ldots, n \).

Now we are ready to state our desired result.

Proposition 2.3. Let \( A \) be a solid subset of \( E(X) \). Then \( \text{conv} A \) is also a solid set in \( E(X) \).

Proof. Assume that \( |F_0| \leq |F| \) where \( F_0 \in E(X) \) and \( F \in \text{conv} A \). Then there exist \( F_i \in A \) and \( \alpha_i \geq 0 \) for \( i = 1, 2, \ldots, n \) with \( \sum_{i=1}^n \alpha_i = 1 \) such that \( F = \sum_{i=1}^n \alpha_i F_i \). Hence by Lemma 2.2 there exist \( F'_i \in E(X) \) for \( i = 1, 2, \ldots, n \) such that \( |F'_i| \leq |\alpha_i F_i| = |\alpha_i |F_i| \) for \( i = 1, 2, \ldots, n \) and \( F_0 = \sum_{i=1}^n F'_i \). Putting \( G_i = \alpha_i^{-1} F_i \) we get \( |G_i| \leq |F_i| \) for \( i = 1, 2, \ldots, n \), so \( G_i \in A \) for \( i = 1, 2, \ldots, n \). Hence \( F_0 = \sum_{i=1}^n \alpha_i G_i \in \text{conv} A \), and this means that \( \text{conv} A \) is solid in \( E(X) \). \( \square \)
3. Mackey topologies on vector-valued functions spaces

One can observe that $(E(X), \tau)^* \subset E(X)_n^\sim$ whenever $\tau$ is a Lebesgue topology on $E(X)$. Moreover, it is known that a locally convex-solid topology $\tau$ on $E(X)$ has the Lebesgue property whenever $(E(X), \tau)^* \subset E(X)_n^\sim$ (see [20, Theorem 2.4]). In [20, Theorem 3.4] it is shown that if an ideal $E$ is perfect and a Banach space $X$ is reflexive, then the Mackey topology $\tau(E(X), E(X)_n^\sim)$ is locally solid and it is the finest Hausdorff locally convex-solid topology on $E(X)$ with the Lebesgue property.

In this section we extend this result to the setting whenever the Mackey topology $\tau(E(X), E(X)_n^\sim)$ is locally solid. This property is characterized by the following result:

**Theorem 3.1.** Let $E$ be an ideal of $L^0$, and let $X$ be a Banach space. Then the following statements are equivalent:

(i) $\tau(E(X), E(X)_n^\sim)$ is locally solid.

(ii) Every absolutely convex $\sigma(E(X)_n^\sim, E(X))$-compact subset of $E(X)_n^\sim$ is contained in a solid absolutely convex $\sigma(E(X)_n^\sim, E(X))$-compact subset of $E(X)_n^\sim$.

**Proof.** It is enough to repeat the reasoning of the proof of [14, Lemma 2.1] and use the fact that the polar sets of subsets of $E(X)$ and $E(X)_n^\sim$ with respect to the dual pair $\langle E(X), E(X)_n^\sim \rangle$ are solid (see [19, Theorem 3.3]).

**Remark.** In Section 4 we note that for $X = l^1$ the Mackey topology $\tau(L^\infty(X), L^\infty(X)_n^\sim)$ is not locally solid.

Now we are in position to prove our main result.

**Theorem 3.2.** Let $E$ be an ideal of $L^0$ and $X$ be a Banach space. Assume that the Mackey topology $\tau(E(X), E(X)_n^\sim)$ is locally solid. Then $\tau(E(X), E(X)_n^\sim)$ is the finest locally convex-solid on $E(X)$ with the Lebesgue property and

$$\tau(E(X), E(X)_n^\sim) = \overline{\tau(E, E_n^\sim)}.$$

**Proof.** We shall show that

$$\tau(E(X), E(X)_n^\sim) = \overline{\tau(E, E_n^\sim)}.$$

Indeed, assume that $\tau(E, E_n^\sim)$ is generated by a family $\{p_t : t \in T\}$ of Riesz seminorms on $E$. In view of [9, Theorem 5.7] $\overline{\tau(E, E_n^\sim)}$ is the finest locally convex Hausdorff Lebesgue topology on $E(X)$. It follows that $\tau(E(X), E(X)_n^\sim) \subset \overline{\tau(E, E_n^\sim)}$.

To prove that $\tau(E, E_n^\sim) \subset \tau(E(X), E(X)_n^\sim)$ it is enough to show that $(E(X), \tau(E, E_n^\sim))^* = E(X)_n^\sim$. Since $\tau(E, E_n^\sim)$ is a Lebesgue topology, it is
enough to prove that \( E(X)_n^\sim \subset (E(X), \tau(E, E_n^\sim))^* \). Indeed, let \( F \in E(X)_n^\sim \), i.e., \( F(f) = F_\delta(f) = \int_\Omega (f(\omega), g(\omega)) \, d\mu \) for some \( g \in E'(X^*, X) \) and all \( f \in E(X) \). Since \( \varphi_{\partial(\delta)} \in E_n^\sim = (E, \tau(E, E_n^\sim))^* \) there exist \( c > 0 \) and \( t_i \in T \) \((i = 1, 2, \ldots, n)\) such that for \( f \in E(X) \)

\[
|F(f)| \leq \int_\Omega \tilde{f}(\omega) \vartheta(g)(\omega) \, d\mu \leq c \max_{1 \leq i \leq n} p_{t_i}(\tilde{f}) = c \max_{1 \leq i \leq n} \overline{p}_{t_i}(f_i).
\]

This means that \( F \) is \( \overline{\tau(E, E_n^\sim)} \)-continuous, as desired. \( \blacksquare \)

As a consequence of Theorem 3.2 and [20, Theorem 2.6] we get the following result.

**Theorem 3.3.** Let \( E \) be a perfect ideal of \( L^0 \), and let \( X \) be a Banach space. Assume that the Mackey topology \( \tau(E, X, E_n^\sim) \) is locally solid. Then the space \( (E(X), \tau(E(X), E(X)_n^\sim)) \) is complete.

The topological dual of \( (E(X), \tau(E(X), E(X)_n^\sim)) \) is characterized by the next theorem.

**Theorem 3.4.** Let \( E \) be an ideal of \( L^0 \), and let \( X \) be a Banach space. Assume that the Mackey topology \( \tau(E(X), E(X)_n^\sim) \) is locally solid. Then the following statements are equivalent:

(i) \( F \) is order continuous, i.e., \( F \in E(X)_n^\sim \).

(ii) \( F \) is sequentially order continuous (i.e., \( F(f_n) \to 0 \) whenever \( \tilde{f}_n \overset{(0)}{\to} 0 \) in \( E \) for a sequence \( (f_n) \) in \( E(X) \)).

(iii) \( F \) is \( \tau(E(X), E(X)_n^\sim) \)-continuous.

(iv) \( F \) is sequentially \( \tau(E(X), E(X)_n^\sim) \)-continuous.

**Proof.** (i) \( \Leftrightarrow \) (ii): This assertion follows from [19, Theorem 2.3].

(i) \( \Leftrightarrow \) (iii) and (iii) \( \Rightarrow \) (iv) are obvious.

(iv) \( \Rightarrow \) (ii): Assume that \( F \) is sequentially \( \tau(E(X), E(X)_n^\sim) \)-continuous, and let \( \tilde{f}_n \overset{(0)}{\to} 0 \) in \( E \) for a sequence \( (f_n) \) in \( E(X) \). Then \( f_n \to 0 \) for \( \tau(E(X), E(X)_n^\sim) \) because \( \tau(E(X), E(X)_n^\sim) \) is a Lebesgue topology. Hence \( F(f_n) \to 0 \), as desired. \( \blacksquare \)

Recall that a Hausdorff locally convex space \( (L, \xi) \) is said to be sequentially barreled whenever every \( \sigma(L^*_\xi, L) \)-convergent to 0 sequence in \( L^*_\xi \) is equicontinuous (see [25]).

**Theorem 3.5.** Let \( E \) be a perfect ideal of \( L^0 \), and let \( X \) be a Banach space. Assume that the Mackey topology \( \tau(E(X), E(X)_n^\sim) \) is locally solid. Then the space \( (E(X), \tau(E(X), E(X)_n^\sim)) \) is sequentially barreled.
Proof. In view of Theorem 3.4 we have

\[(E(X), \tau(E(X), E(X)_n^\sim))^* = (E(X), \tau(E(X), E(X)_n^\sim))^+ = E(X)_n^\sim\]

(here \((E(X), \tau(E(X), E(X)_n^\sim))^+\) denotes the sequential topological dual of \((E(X), \tau(E(X), E(X)_n^\sim))\). Since the space \((E(X), \tau(E(X), E(X)_n^\sim))\) is complete (see Theorem 3.3), by [25, Proposition 4.3] the space \((E(X), \tau(E(X), E(X)_n^\sim))\) is sequentially barreled.

Note that if \((E, \| \cdot \|_E)\) is a Banach function space with the norm \(\| \cdot \|_E\) satisfying the \(\sigma\)-Fatou property (i.e., \(0 \leq u_n \uparrow u \) in \(E\) implies \(\| u_n \|_E \uparrow \| u \|_E\)), then the space \((E(X), \tau(E(X), E(X)_n^\sim))\) is barreled if and only if \(\| \cdot \|_E\) is order continuous (see [21, Corollary 3.9]).

It is well known that the space \(E_n^\sim\) is \(\sigma(E_n^\sim, E)\)-sequentially complete (see [2, Theorem 20.23], [10, Corollary 10.3.1]). Now, by making use of Theorem 3.5, Theorem 3.4 and [25, Proposition 4.4] we obtain the vector-valued version of this result.

**Theorem 3.6.** Let \(E\) be a perfect ideal of \(L^p\), and let \(X\) be a Banach space. Assume that the Mackey topology \(\tau(E(X), E(X)_n^\sim)\) is locally solid. Then the space \(E(X)_n^\sim\) is \(\sigma(E(X)_n^\sim, E(X))\)-sequentially complete.

As an application of Theorem 3.6 and (1.1) we get immediately the Hahn-Vitali-Saks type theorem for sequences in \(E(X)_n^\sim\):

**Corollary 3.7.** Let \(E\) be a perfect ideal of \(L^p\), and let \(X\) be a Banach space. Assume that the Mackey topology \(\tau(E(X), E(X)_n^\sim)\) is locally solid. Let \((g_n)\) be a sequence in \(E'(X^*, X)\) such that for each \(f \in E(X)\), \(\lim_n \int_{\Omega} \langle f(\omega), g_n(\omega) \rangle \, d\mu\) exists. Then there is a \(g \in E'(X^*, X)\) such that

\[
\lim_n \int_{\Omega} \langle f(\omega), g_n(\omega) \rangle \, d\mu = \int_{\Omega} \langle f(\omega), g(\omega) \rangle \, d\mu \quad \text{for every } f \in E(X).
\]

4. Mackey topologies on Orlicz-Bochner spaces

In this section we examine the Mackey topology \(\tau(L^\Phi(X), L^\Phi(X)_n^\sim)\) on Orlicz-Bochner spaces \(L^\Phi(X)\) whenever \(\Phi\) is an Orlicz function (not necessarily convex) and \(X\) is a general Banach space. Throughout this section we will assume that the measure space \((\Omega, \Sigma, \mu)\) is atomless.

First we establish notation and basis results concerning Orlicz spaces (see [13, 24] for more details). By an *Orlicz function* we mean here a map \(\Phi : [0, \infty) \to [0, \infty)\) that is non-decreasing left continuous, continuous at 0, vanishing only at 0 and \(\lim_{t \to \infty} \inf \Phi(t)/t > 0\). Let \(\Phi^*\) stand for the convex Orlicz function complementary to \(\Phi\) in the sense of Young. Then the function
Theorem 4.2. \( \overline{\Phi}(t) = (\Phi^*)^*(t) \) for \( t \geq 0 \) is called a convex minorant of \( \Phi \), because it is the largest convex Orlicz function that is smaller than \( \Phi \) on \( [0, \infty) \).

The Orlicz space \( L^\Phi \) can be equipped with a complete topology \( \tau_\Phi \) of the Riesz \( F \)-norm \( ||u||_{\Phi} := \inf\{\lambda > 0 : \int_{\Omega} \Phi(|u(\omega)|/\lambda)\,d\mu \leq \lambda\} \). It is known that \( (L^\Phi)' = L^{\Phi^*} \) (see [11]). Clearly \( L^\Phi \) is perfect if and only if \( L^\Phi = L^{\overline{\Phi}} \) (i.e., \( \Phi \) is equivalent to some convex Orlicz function). It is seen that \( (L^{\overline{\Phi}})' = L^{\Phi^*} \) because \( \overline{\Phi} = \Phi^* \).

The Orlicz-Bochner space \( L^\Phi(X) = \{f \in L^\Phi(X) : \tilde{f} \in L^\Phi\} \) can be equipped with the complete topology \( \tau_\Phi(X) \) of the solid \( F \)-norm \( ||f||_{L^\Phi(X)} := ||\tilde{f}||_\Phi \) for \( f \in L^\Phi(X) \) (i.e., \( \tau_\Phi(X) = \tau_{\Phi^*} \).

For \( \varepsilon > 0 \) let \( V_\Phi(\varepsilon) = \{f \in L^\Phi(X) : \int_{\Omega} \Phi(\tilde{f}(\omega))\,d\mu \leq \varepsilon\} \). Then the family of all sets of the form:

\[
\bigcup_{n=1}^{\infty} \left( \sum_{i=1}^{n} V_\Phi(\varepsilon_i) \right),
\]

where \((\varepsilon_n)\) is a sequence of positive numbers, forms a local base at 0 (consisting of solid subsets of \( L^\Phi(X) \)) for a linear topology \( \tau_\Phi(X) \) on \( L^\Phi(X) \), called the modular topology (see [9]).

In particular, for \( X = \mathbb{R} \) we will write \( \tau_\Phi^0 \) instead of \( \tau_\Phi^0(\mathbb{R}) \). The basic properties of the modular topology \( \tau_\Phi^0 \) are included in the following theorem (see [15, Theorem 1.1], [17, Theorems 2.5 and 3.2], [18, Theorem 2.2]):

**Theorem 4.1.** Let \( \Phi \) be an Orlicz function. Then:

(i) \( \tau_\Phi^0 = \tau_\Phi \) holds if and only if \( \Phi \) satisfies the \( \Delta_2 \)-condition.

(ii) \( \tau_\Phi^0 \) is the finest Lebesgue topology on \( L^\Phi \).

(iii) The Mackey topology \( \tau(L^\Phi, L^{\Phi^*}) \) is the finest of all locally convex topologies on \( L^\Phi \) that are weaker than \( \tau_\Phi^0 \). Moreover, \( \tau(L^\Phi, L^{\Phi^*}) = \tau_\Phi^0 \) whenever \( \Phi \) is convex.

(iv) \( \tau(L^\Phi, L^{\Phi^*}) \) coincides with the restriction of the Mackey topology \( \tau(L^{\overline{\Phi}}, L^{\Phi^*}) \) on \( L^\Phi \), i.e., \( \tau(L^\Phi, L^{\Phi^*}) = \tau(L^{\overline{\Phi}}, L^{\Phi^*})|_{L^\Phi} \).

(v) The completion of \( (L^\Phi, \tau(L^\Phi, L^{\Phi^*})) \) equals \( (L^{\overline{\Phi}}, \tau(L^{\overline{\Phi}}, L^{\Phi^*})) \).

Now we pass on to Orlicz-Bochner spaces. Then \( L^\Phi(X)_n = \{F_\Phi : g \in L^{\Phi^*}(X^*, X)\} \) and we can write \( \tau(L^\Phi(X), L^{\Phi^*}(X^*, X)) \) instead of \( \tau(L^\Phi(X), L^\Phi(X)_n) \).

**Theorem 4.2.** Let \( \Phi \) be an Orlicz function and \( X \) be a Banach space. Assume that the Mackey topology \( \tau(L^\Phi(X), L^{\Phi^*}(X^*, X)) \) is locally solid. Then:

(i) \( \tau_\Phi^0(X) \) is the finest Lebesgue topology on \( L^\Phi(X) \).

(ii) \( \tau_\Phi^0(X) = \overline{\tau_\Phi} \).

(iii) \( (L^\Phi(X), \tau_\Phi^0(X))^* = L^\Phi(X)_n \),
Proof. (i): The assertion follows from [9, Theorem 6.3].

(ii): Since $\tau_0^\Phi$ is the finest Lebesgue topology on $L^\Phi$ (see Theorem 4.1(ii)), by making use of [9, Theorem 5.7] $\tau_0^\Phi$ is the finest Lebesgue topology on $L^\Phi(X)$. Hence, in view of (i) $\tau_0^\Phi(X) = \tau_0^\Phi$ as desired.

(iii): In view of (i) we have that $(L^\Phi(X), \tau_0^\Phi(X))^* \subset L^\Phi(X)^\sim$. On the other hand, by making use of Theorem 3.2 and Theorem 4.1(iii) we get

$\tau(L^\Phi(X), L^\Phi(X)^* \times X)) = \tau(L^\Phi, L^\Phi^*) \subset \tau_0^\Phi = \tau_0^\Phi(X)$.

It follows that $L^\Phi(X)^\sim \subset (L^\Phi(X), \tau_0^\Phi(X))^*$, and the proof is complete.

Now we are ready to characterize the Mackey topology $\tau(L^\Phi(X), L^\Phi(X)^\sim)$.

**Theorem 4.3.** Let $\Phi$ be an Orlicz function and $X$ be a Banach space. Assume that the Mackey topology $\tau(L^\Phi(X), L^\Phi(X)^* \times X)$ is locally solid. Then:

(i) $\tau(L^\Phi(X), L^\Phi(X)^* \times X))$ is the finest of all locally convex topologies on $L^\Phi(X)$ that are weaker than $\tau_0^\Phi(X)$. In particular, $\tau(L^\Phi(X), L^\Phi(X)^* \times X)) = \tau_0^\Phi(X)$ whenever $\Phi$ is convex.

(ii) $\tau(L^\Phi(X), L^\Phi(X)^* \times X)) = \tau(L^\Phi(X), L^\Phi(X)^* \times X)|_{L^\Phi(X)} = \tau_0^\Phi(X)|_{L^\Phi(X)}$.

**Proof.** (i): We know that $\tau(L^\Phi(X), L^\Phi(X)^\sim) \subset \tau_0^\Phi(X)$ (see the proof of (iii) of Theorem 4.2). Now, let $\eta$ be a locally convex topology on $L^\Phi(X)$ that is weaker than $\tau_0^\Phi(X)$. Then $(L^\Phi(X), \eta)^* \subset (L^\Phi(X), \tau_0^\Phi(X))^*$ $L^\Phi(X)^\sim$ (see Theorem 4.2 (iii)). Hence $\sigma(L^\Phi(X), (L^\Phi(X), \eta)^*) \subset \sigma(L^\Phi(X), L^\Phi(X)^\sim)$ and it follows that $\eta \subset \tau(L^\Phi(X), L^\Phi(X)^\sim)$ (see [23, Proposition 3.7.14]).

Moreover, if $\Phi$ is convex, then by Theorem 4.1 (iii) we get

$\tau(L^\Phi(X), L^\Phi(X)^* \times X)) = \tau(L^\Phi, L^\Phi^*) = \tau_0^\Phi = \tau_0^\Phi(X)$.

(ii): By making use of Theorem 3.2 and Theorem 4.1 (iv) we get:

$\tau(L^\Phi(X), L^\Phi(X)^* \times X)) = \tau(L^\Phi, L^\Phi^*)$ $= \tau(L^\Phi^*, L^\Phi^*)|_{L^\Phi}$ $= \tau(L^\Phi^*, L^\Phi^*)|_{L^\Phi(X)}$ $= \tau(L^\Phi(X), L^\Phi(X)^* \times X)|_{L^\Phi(X)}$ $= \tau_0^\Phi(X)|_{L^\Phi(X)}$.

**Theorem 4.4.** Let $\Phi$ be an Orlicz function and $X$ be a Banach space. Assume that the Mackey topology $\tau(L^\Phi(X), L^\Phi(X)^* \times X)$ is locally solid. Then:

(i) The completion of $(L^\Phi(X), \tau(L^\Phi(X), L^\Phi(X)^* \times X))$ equals $(L^\Phi(X), \tau(L^\Phi(X), L^\Phi(X)^* \times X))$. 


(ii) The space $(L^\Phi(X), \tau(L^\Phi(X), L^{\phi^*}(X^*, X)))$ is complete if and only if $L^\Phi$ is perfect.

Proof. (i): We know that the space $(L^\Phi(X), \tau(L^\Phi(X), L^{\phi^*}(X^*, X)))$ is complete, because $L^\Phi$ is perfect (see Theorem 3.3). In view of Theorem 4.3 it is enough to show that $L^\Phi(X)$ is dense in $(L^\Phi(X), \tau^*_\Phi(X))$. Indeed, let $f \in L^\Phi(X)$. Then there exists a sequence $(\Omega_n)$ such that $\Omega_n \uparrow \Omega$, $\mu(\Omega_n) < \infty$ and $\chi_{\Omega_n} \in L^\Phi$ for $n \in \mathbb{N}$ (see [27, Theorem 86.2]). For $n \in \mathbb{N}$ let us put

$$f_n(\omega) = \begin{cases} f(\omega) & \text{if } \hat{f}(\omega) \leq n \text{ and } \omega \in \Omega_n \\ 0 & \text{elsewhere.} \end{cases}$$

Then $f_n \in L^\Phi(X)$ for $n \in \mathbb{N}$ and $\hat{f}(\omega) \uparrow \hat{f}(\omega)$ for $\omega \in \Omega$. Moreover, we have

$$\hat{f} - f_n(\omega) = \hat{f}(\omega) - \hat{f}_n(\omega) = \begin{cases} 0 & \text{if } \hat{f}(\omega) \leq n \text{ and } \omega \in \Omega_n \\ \hat{f}(\omega) & \text{elsewhere.} \end{cases}$$

Hence $\hat{f} - f_n \downarrow 0$ in $E$, and since $\tau^*_\Phi(X)$ is a Lebesgue topology on $L^\Phi(X)$ we get $f_n \rightharpoonup f$ for $\tau^*_\Phi(X)$, as desired.

(ii): Assume that the space $(L^\Phi(X), \tau(L^\Phi(X), L^{\phi^*}(X^*, X)))$ is complete. Then by (i) $L^\Phi(X) = L^\Phi(X)$, and this means that $L^\Phi$ is perfect. Hence in view of Theorem 3.3 the proof is complete.

Now we consider the Mackey topology $\tau(L^\infty(X), L^\infty(X))$. The Riesz $F$-norm

$$\|u\|_o = \int_{\Omega} \frac{|u(\omega)|}{1 + |u(\omega)|} w(\omega) \, d\mu$$

for $u \in L^\nu$, where $w : \Omega \to (0, \infty)$ is a $\Sigma$-measurable function with $\int_{\Omega} w(\omega) \, d\mu = 1$, determines the Lebesgue topology $\tau_0$ on $L^\nu$ of the convergence in measure on subsets of finite measure. Recall the mixed topology $\gamma[\tau_\infty, \tau_0, L^\nu]$ (briefly $\gamma$) is the finest Hausdorff locally convex-solid topology with the Lebesgue property on $L^\infty$, i.e., $\gamma$ coincides with the Mackey topology $\tau(L^\infty, L^1)$ (see [16]).

Now we consider the mixed topology $\gamma[\tau_\infty(X), \tau_0(X)|_{L^\infty(X)}]$ (briefly $\gamma_X$) on $L^\infty(X)$ (here $\tau_\infty(X)$ stands for the topology of the norm $\|f\|_{L^\infty(X)} := \|\hat{f}\|_{\infty} = \text{ess sup}_{\omega \in \Omega} \hat{f}(\omega)$ and $\tau_0(X)$ denotes the topology of the $F$-norm $\|f\|_{L^0(X)} := \|\hat{f}\|_o$ on $L^0(X)$). For a sequence $(\varepsilon_n)$ of positive numbers and $r > 0$ let

$$W(\varepsilon_n, r) = \bigcup_{n=1}^{\infty} \left( \sum_{i=1}^{n} (V_X(\varepsilon_i) \cap iB_X(r)) \right)$$
where $B_X(r) = \{ f \in L^\infty(X) : \|f\|_{L^\infty(X)} \leq r \}$ and $V_X(\varepsilon) = \{ f \in L^\sigma(X) : \|f\|_{L^\sigma(X)} \leq \varepsilon \}$. Then the family of all such $W(\varepsilon_n, r)$ forms a local base at 0 for $\gamma_X$ (see [20, 26] for more details). One can show that $\gamma_X = \overline{\gamma}$ (see [20, Theorem 4.2]).

Hence, by Theorem 3.2 we get:

**Theorem 4.5.** Assume that the Mackey topology $\tau(L^\infty(X), L^1(X^*, X))$ is locally solid. Then $\tau(L^\infty(X), L^1(X^*, X))$ coincides with the mixed topology $\gamma_X$.

**Remark.** The Mackey topology $\tau(L^\infty(X), L^1(X^*, X))$ and the mixed topology $\gamma_X$ on $L^\infty(X)$ are closely related to the theory of operator valued measures $m : \Sigma \to B(X, Y)$, where $Y$ is a Banach space and $B(X, Y)$ stands for the space of all bounded linear operators from $X$ to $Y$. One can show (see [22]) that if $\tau(L^\infty(X), L^1(X^*, X))$ is locally solid (i.e., $\tau(L^\infty(X), L^1(X^*, X)) = \gamma_X$) then for every Banach space $Y$ an operator valued measure $m : \Sigma \to B(X, Y)$ is countably additive in the uniform operator topology if and only if $m$ is variationally semiregular (see [11] for more details).

On the other hand, I. Dobrakov [6, Example 7] defined a measure $m : 2^N \to B(l^1, c_0)$ which is countably additive in the uniform operator topology but it is not variationally semiregular. It follows that for $X = l^1$ the Mackey topology $\tau(L^\infty(X), L^1(X^*, X))$ is not locally solid.

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**References**


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