Bifurcation and Stability of Capillary-Gravity Waves

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1. Introduction

1.1. Introductory remarks. Starting with Zeidler [7], where a large class of wave problems in two dimensions is studied by one method (conformal mapping on the unit circle), we will give an example treating uniformly three-dimensional problems. Consider the three-dimensional irrotational stationary motion of an inviscid incompressible fluid of constant density \( \varrho \) and of finite depth \( B \) in the presence of gravity \((0, 0, -g)\) with surface tension \( \beta \) acting at the free surface \( \Gamma \). A trivial configuration of \( \Gamma \) for constant velocity \((U, 0, 0)\) is a horizontal plane, which we take as \( Z = 0 \).

We are seeking for periodic and small amplitude functions \( Z = H(X, Y) \) as solutions for \( \Gamma \). To do this, we formulate the variational problem for the boundary value problem with given surface \( \Gamma \). By normalization the solution \( \Phi \) is uniquely determined.

Inserting \( \Phi \) in the energy functional \( E: H \rightarrow E(H) \) of the system with free surface, we get equilibrium states by minimization of \( E \) in dependence on \( H \). We will see that stationary points of \( E \) satisfy the Bernoulli equation of the boundary value problem with free \( \Gamma \). With vanishing second variation we get the set of critical Froude numbers \( F_\epsilon \) and critical Bond numbers \( b_\epsilon \). In two dimensions, this set was studied by Kirchgaessner [1]. For uncritical Froude numbers \( F \) and Bond numbers \( b \), the problem in three dimensions but in the case of periodic bottom was studied by Shinbrot [5].

We are studying the smallest critical Froude numbers. With the results of Beyer [2] we show: in the corresponding Sobolev spaces the minimum of \( E \) and its first and second variation are analytic in a small neighbourhood of critical points. We get the bifurcation equation via Ljapunov-Schmidt procedure. The symmetries underlying the physical problem give a nice structure of the equation. So we can solve it only with the Implicit Function Theorem. The solution is a two-dimensional wave \( H(X) = a(\epsilon, \mu) \cos (mX/l + \delta) \) with \( a \) depending on small parameters describing a neighbourhood of \( b_\epsilon \), \( b = b_\epsilon(1 - \mu) \), and \( F_\epsilon, F = F_\epsilon(1 + \epsilon) \); \( m \) is determined by \( b_\epsilon \), \( l \) follows from the starting periodicity \( 2\pi l \) of \( H(X, Y) \), and \( \delta \) is free as the consequence...
of symmetry properties. But our variational approach also gives local stability results. We find that the second variation is positive if \(|\epsilon|\) and \(|\mu|\) are small enough and if the wave-length \(l/m\) is less than the product of the mean depth \(B\) and a constant \(c \approx 4/5\).

1.2. The boundary value problem. We have to solve the boundary value problem

\[
\Delta \Phi = 0 \quad \text{for } -B < Z < H(X, Y),
\]

\(\Phi_X H_X + \Phi_Y H_Y = 0\) on \(\Gamma\), and \(\Phi_Z = 0\) for \(Z = -B\).

Moreover, on \(\Gamma\) we have to fulfil the Bernoulli equation \((\nabla = (\partial/\partial X, \partial/\partial Y, \partial/\partial Z)\) and \(\nabla' = (\partial/\partial X, \partial/\partial Y)\))

\[
|\nabla \Phi|^2/2g + H - \text{div} (\nabla H/1 + |\nabla H|^2) \beta/\rho g \text{ const.}
\]

Let \(Z\) denote the integers and \(\mathbb{R}^+\) the positive real numbers. For fixed \(l, k \in \mathbb{R}^+\) we define a lattice \(\Lambda' = \{\omega' = k_1 \omega_1 + k_2 \omega_2 : k_1, k_2 \in \mathbb{Z}, \omega_1 = 2\pi l(1, 0), \text{ and } \omega_2 = 2\pi k(0, 1)\}.\) Let \(R'\) be the rectangle which is formed by \(\omega_1\) and \(\omega_2\) and let \(|R'|\) be its area. The dual lattice \(\Lambda^*\) of \(\Lambda'\) is given by \(\Lambda = \{\omega : \omega/2\pi e \in \mathbb{Z} \text{ for all } \omega' \in \Lambda'\}\). A basis in \(\Lambda\) is

\[
\omega_1 = (0, 1)/k \quad \text{and} \quad \omega_2 = (1, 0)/l.
\]

\(H_m\) denotes the Sobolev space of \(\Lambda\)-periodic functions \(H(X, Y) = \sum_{\omega \in \Lambda} H_\omega e^{i\omega x}\) where \(\omega x = \omega_1 X + \omega_2 Y\) for \(\omega = (\omega_1, \omega_2)\), with finite norm \(\|H\|_{m}^2 = \|H_0\|^2 + \sum_{\omega \in \Lambda} |\omega|^2 \|H_\omega\|^2\). For the sake of incompressibility we assume

\[
\int_{R'} H(X, Y) \, dX \, dY = 0.
\]

Let \(\Phi(X, Y, Z) = U[\Phi(X, Y, Z) + X]\) and assume \(\Phi\) to be \(\Lambda\)-periodic, which means that the \(X\)-axis has common direction with the mean flow \(U = \int \nabla \Phi|_{Z = -H(X, Y)} dX dY/|R'|\), hence, \(U = |U|\). So we consider our problem only in \(\Omega = R' \times \{-B < Z < H(X, Y)\}\). By the transformation \(X = Bx_1, Y = Bx_2, Z = B(x_3 + v(x_1, x_2, x_3))\), where

\[
v(x_1, x_2, x_3) = \sum h_\omega e^{i\omega x} \sinh (|\omega x_2 + |\omega| x_3)/\sinh |\omega|,
\]

\(H(X, Y) = B h(x_1, x_2), \quad \Phi(X, Y, Z) = B v(x_1, x_2, x_3),\)

we map the fluid region \(\Omega\) on a region \(S = R \times \{-1 < x_3 < 0\}\) (\(R\) denotes the rectangle formed by \(\omega_1 B\) and \(\omega_2 B\), with fixed boundary and handle with dimensionless quantities. We remark that \(v\) belongs to some function class (later see the proof of Lemma 2), whereas a simpler transformation \(v(x_1, x_2) = \sum h_\omega e^{i\omega x}\), for instance, would not have this property. The transformed dual lattice we call \(\Lambda = \{\omega = (m/l, \text{ for } m/n/k/B: m, n \in \mathbb{Z}\}\). Then \(h\) belongs to \(H_m\), the corresponding Sobolev space over the transformed lattice \(\Lambda\). We force the uniqueness of the solution of (1) by normalizing \(\int v(x_1, x_2, 0) \, dx_1 \, dx_2 = 0\) (\text{STINTHOBROT [5]}). Finally we define the Bond number

\[
b = \beta/\rho g B^2\]

and the Froude number \(F = U^2/gB\).
2. Energy functional, properties and variations

2.1. The equivalent variational problem. The potential energy of our physical system is $\mathcal{E}$ with

$$\mathcal{E} = bB^2 \int \sqrt{1 + |\nabla H|^2} \, dx \, dy + \int \frac{H^2}{2} \, dx \, dy + \frac{FB}{2U^2} \int |\nabla \phi|^2 \, dz \, dx \, dy,$$

where $\Phi(X, Y, Z) = U\{\Phi(X, Y, Z) + X\}$ is assumed to be $\Lambda$-periodic and has to solve the boundary value problem (1) for given $H \in \Omega$. Then $\mathcal{E}$ defines a functional over $\Omega$. For $t \in \mathbb{R}$ we consider a family of surfaces $F_t : Z = H(X, Y) + t\zeta(X, Y)$ with $\zeta$ is $\Lambda$-periodic and satisfies (4). The potentials we call $\Phi$ again and write $\partial \Phi(X, Y, Z, t)/\partial t = \Phi(X, Y, Z, t)$. As first variation of $\mathcal{E}$ we then have (denoting the Gateaux derivative)

$$\langle \mathcal{E}, \zeta \rangle = \frac{d \mathcal{E}(H + t\zeta)}{dt} |_{t=0} = bB^2 \int \nabla H \nabla \zeta \sqrt{1 + |\nabla H|^2} \, dx \, dy$$

$$+ \int \nabla \zeta \, dx \, dy + \frac{FB}{2U^2} \int \zeta |\nabla \phi|^2 \, dx \, dy$$

$$+ \frac{FB}{U^2} \int \nabla \phi \nabla \phi \, dz \, dx \, dy.$$

Because

$$0 = \int \nabla \phi \nabla \zeta \, dx \, dy + \int \nabla \phi \zeta \, dx \, dy = \int (\phi_{xx} \phi + \phi_x \phi_x) \, dy$$

partial integration gives

$$\langle \mathcal{E}, \zeta \rangle = \int \zeta \left(-bB^2 \text{div} \left(\sqrt{1 + |\nabla H|^2}\right) + H + \left(\frac{FB}{2U^2} |\nabla \phi|^2\right)\right) \, dx \, dy$$

$$+ \left(\frac{FB}{U^2}\right) \left(\int \phi \Delta \phi \, dx \, dy + \int \phi \zeta \, dx \, dy - \int \phi \phi \, dx \, dy\right)$$

where $\partial/\partial n$ denotes the derivative in normal direction. Since $\Phi$ solves (1), the integrals containing $\Phi$ are vanishing. So by suitable choice for $\zeta$ the equation $\langle \mathcal{E}, \zeta \rangle = 0$ implies (2) (see also Whitham [6; p. 435ff.]).

Finally we note $\mathcal{E}$, the transformed energy functional divided by $\rho g B^2$. In the following we write $\partial/\partial x_i = f_i$ for $i = 1, 2, 3$. Since there can be no confusion, we call $\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3 = \nabla$ and $\partial/\partial x_1, \partial/\partial x_2 = \nabla'$. Again the relation (5) transforms the problem $\mathcal{E} \to \min$ into the problem

$$\mathcal{E} = b \int \sqrt{1 + |\nabla H|^2} \, dx_1 \, dx_2 + \frac{1}{2} \int h^2 \, dx_1 \, dx_2 + FJ \to \min,$$

where

$$2J = \int \left(|\nabla \phi|^2 \, (1 + v_{,3}) \right) \, dV$$

$$- \int \phi \, (x_1, x_2, 0) \, h_1 \, dx_1 \, dx_2.$$

with $dV = dx_1 \, dx_2 \, dx_3$. Taking (1) into account with $\zeta(X, Y) = B \zeta(x_1, x_2)$, the transformed first variation is

$$\langle \mathcal{E}, \zeta \rangle = \int \zeta \left(-b \text{div} \left(\sqrt{1 + |\nabla H|^2}\right) + \frac{h}{b} \right) \, dx_1 \, dx_2 + F \langle J, \zeta \rangle.$$
2.2. Critical points. We compute the second variation of $E$ at $h = 0$. From the stability theory it is clear that bifurcations may occur at points where the second variation loses its positivity. We have

$$E_{hh}(\zeta, \zeta)|_{h=0} = \iint \left( \frac{b}{2} |\nabla \zeta|^2 + \zeta^2 \right) dx_1 dx_2 - F J_{hh}(\zeta, \zeta)|_{h=0}$$

with

$$J_{hh}(\zeta, \zeta)|_{h=0} = \iint |\nabla \phi|^2 dV - 2 \iint \phi(x_1, x_2, 0) \zeta_1 dx_1 dx_2,$$ (8)

where $\phi$ is determined by the Euler equation

$$-\int \nabla \phi_1 dV + \int \nabla(\phi_{1,3} - \zeta_1) dx_1 dx_2,$$

whose solution is $\phi_1 = \sum \phi_{1w} e^{iwx}$, $\phi_{1w} = \omega_1 \zeta_1 \cosh (|\omega| x_3 + |\omega|)/(|\omega| \sinh |\omega|)$, and from now on $\omega x = \omega_1 x_1 + \omega_2 x_2$. This we put into (8), obtaining

$$E_{hh}(\zeta, \zeta)|_{h=0} = \sum_{\omega \in A} \gamma(b, F, \omega) |\zeta_\omega^2|

with $\gamma(b, F, \omega) = 1 + b |\omega|^2 - F \omega^2 (\coth |\omega|)/|\omega|$. Denote $\bar{F} = |\omega| (1 + b |\omega|^2) \times (\tanh |\omega|)/\omega_1^2$, then $E_{hh}(\zeta, \zeta)|_{h=0}$ remains positively definite, until $F < \bar{F} = \min \{\bar{F}(\omega) : \omega \in A \setminus \{0\} \}$. For these $F$ the trivial solution of (6), $h = 0$, is stable with respect to the $A'$-periodic perturbations. Since the linear map $L = E_{hh} F_{\bar{F}}$ may possess nontrivial kernel $N(L)$, we call the $\bar{F}$ critical. Here we are only studying $\bar{F}$ described by

**Lemma 1:** The following assertions are true:

(i) If $b \geq 1/3$, $N(L)$ is trivial and $F_\epsilon = 1$.

(ii) If $b < 1/3$, $\bar{F}$ attains its minimum only at points $(r, 0)$, $r = m/(B1)$, $m \in \mathbb{Z}$.

Every $r \neq 0$ uniquely determines the critical points.

$$b_\epsilon = \frac{-\sinh (2r) - 2r}{r^2 (\sinh (2r) + 2r)}, \quad F_\epsilon = \frac{4 \sinh^2 r}{r (\sinh (2r) + 2r)}$$

and $h_1 = e^{ix}$ and $h_2 = e^{-ix}$ are a basis in $N(L)$.

(iii) If $|\omega| = r$, then $\gamma(|\omega|, b_\epsilon(r), F_\epsilon(r)) > 0$.

Proof: $\bar{F}$ attains its minimum for $\omega_1^2 = |\omega|^2$. So we are studying the case $\omega = (s, 0)$, $s \in \mathbb{R}$. Consider

$$\bar{F}_d = \partial^2 \bar{F}/\partial s = (bs^2 - 1) \sinh (2s) + (bs^2 + 1) 2s)(2s^2 \cosh^2 s)$$

$\bar{F}(0)$ is a minimum of $\bar{F}$ if $\bar{F}_d(0) = 2(b - 1/3) > 0$. So $F_\epsilon = \bar{F}(0, b) = 1$ for all $b \geq 1/3$. $N(L)$ has the basic elements $e^{iwx}$ with $\omega \in A$ and $\gamma(b_\epsilon, F_\epsilon, \omega) = 0$.

a) $b \geq 1/3$. We have $\gamma(b, 1, 0) = 0$. Taking the power series expansion of the hyperbolic functions, we get $\gamma(b, 1, (\omega_1, \omega_2)) \sinh r \geq \gamma(b, 1, (\omega_1, 0)) \sinh r \geq \sum |\omega_1|^2 \times ((4n^2 + 10n + 6) b - 2n - 1)/(2n + 3)! \geq \Sigma |\omega_1|^2 (4n^2 + 4n + 3)/(3(2n + 3)!)$.

So $\omega_1 = 0$ is the only solution of $\gamma(b; 1, (\omega_1, \omega_2)) = 0$. Remarking that the incompressibility condition (4) holds we have (i).

b) $b < 1/3$. Vanishing $\bar{F}$, for $s \neq 0$ determinate $b = b_\epsilon$, whence $\bar{F} = F_\epsilon$ follows, and power series expansion of the second derivative gives $\bar{F}_\epsilon(s) > 0$. In correspondence with (i) the limit case $s \to 0$ gives $b_\epsilon = 1/3$ and $F_\epsilon = 1$. In order to get $N(L)$ we set $|\omega| = w$. Hence, $f(w) = \gamma(b, F_\epsilon, w) \geq (const)^2 (w^2 (\sinh (2s) - 2s) + s^2 (\sinh (2s) + 2s) + 2ws(1 - \cosh (2s)) \cosh w)$. But $\gamma(b_\epsilon, F_\epsilon, s) = 0$ and $f_\epsilon = (g(s) - g(w))/w$ with $g(w) = (const)^2 (\sinh (2w) - 2w)/(w \sinh^2 w)$. Since $g$ is a strictly monotone
decreasing function, $s = w$ is the only solution of $\gamma(b_c, F_c, \omega) = 0$. Further, $F_c(s)$ is a strictly monotone decreasing function. So different $s$ really give different $F_c$. This completes (ii). Since $g$ is decreasing, $f$ is increasing (decreasing) for $w > s$ ($w < s$). Therefore, $\gamma(b_c, F_c(s), \omega) > 0$ if $w \neq s$. The restriction $\omega = (s, 0) \in A$ gives $s = m/(Bl)$ with $m \in \mathbb{Z}$.

Remark: Consider $r$ in (9) as a continuous parameter. Then (9) describes a curve $C$ parametrized with respect to $r$ (Figure 1). Below $C$ the trivial solution $h' = 0$ of (1), (2) is stable, no bifurcations occur. At points $F_c = 1$, $b > 1/3$ by our additional incompressibility condition (4) the bifurcating waves $h = \text{const}$ are eliminated. So the bifurcation points $F_c < 1$ and $b_c < 1/3$ have to be studied.

2.3. Properties of the energy functional. Studying the energy functional and its first and second variation as maps in Sobolev spaces, we can use some results of Beyers [1]. We need some definitions given also in [1: § 3]. Let $H_m$ be the subspace of $\Pi_m$ whose functions satisfy (4). For any open interval $I$ on the $x_3$-axis we denote by $\|\varphi\|_I$ the norm of $\varphi$ in $L_2(I)$.

$$W_{m,1} = \{\varphi \in L_2(\mathbb{I}, H_m) : \varphi^{(m)} = \partial^m \varphi/\partial x_3^m \in L_2(\mathbb{I}, H_0)\}$$

be the Sobolev space of $A'$-periodic functions $\varphi(x_1, x_2, x_3) = \sum \varphi_\omega(x_3) e^{i\omega x}$ with distributional derivatives up to order $m$ in $L_2(\mathbb{R} \times I)$. The derivatives up to order $m - 1$ should be $A'$-periodic, too. Let $\|\varphi\|_{m,1}^2 = \|\varphi_0\|^2 + \sum_{\omega \in A} (|\omega|^m \|\varphi_\omega\|_I)^2 + \|\varphi^{(m)}\|_I^2$ be the norm in $W_{m,1}$. If $m \geq 1$, we further define

$$V_{m,1} = \{\varphi \in D'(\mathbb{I}, H_m) : \varphi', \varphi^{(m)} \in L_2(\mathbb{I}, H_0)\}$$

with the norm $\|\varphi\|_{m,1}^2 = \|\varphi_0\|^2 + \sum_{\omega \in A} (|\omega|^m \|\varphi_\omega\|_I^2 + \|\varphi^{(m)}\|_I^2)$. Henceforth, we set $b = b_c(1 - \mu)$ and $F = F_c(1 + \varepsilon)$. So we set $E = E(h, \varepsilon, \mu)$.

Lemma 2: Assume $\varepsilon \geq 5/2$. Then

(i) $E(h, \mu, \varepsilon)$ maps a neighbourhood of $(0, 0, 0) \in H_s \times \mathbb{R}^2$ analytically into $\mathbb{R}$,

(ii) $E_h(h, \mu, \varepsilon)$ considered as a map from $\hat{H}_s \times \mathbb{R}^2$ into $\hat{H}_s - 2$ is analytic at $(0, 0, 0)$,

(iii) $E_h(h, \mu, \varepsilon)$ - originally considered as a map on $\hat{H}_s \times \hat{H}_s$ - is continuous on $H_s \times H_s$, and its continuous extension on $\hat{H}_1 \times \hat{H}_s$ as a map from $\hat{H}_s \times \mathbb{R}^2$ into $\mathcal{L}(\hat{H}_s \times \hat{H}_s, \mathbb{R})$ is analytic at $(0, 0, 0)$.

Proof: Setting

$$f' = -\nabla \varphi \varphi_3 + \varphi_3 \nabla \varphi + (\nabla \varphi \nabla \varphi - \|
abla \varphi\|^2 \varphi_3/(1 + \varphi_3))(0, 0, 1)$$
and transforming the variational equation \( \int \nabla \Phi \nabla \psi \, dZ \, dX \, dY = 0 \) we get

\[
\int_{\mathcal{S}} \nabla \psi f' \, dV = \int_{\mathcal{S}} \nabla \Phi \nabla \psi \, dV - \int_{\mathcal{S}} \psi(0) \, h_{i} \, dx_{1} \, dx_{2} \quad \text{for all } \psi \in V_{1}.
\]

If we take \( \psi = \varphi_{1} + \bar{\varphi} \) with \( \Delta \varphi_{1} = 0 \) in \( S \), \( \varphi_{1,3} = h_{1} \) at \( x_{3} = 0 \), and \( \varphi_{1,3} = 0 \) at \( x_{3} = -1 \), then \( \varphi_{1} \) is given by \( \varphi_{1} \) in Subsection 2.2 and

\[
\int_{\mathcal{S}} \nabla \psi f' \, dV = \int_{\mathcal{S}} \nabla \Phi \nabla \psi \, dV \quad \text{for all } \psi \in V_{1}, \quad \text{with } f' = f'(\varphi_{1}) + f'(\bar{\varphi}).
\]

For the Fourier series \( f = \sum f_{n}(x_{3}) e^{i\omega_{n} x} \) this implies

\[
\int (\varphi_{n,3} \varphi_{n,3} + |\omega|^{2} \varphi_{n} \bar{\varphi}) \, dx_{3} = \int (f_{n} \varphi_{n,3} - i |\omega| f_{n} \bar{\varphi}) \, dx_{3}.
\]

Choosing \( \varphi_{n} = \varphi_{n} \) and applying Schwarz's inequality, we obtain

\[
|\varphi_{n,3}|^{2} + |\omega|^{2} |\varphi_{n}|^{2} \leq |f_{n}||\varphi_{n,3}| + |\omega||\varphi_{n}|,
\]

hence \( |\varphi_{n,3}|^{2} + |\omega|^{2} |\varphi_{n}|^{2} \leq 2 \|f_{n}\|^{2} \). So, for \( m = 0 \) we have shown that the unique solution \( \varphi \in V_{1} \) of (10) belongs to \( V_{m+1} \) and satisfies \( |\varphi|_{m+1} \leq c \|f\|_{m} \), with \( c \) independent of \( f \). Differentiating the Euler equation of (11) \( m \geq 1 \)-times, we have \( -\varphi_{n}^{(m+1)} + |\omega|^{2} \varphi_{n}^{(m-1)} = -i |\omega| f_{n}^{(m-1)} - f_{n}^{(m)} \) for \( m > 0 \), whence

\[
|\varphi_{n}^{(m+1)}| \leq \|f_{n}^{(m-1)}\| + \|f_{n}^{(m)}\|
\]

follows. With the same argument as for \( m = 0 \) it follows that

\[
|\varphi_{n}^{(m+1)}| \leq 2 \|f_{n}^{(m-1)}\| + \|f_{n}^{(m)}\|
\]

Taking notice of \( |\varphi_{n}^{(m)}|^{2} \leq \text{const} \left( \xi^{k} \|\varphi_{n}^{(m+k)}\|^{2} + \|\varphi_{n}^{(m)}\|^{2}\xi^{m-k} \right) \) for \( \xi > 0 \), we get the proposition for \( m > 0 \).

Let \( h \in H_{m+1/2} \). Obviously our transformation function \( v \) in (5) belongs to \( V_{m+1} \) and as map of \( h \) it is analytic in 0: Notice that \( |\varphi_{1}|_{m+1} \leq \sqrt{2} \|h\|_{m+1/2} \). Since the \( W_{n} \) are Banach algebras if \( s > 3/2 \),

\[
f : (h, \varphi_{1}, \bar{\varphi}) \in \mathcal{H}_{m+1/2} \times V_{m+1} \times V_{m+1} \rightarrow f(h, \varphi_{1}, \bar{\varphi}) \in W_{m}
\]

is analytic at \( (0, 0, 0) \) if \( m \geq 2 \). So we can solve (10) via a fixed point theorem for \( \varphi \in V_{m+1} \), obtaining \( \bar{\varphi}(h) \) analytic in 0, and then \( \varphi \) is analytic. Since \( J \) in (6) depends analytically on \( (\varphi, v) \in V_{1} \times V_{m+1} \), power series expansion of \( \varphi \) and \( v \) gives the analyticity of \( J \) in \( h = 0 \). Obviously the other terms in \( E \) are analytic, too, which gives (i). Now it is time to justify (7). The integrand in \( J \) is an analytic function in \( h, \nabla \varphi |_{x_{3}=0} \), and \( v |_{x_{3}=0} \). Since \( \nabla \varphi |_{x_{3}=0}, \nabla v |_{x_{3}=0} \in H_{m-1/2} \), the integrand belongs to \( H_{m-1/2} \), too. The term due to the Bond number belongs to \( H_{m-3/2} \). This completes (ii). Proposition (iii) is independent of the concrete problem: It follows solely from the analyticity of \( E_{h} \). Proposition (iii) is proved by estimating the coefficients of the power series expansion of \( E \) via interpolation theory. The proof is given in BEYER [1: Corollary 2.2], so we omit it.

3: The bifurcation equations

3.1. The Lyapunov-Schmidt procedure. Let \( h \in H_{s} \). In a small neighbourhood of \( (h, \mu, \varepsilon) = (0, 0, 0) \) and for all \( \zeta \in H_{s} \) we have to solve \( (E_{h}(h, \mu, \varepsilon), \zeta) = 0 \). With (4) we get

\[
E_{h}(h, \mu, \varepsilon) = -Lh + N(h, \mu, \varepsilon).
\]
where $N(h, \mu, \varepsilon)$ denotes the nonlinear part of $E_h(h, \mu, \varepsilon)$. Let $Q$ be the projection from $H$ onto the range of $L$. Then $I - Q$, with $I$ being the identity, projects onto the orthogonal complement of the range of $L$, that means onto $N(L^*)$. We have $I - Q = \langle \cdot, h \rangle h^* + \langle \cdot, h \rangle h^*$ with $h_1, h_2 \in N(L)$ and $h^*_1, h^*_2 \in N(L^*)$. Recall that the norms in $N(L)$ and $N(L^*)$ are different. But, since $L$ is formally self-adjoint, the basic elements in $N(L)$ and $N(L^*)$ are the same ($h_1 = h^*_1$ and $h_2 = h^*_2$). Now, (12) is equivalent to

$$QE_h(h, \mu, \varepsilon) = 0$$

and

$$(I - Q) E_h(h, \mu, \varepsilon) = 0.$$  

Decomposing $h = h' + \eta$ with $h' \in N(L)$ and $\eta \in N(L)^\perp$, we find that (13) has the unique solution $\eta(h', \mu, \varepsilon)$. Recall from Lemma 2 that $\eta$ is analytic. Inserting this in (14), together with the linear independence of the $h^*$, we get $\langle E_h(h' + \eta(h'; \mu, \varepsilon), \mu, \varepsilon), \zeta_i \rangle = 0$ for $\zeta_i \in N(L)$ with $i = 1, 2$. Then, regarding $\langle \eta_i, \zeta_i \rangle = 0$ for $i = 1, 2$ and (12) we get the bifurcation equations

$$G_i \equiv \langle N(h' + \eta(h', \mu, \varepsilon), \mu, \varepsilon), \zeta_i \rangle = 0 \text{ for } \zeta_i \in N(L), i = 1, 2,$$

where $\eta$ has to be determined from

$$\langle \eta_i, \zeta \rangle = -\langle QN(h' + \eta(h', \mu, \varepsilon), \mu, \varepsilon), \zeta \rangle \text{ for all } \zeta \in N(L)^\perp.$$

3.2. Symmetries. Since we are looking for real $h'$, set $h' = z e^{i\omega x} + \bar{z} e^{-i\omega x}$, where $z = a \bar{e}^{i\delta} \varepsilon$ with $a, \delta \in \mathbb{R}$. Taking into consideration the symmetries underlying the physical problem, we are able to say what terms $a^m \bar{e}^{in}, m \in \mathbb{N}$ and $n \in \mathbb{Z}$, only can occur in our equation. Here we suppress the dependence on the parameters in $E_h$.

Definition: We call $\langle E_h(h), \zeta \rangle = 0$, $E_h$ acting in a Hilbert space, covariant with respect to a unitary representation $T_g$ of a compact group $G$ if $\langle E_h(h), \zeta \rangle = \langle E_h(T_g h), T_g \zeta \rangle$ for all $g \in G$.

In our case we consider the translations in the $(x_1, x_2)$-plane via a vector $a$ with the representation $T_a$ and the rotation through $\theta$ with the representation $T_{\theta}$. If $g$ is an element of the translation or rotation group, we take the representation $(T_g x_1, x_2) = h g^{-1}(x_1, x_2)$. Consider (12) with $E_h$ as analytic operator over a Hilbert space. We have $E_h(0, 0, 0) = 0$. $L$ is a Fredholm operator of zero index. So we have the following.

Lemma 3: [4: Theorem 4.4]. Let $\langle E_h(h), \zeta \rangle = 0$ be covariant with respect to $T_g$, then

(i) $T_g$ leaves $N(L)$ invariant;

(ii) $L$ commutes with $T_g$;

(iii) the bifurcation equations are covariant with respect to the finite-dimensional representation $T_g$ restricted to $N(L)$.

This we apply to prove the following.

Lemma 4: The bifurcation equations (15) are reduced to one equation

$$G_1 = \sum_{m+k+l+j>0} c_{mkj} \mu^{k+2m+1} \zeta = 0, \quad c_{mkj} \text{ constants}.$$

Proof: Because of the analyticity of $E_h$ and $\eta(h')$ our bifurcation equations (15) are

$$G_i = \sum_{m+k+l+j>0} \sum_{n+m=n} g_{mk}^l(\mu, \varepsilon) \zeta^m \bar{z}^k = 0 \quad (i = 1, 2).$$
The action of $T_a$ restricted to $N(L)$ is $T_a h_1 = e^{i\omega a} h_1$ and $T_a h_2 = e^{-i\omega a} h_2$. Applying this to the terms of $n$th order in (17), we get

$$e^{i\omega a} \sum_{k+m=n} g_{mk}^i(\mu, e) z^{m-k} = \sum_{k+m=n} g_{mk}^i(\mu, e) e^{i\omega (m-k)} z^{m-k}.$$ 

This must hold for all $a \in \mathbb{R}^2$, whence $m = k + 1$ follows. So we have $G_i = \sum_{k=0}^\infty g_{ki}^i(\mu, e)$.

3.3. Concrete coefficients of the bifurcation equation. Here we will solve the bifurcation equation. We will see that it is enough to know the coefficients $c_{mkj}$ up to $m + k + j = 4$. At first we have to compute some Fourier coefficients of the next approximations of $\varphi = \sum_{k \geq 0} \varphi_k(h^k)$.

Lemma 5: Let

$$h = h' + \eta z = z e^{i\omega x + z c^{-i\omega x}} + \eta'(z^2 e^{2i\omega x} + z^2 e^{-2i\omega x}).$$

and $|x_3| = u$. Then for the power series expansion

$$\varphi(h) = \varphi_0(h') + \varphi_1(\eta z) + \varphi_2(h^2) + \varphi_3(\eta z, h') + \varphi_3(\eta z, h') + \sum_{2k+m>3} \varphi_k(h^{m+k})$$

we get

(i) $\varphi_1(h') = \varphi_0 e^{i\omega x} + \varphi_0 e^{-i\omega x}$ with $\varphi_0 = i\omega \cosh (u + r)/\sinh r$, $\varphi_1(\eta z) = \eta z e^{2i\omega x}$

(ii) $\varphi_2(h^2) = 0$ as Fourier coefficient of $e^{i\omega x}$ in the Fourier expansion of $\varphi_2$;

(iii) the Fourier coefficient of $e^{i\omega x}$ at $x_3 = 0$.

Proof: Transformation of (1) with (5) leads to (h.o.t. denotes higher order terms)

$$\Delta \varphi = A(\varphi v) + B(v^3) + \text{h.o.t.}$$

in $S$ with

$$A(\varphi v) = 2v_{13}\varphi_1 + 2v_{33}\varphi_3 + 2v_{11}\varphi_{13} - \varphi_{11}v_3 + v_3\varphi_{33},$$

$$B(v^3) = 4v_{13}v_{33} + 5v_{11}v_{13} + v_{33}v_{13},$$

to $\varphi, \varphi_3 = \varphi_0 e^{-i\omega x}$, $C(\varphi v) + D(v^3)$ + h.o.t. on $x_3 = 0$ with $C(\varphi v) = 2\varphi_{11}v_3 + \varphi_3 v_3$ and $D(v^3) = -2v_{11}(v_3^2 + v_3^2)$, and to $\varphi, \varphi_3 = 0$ on $x_3 = -1$. The first approximation of $\varphi$ with linear right sides is given in (11). We get $\varphi_2(h^2)$ from $\Delta \varphi_2(h^2) = A(\varphi_1(h') \varphi(h'))$ in $S$ and $\varphi_2(h^2) = C(\varphi_1(h') \varphi(h'))$ on $x_3 = 0$. This implies (i) since the right sides are zero.

With (18) equation (5) reads $\varphi(h) = v_1(h') + v_0(\eta z) + \text{h.o.t.}$ Solving

$$\Delta \varphi_2(h\eta z) = A(\varphi_2(h\eta z) v_1(h') + \varphi_1(h') v_2(\eta z))$$

in $S$,

$$\varphi_3(h\eta z) = C(\varphi_3(\eta z) v_1(h') + \varphi_2(\eta z) v_2(\eta z))$$

on $x_3 = 0$,

notice that $\varphi_{i,j} = -v_{i,j}$ and $\varphi_{i,j} = v_{i,j}$ for $i = 1, 2$. This implies in that case $A = C = 0$, which gives (ii).

Finally the solution of

$$\Delta \varphi_2(h\eta z) = B(v^3) = i\omega z^2(5 \cosh (3u + 3r) - 4 \cosh (u + r))/\sinh^3 r$$

in $S$ and $\varphi_0(h^{3,1}) = D(v^3) = -2i\omega z^2(3 + \coth^2 r)$ on $x_3 = 0$ gives (iii)
Studying in which manner \( \eta \) from (16) contributes to the lower terms in the bifurcation equations (15), we will also justify (18).

**Lemma 6:** The solution \( \eta = \sum_{k+l+m \geq 0} \eta_{klm}(h^{k+2l}e^{l \mu^m}) \) of (16) contains the terms

\[
\eta_2 = \eta_{200} = \eta'(z^2 e^{i \omega x} + \tanh(2r)), \quad \eta' = -\omega \frac{\cosh(2r)}{\sinh r \cosh r + 2 \tanh r - 3r}.
\]

The nontrivial Fourier coefficients in \( \eta_{210} \) and \( \eta_{201} \) are only \( \langle \eta_{210} \rangle_2 \) and \( \langle \eta_{201} \rangle_2 \).

**Proof:** Let \( \hat{\eta} \) be the approximation of \( \eta \), depending on \( h^2 \). Then \( \varphi = \sum_{k \geq 0} \varphi_k(h^k) \) and \( \hat{h} = h' + \hat{\eta} + \sum_{k+l+m \geq 0} \eta_{klm}(h^{k+2l}e^{l \mu^m}) \) implies \( \varphi = \varphi_1(h') + \varphi_2(\hat{\eta}) + \sum_{k \geq 0} \varphi_k(h^k) \).

Recall from (7) that (16) is

\[
\langle L \eta, \zeta \rangle = -\frac{1}{2} F_\epsilon(v + \epsilon) \langle |\nabla \varphi_1|^2 - 2 \varphi_1, h', \zeta \rangle = \epsilon F_\epsilon(\varphi_2, \zeta)
\]

\[
+ \mu b \langle \delta \eta, \zeta \rangle + \text{h.o.t.}
\]

**Lemma 5(ii) yields**

\[
\langle L \eta, \zeta \rangle = \left(-F_\epsilon \omega^2 \cosh(2r)/(2 \sinh^2 r) + \theta(r, \epsilon, \mu)\right) \langle \z^2 e^{i \omega x} + \tanh(2r), \zeta \rangle + \text{h.o.t.}
\]

with some \( \theta \) depending on \( \epsilon \) and \( \mu \). So, \( \theta \) determines only \( \langle \eta_{210} \rangle_2 \) and \( \langle \eta_{201} \rangle_2 \). The last equation must hold for all \( \z > \z_0 = r_0 \). Choosing \( \z = \z_2 e^{i \omega x} + \z_0 e^{i \omega x} \) we get together with \( L(\eta_{210}) = (1 + 4 \omega^2 b_\epsilon - 2 \omega F_\epsilon \coth(2r)) \langle \eta_{200} \rangle_2 \) and with (9)

\[
\zeta = \eta_{200}.
\]

At this time we have determined all unknown functions which are important for our bifurcation equation.

**Theorem 1:** The following assertions are true:

(i) The bifurcation equation (15) reads as

\[
-(\mu' + \epsilon') a + (w(r) + 4 \eta') a^3 + \sum_{k+l+m \geq 0} c_{klm}2^{k+3}e^{l \mu^m} = 0,
\]

where \( \mu' = \left(\sinh(2r) - 2r\right)/\omega^2 \), \( \epsilon' = 2 \sinh(2r)/\omega^2 \), and

\[
w(r) = -\frac{3}{2} \left(\sinh(2r) - 2r\right) + \left(7 \sinh(4r) - 8 \sinh(2r) - 8r\right)/\sinh^2 r.
\]

(ii) The nontrivial solution of (19), which only exists if \( r > r_0 \) for some \( r_0 \approx 0.8 \), is

\[
a^2 = (\epsilon' + \mu')/(w(r) + 4 \eta') + \sum_{l+j \geq 1} a_{l^j} e^{l \mu^j}.
\]

**Proof:** The bifurcation equation (15) with the concrete terms from (7) reads as

\[
\langle b_{\epsilon \mu} \Delta h' + \epsilon F_\epsilon \varphi_1, \zeta_1 \rangle + 3b \langle b_{1,1}^2, h_{1,1}, \zeta_1 \rangle
\]

\[
= -F_\epsilon \langle \varphi_3^2, \sqrt{\varphi_1 \varphi_2 - v_{1,2} \varphi_3 - v_{1,3}, \varphi_1 + v_{1,2} v_{1,3}, \zeta_1 \rangle + \text{h.o.t.,}
\]

where h.o.t. denotes higher order terms in \( a^k \) with \( k \geq 4 \). A simple computation with the functions given in Lemma 5 and Lemma 6 and the \( b_{\epsilon} \) and \( F_\epsilon \) from (9) together with Lemma 4 gives (i). The Implicit Function Theorem together with the condition \( w(r) > 0 \) finally gives (ii).
Remark: We have considered our problem in three dimensions. However, we have shown that at smallest critical Froude numbers the bifurcating wave is a two-dimensional one. Indeed, it is that wave which was derived in Zeidler [7: Chapter 4]. Therefore, we have also justified the two-dimensional approach of [7].

Now, in analogy to Beyer [2], the variational approach gives the stability of the wave in a natural way.

4. Stability

4.1. The second variation of $E$. Stability intervals we get by the positive definiteness of the second variation of the potential-energy. By Lemma 2, $E_{hh}$ has the structure $E_{hh}(\xi, \xi) = (L\xi, \xi) + \sum E_i(\xi^2h^4)$. If we set $\xi = \xi_1 + \xi_2$, where $\xi_1 \in N(L)$ and $\xi_2 \in N(L)^{\perp}$, this yields $\lambda > 0$.

Lemma 7: For $|\varepsilon|$ and $|\mu|$ small enough the stability of the solution (20) is determined by the positive definiteness of

$$E_0 = (L\xi_2, \xi_2) + E_1(\xi_1^2h^4) + E_2(\xi_2^2h^4),$$

where $\xi_1 = X_1(e^{iwx+\delta} + e^{-iwx-\delta})$ and $\xi_2 = X_2(e^{2iwx+\delta} + e^{-2iwx-\delta})$.

Proof: 1. The inequality $(L\xi, \xi) \geq ||\xi||^2$ follows by Lemma 1/(iii). 2. Recall from (21) that $h' = O(\varepsilon, \mu)$ as $\varepsilon, \mu \to 0$. Since $||\xi_1||^2 + ||\xi_2||^2 \leq \lambda ||\xi_1||^2 + ||\xi_2||^2$, where we take $\lambda = \varepsilon$ or $\lambda = \mu$, respectively, together with Part 1 we get $|E_1(\xi_2^2h^4)| \leq c ||\xi||^2 (|\varepsilon| + |\mu|)$. The same argument yields $|E_2(\xi_2^2h^4)| \leq c ||\xi||^2 (|\varepsilon| + |\mu|)$. But integration of the $e^{ikx}$ terms ($k = 1, 3$) shows $E_1(\xi_2^2h^4) = 0$. Hence, $E_{hh}(\xi, \xi)$ is given by the sum of a positive part $L\xi, \xi$, $xi \perp \xi_2$ and $\xi \in N(L)^{\perp}$, of $E_0$, and of other terms of higher order if $|\varepsilon|$ and $|\mu|$ are small enough. 3. With respect to translations through any $a = (a_1, a_2)$ in the $(x_1, x_2)$-plane the covariance of $E_{hh}(h)$ was given by $T_aE_{hh}(h) = E_{hh}(T_aT_a)$.

Next we compute $E_0$. We know from the splitted Euler equation (15) and (16) that

$$\langle E_h, \xi \rangle = \langle (\xi, \xi) + \langle (N(h' + \eta), \xi) + \langle (N(h' + \eta), \xi_1)\rangle, \xi \rangle\rangle,$$

where $\xi \in N(L)^{\perp}$, (19) and (21) taken into account, this reads as

$$\frac{1}{2} \langle E_h, \xi \rangle = (\sinh (2r) - 6r + 4 \tanh r) \eta' a^2X_2 + 2 \cosh (2r) a^2X_2$$

$$- (\varepsilon' + \mu') aX_1 + w(r) a^2X_1 + 4\eta'a^3X_1 + h.o.t.$$

But varying $h$ really means varying every Fourier coefficient. So we have to take $a(t) = a + tX_1$ and $\eta(t) = \eta' + tX_2$, which yields $E_0 = E_{hh}X_1 + E_{hh}X_2$. Therefore, $E_0 = C_1X_1^2 + 2C_2X_1X_2 + C_3X_2^2$ with $C_1 = 4(\varepsilon' + \mu')$, $C_2 = 2a^2(\sinh (2r) - 6r + 4 \tanh r)$, and $C_3 = 4a^3$. This quadratic form is positive definite if $C_1 + C_3 > 0$. 

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and \( C_1C_2 - C_3^2 > 0 \). The first condition is fulfilled for all \( \epsilon, \mu > 0 \), the second means \([\sinh(2r) - 2r]/\sinh(2r)\) \( \mu + 2\epsilon < g(r) \) with a function \( g, g(r) > 0 \) for \( r > 0 \) and \( g(0) = 0 \), with \( r = m/(Bl) \) where \( B \) is the mean depth and \( m/l \) the wave-length determined by the choice of the critical parameters. The inequality shows that for small wave-length the dependence on \( \mu \) becomes important. This finally yields

**Theorem 2:** The bifurcating wave \( h = a(\epsilon, \mu) \cos (x, m/(Bl) + \delta) \) is stable with respect to perturbations, which belong to \( \mathbb{H}_m \), if \( |\epsilon'| \) and \( |\mu'| \) are small enough and \( r > r_0 \).

**REFERENCES**


