On a Coupled System of Partial Differential Equations

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We investigate the following coupled system of stationary partial differential equations:

\[ \begin{align*}
\nabla \cdot (D_i \nabla u_i) + \nabla \cdot (k_i z_i u_i (a + bu_0)^{-1/2} \nabla \Psi) &= f_i, \\
-\nabla \cdot (D_0 \nabla \Psi) &= k(N + u_0),
\end{align*} \]

where \( u_i, f_i, \Psi \) and \( N \) are real functions on a bounded domain \( G \) of the \( n \)-dimensional Euclidean space \( R^n \) allowing partial integration and the Sobolev imbedding theorems. \( \nabla \cdot (D \nabla u) \) means \( \text{div} (D \text{grad} u) \). Let further \( u_0 = z_i u_i + \cdots + z_M u_M \), \( z_i = \pm 1 \), \( U_0 = 1/(a + bu_0^2)^{1/2} \), \( M \) a natural number, \( k, k_i, a, b \) positive constants.

The diffusion coefficients may be of the form \( D_i = D_i(\nabla u_i) \) and \( D_0 = D_0(\nabla \Psi) \). For all functions \( u_i \) and \( \Psi \) we suppose the same type of boundary conditions:

\[ \begin{align*}
|u_i|_{\partial G_1} &= \gamma_i, \quad i = 1, \ldots, M, \\
\Psi|_{\partial G_1} &= \gamma_0, \\
J_i \cdot \vec{n}|_{\partial G_2} &= 0, \quad i = 0, 1, \ldots, M
\end{align*} \]

where \( J_0 = -D_0 \nabla \Psi \) and \( J_i = -D_i \nabla u_i - k_i z_i u_i (a + bu_0^2)^{-1/2} \nabla \Psi \) are the flow densities, \( \partial G_1 \) and \( \partial G_2 \) are portions of the boundary of \( G \) with the properties \( \partial G_1 \cap \partial G_2 = \emptyset \), \( \partial G = \partial G_1 \cup \partial G_2 \), mes \( \partial G_1 > 0 \) with respect to the boundary measure, and \( \vec{n} \) is the outer normal to the boundary of \( G \).

The system (1) is similar to the stationary equations of carrier transport in semiconductors (cf. [3, 7]). Instead of two equations, here an arbitrary number of carriers is supposed. The mobilities are replaced by the terms \( z_i k_i z_i u_i (a + bu_0^2)^{-1/2} \) and the recombination rates are lacking. The coupling of the quantities \( u_i \) by the last equation of (1) here takes place in an
analogous way. The system (1) has its origin in the models describing the implantation of impurities into semiconductor devices as they were used by Dutton (cf. [5]) and others for numerical simulations.

It is the purpose of this paper to prove the existence of a solution of the problem (1), (2) by functional analytical means. We proceed in a similar way as in [3, 7].

In Section 1 we consider first the problem with homogeneous boundary conditions

\[ u_i \big|_{\partial G} = 0, \quad i = 1, \ldots, M, \quad \psi_i \big|_{\partial G} = 0 \]

\[ J_i \cdot n \big|_{\partial G} = 0, \quad i = 0, 1, \ldots, M. \]

The closure of \( \{ v \in C^2(\bar{G}) : v \big|_{\partial G} = 0 \} \) with respect to the norm

\[ \| v \|_{1,q} = \left( \int_{\bar{G}} \left[ |v|^2 + \left| D^2 v \right|^q \right] \right)^{1/q}, \quad q > 1, \]

of the Sobolev space \( W_q^1(G) = \{ v \in L_q(G) : D^2 v \in L_q(G) \} \) is denoted by \( W_q^1(G) \). Let us assume that \( G \) has a Lipschitzian boundary (cf. [9]). Then

\[ \| v \|_{0,q} = \left( \int_{\bar{G}} |\nabla v|^q \right)^{1/q} \]

and \( \| \cdot \|_{1,q} \) are equivalent norms on \( W_q^1(G) \). We ask for solutions \( u_1, \ldots, u_M, \psi \) of the problem (1), (3) in the sense of the notion of the solution in variation (weak solution) in the space \( W^1_x(\Omega) \), where \( W^M = W \times \cdots \times W \) is the \( M \)-fold product of \( W = W_p(G) \), \( r, p \geq 2 \). For the sake of simplicity we suppose \( p > n \) in order to be sure of the compact imbedding of \( W^1_p(\Omega) \) into the space \( C(\bar{G}) \) of continuous functions on \( G \). We assume further that \( N \in L_r(G), 1/r' + 1/r = 1, f_i \in L_{r'}(G), 1/p' + 1/p = 1 \). Concerning the diffusivities we assume

\[ D_i(|\nabla u_i|) = d_i(|\nabla u_i|)|\nabla u_i|^{p-2} \quad \text{and} \quad D_0(|\nabla \psi|) = |\nabla \psi|^{r-2}, \]

where the \( d_i \) are defined on \( \mathbb{R}_+ = [0, +\infty) \), continuous, non-decreasing, and \( 0 < d_{i0} \leq d_i(s) \leq d_{i1}, \quad s \in \mathbb{R}_+ \). Under these assumptions the diffusivity terms generate uniformly monotone operators in \( W_p(G) \) and \( W_{r}(G) \), respectively. An operator \( T \) of the whole of a real, reflexive Banach space \( B_0 \) into its adjoint space \( (B_0)^* \) is called uniformly monotone if there is a function \( \delta \in \mathbb{R}_+ \) with \( \delta(1) > 0 \) such that \( (Tu - Tv, u - v) \geq \delta(\|u - v\|) \) holds for \( u, v \in B_0 \). In our cases either \( \delta(s) = c_0 s^p \) or \( \delta(s) = e_0 s^r \) hold. We will show that the operator equation \( Tu = 0 \) corresponding to the homogeneous problem has a bounded, coercive, pseudomonotone operator \( T \), and therefore a solution exists by a well-known theorem of Brézis [1].

In Section 2 the inhomogeneous problem (1), (2) will be investigated, reducing it to the homogeneous case. In this way a "disturbed" operator equation will arise that will then be solved by the same means as the undisturbed one considered in Section 1.

1. The homogeneous boundary value problem

Here we consider the homogeneous boundary value problem (1), (3). Let us begin with the following useful abstract lemma, which is partially contained in many papers (cf. e.g. [2, 4, 6]).
Lemma: Let \( (H, (\cdot, \cdot)) \) be a Hilbert space, \( \|\cdot\| = (\cdot, \cdot)^{1/2} \) the norm and \( \alpha \) a real number. For \( u, v \in H \) we have

\[
(\|u\|^{\alpha} u - \|v\|^{\alpha} v, u - v) \geq \begin{cases} K \|u - v\|^{\alpha + 2} & \text{for } \alpha \geq 0 \\ K((\|u\|^{\alpha} + \|v\|^{\alpha})^{-1} \|u - v\|^{2} & \text{for } -1 < \alpha \leq 0. 
\end{cases}
\]

(1.1)

and

\[
\|u\|^{\alpha} u - \|v\|^{\alpha} v \leq \begin{cases} G(\|u\|^{\alpha} + \|v\|^{\alpha}) \|u - v\| & \text{for } \alpha \geq 0 \\ G \|u - v\|^{\alpha + 1} & \text{for } -1 < \alpha \leq 0
\end{cases}
\]

(1.2)

where \( K \) and \( G \) are positive constants depending on \( \alpha \).

Proof: We are going to prove the Lemma in an elementary algebraic manner. We begin with (1.1). Division of (1.1) by \( \|u\|^{\alpha + 2} \) (we assume \( \|u\|^{\alpha + 2} > 0 \)) yields

\[
\begin{align*}
\frac{u}{\|u\|} &\quad - \frac{(\|v\|/\|u\|)^{\alpha} v}{\|u\|} = \frac{u}{\|u\|} - \frac{v}{\|u\|} \\
&\quad \geq \begin{cases} K(\|u\|^{\alpha} - \|v\|^{\alpha}) & \text{for } \alpha \geq 0 \\ K((1 + (\|v\|/\|u\|)^{\alpha})^{-1} \|u\|/\|v\|^{\alpha} & \text{for } -1 < \alpha \leq 0.
\end{cases}
\end{align*}
\]

Consequently, it suffices to show for \( ||x|| \leq 1 \) and \( ||y|| \leq 1 \) that

\[
(x - |y|^{\alpha} y, x - y) \geq \begin{cases} K \|x - y\|^{\alpha + 2} & \text{for } \alpha \geq 0 \\ K(1 + |y|^{-\alpha})^{-1} \|x - y\|^{2} & \text{for } -1 < \alpha \leq 0.
\end{cases}
\]

Setting \( t = ||y|| \) and \( s = (x, y) \) with \( ||s|| = 1 \) we get

\[
\begin{align*}
(1 - t) (1 - t^{\alpha + 1}) &\quad + (t^{\alpha + 1} + 1) (t - s) \\
&\quad \geq \begin{cases} K((1 - t)^{2} + 2(t - s))^{\alpha/2 + 1} & \text{for } \alpha \geq 0 \\ K((1 + t^{-\alpha})^{-1} [(1 - t)^{2} + 2(t - s)] & \text{for } -1 < \alpha \leq 0
\end{cases}
\end{align*}
\]

(1.3)

for \(-t \leq s \leq t \leq 1 \) with \( t \geq 0 \). This is what we have to show.

First let us consider the special cases \( t = 1 \) and \( t = s \). The case \( t = 1 \) is easy and clear. In the case \( t = s \) and \( \alpha \geq 0 \) we argue that, because of \( 1 \geq (1 - t)^{\alpha + 1} + t^{\alpha + 1} \), \( (1 - t)^{\alpha + 1} \leq 1 - t^{\alpha + 1} \) holds, i.e. \( (1 - t) (1 - t^{\alpha + 1}) \geq (1 - t)^{\alpha + 2} \). For \( 0 \leq \varepsilon < 1 \) we have \( (1 - \varepsilon)/(1 - t) \geq \varepsilon \), as the difference quotient equals \( f'(t_0) \) for \( f(t) = t' \) and some \( t_0 \leq 1 \). Because of the monotonicity of \( f' \) then \( f'(t_0) \geq f'(1) = \varepsilon \). Then, in the case \( t = s \) and \(-1 < \alpha \leq 0, \)

\[
(1 + t^{-\alpha})^{-1} (1 - t^{\alpha + 1})/(1 - t) \geq (1 - t^{\alpha + 1})/(1 - t) \geq \alpha + 1 \Rightarrow K > 0.
\]

Now we are going to show (1.3) in case \( t \neq s \) and \( t + 1 \). Because of \( (1 - t)^{\alpha + 1} \leq 1 - t^{\alpha + 1} \) we have for \( \alpha \geq 0, \)

\[
(1 - t^{\alpha + 1})/(1 - t) \geq (1 - t^{\alpha + 1})/(t^{\alpha + 1} + (t^{\alpha + 1} - s)(t - s)]
\]

\[
\leq 2t^{\alpha/2}[(1 - t)^{\alpha + 2} + 2t^{\alpha + 1}(t - s)(t^{\alpha/2 + 1})]/[(1 - t)(1 - t^{\alpha + 1}) + (t^{\alpha + 1} + 1) t - s)]]
\]

\[
\leq 2t^{\alpha/2}[(1 - t)^{\alpha + 2}/(1 - t)(1 - t^{\alpha + 1}) + 2t^{\alpha/2 + 1}(t - s)(t^{\alpha/2 + 1})/(t^{\alpha + 1} + 1) (t - s)]
\]

\[
\leq 2t^{\alpha/2}(1 + 2t^{\alpha + 1} 2t^{\alpha/2}] = 1/K.
\]
At last, for \(-1 < \alpha \leq 0\), using \((1 - t')/(1 - t) \geq \varepsilon\) for \(0 \leq \varepsilon < 1\), we have

\[
(1 - t)^2/(1 - t)^2(1 + t - s) - 2/(1 + t - s)(t + 1)(t - s) \\
\leq (1 - t)^2/[1 + t - s - s] - 1/(1 + t - s)(t + 1)(t - s).
\]

The proof of (1.1) is complete.

As to (1.2), we first observe that the inverse of the operator \(T_\alpha u = |u|^\alpha u\) is just the operator \(T_\alpha^{-1} u = T_{-\alpha}^{-1} u = \alpha u^{-\alpha/\alpha+1} u = T_\beta u\), and \(\alpha \leq 0\) iff \(-1 < \beta \leq 0\), \(-1 < \alpha \leq 0\) iff \(\beta \geq 0\) hold. This way, to prove (1.2) for \(T_\alpha\) we use (1.1) for \(T_\alpha^{-1} = T_\beta\).

Let \(u_1 = T_\alpha u\) and \(v_1 = T_\beta v\), i.e. \(u = T_\beta u_1\) and \(v = T_\beta v_1\). Then we have

\[
|T_\alpha u_1 - T_\beta v_1| \leq (T_\beta u_1 - T_\beta v_1, u_1 - v_1)
\]

\[
\leq \left \{ \begin{array}{ll}
K(\beta) (|u_1|^\beta + |v_1|^\beta)^{-1} |u_1 - v_1|^2 & \text{for } \alpha \geq 0 \\
K(\beta) |u_1 - v_1|^{\beta+1} = K(\beta) |u_1 - v_1|^{1+(\alpha+1)} & \text{for } \alpha \leq 0,
\end{array} \right.
\]

\[
|T_\alpha u - T_\alpha v| = |u_1 - v_1| \leq \left \{ \begin{array}{ll}
|u - v| K(\beta)^{-1} (|u_1|^\beta + |v_1|^\beta) \\
(|u - v| K(\beta)^{-1})^{\alpha+1}.
\end{array} \right.
\]

This is (1.2), taking into account that \(|u_1|^\beta = |T_\alpha u|^\beta = |u|^\alpha u^{\alpha/\alpha+1} = |u|^\alpha\) holds.

Using the boundary conditions (3) and partial integration we come to the weak formulation of the problem (1). The function \([u_1, \ldots, u_M, \Psi'] \in W^M \times W_r(G)\) is called solution of our problem if

\[
\int_D D_i \nabla u_i \nabla w_i dG + \int \kappa_i z_i u_i U_0 \nabla \Psi \nabla w_i dG + \int f_i w_i dG = M, \quad i = 1, \ldots, 0,
\]

and

\[
\int_D D_0 \nabla \Psi \nabla w_0 dG = \int k(N + u_0) w_0 dG
\]

hold for all \([w_1, \ldots, w_M, w_0] \in W^M \times W_r(G)\). If \(D_0 = |\nabla \Psi|^{-2}\), the left-hand side of (1.5) generates a uniformly monotone, coercive operator on \(W_r(G)\). Hence (1.5) is uniquely solvable for fixed \(N\) and \(u_0\). Since this solution (for generally fixed \(N\)) only depends on \(u = [u_1, \ldots, u_M]\) we denote it by \(S u\). Substituting this into (1.4), we obtain the problem: Find \(u = [u_1, \ldots, u_M] \in W^M\) such that

\[
\int_D D_i \nabla u_i \nabla w_i dG + \int \kappa_i z_i u_i U_0 \nabla S u \nabla w_i dG + \int f_i w_i dG = 0
\]

hold for \(i = 1, \ldots, M\). In the sequel we will transform this problem into the operator equation \(Au + Bu = f\) in \((W^M)^*\) and investigate the properties of the operators \(A\) and \(B\).

Now, let \langle \cdot, \cdot \rangle be the dual pairing between \(W^*\) and \(W = W_p(G)\). We generate operators \(A_i\) and \(B_i\) mapping \(W^M\) into \(W^*\):

\[
\langle A_i v, w \rangle = \int_D D_i \nabla v_i \nabla w dG,
\]

\[
\langle B_i v, w \rangle = \int \kappa_i \nabla (a + b_v)^{3/2} \nabla S v \nabla w dG.
\]
Under suitable conditions on $D$, the operator $A_i$ will be uniformly monotone \((\text{cf. } (8) \text{ and } [3]):\)

$$-\langle A_i u - A_i v, u_i - v_i \rangle \geq c_i \|u_i - v_i\|^p, \quad c_i > 0. \tag{1.6}$$

Now, after proving some estimates for $\|B_i u - B_i v\|_w$ and $\|B_i u\|_w$, we will show that $B_i$ is increasing continuous. First we estimate $b_i = \langle B_i u - B_i v, w \rangle$. We have

$$b_i \leq |k| \int_G |u_0 U_0 \nabla Su - v_0 V_0 \nabla Sv| |\nabla w| \, dG,$$

where $U_0 = |a + bu_0^2|^{-1/2}$ and $V_0 = |a + bv_0^2|^{-1/2}$. As the derivative of the function $(a + bx^2)^{-1/2}$ is bounded we have $|U_0 - V_0| \leq k(a, b) |u_0 - v_0|$. We calculate

$$|u_0 U_0 \nabla Su - v_0 V_0 \nabla Sv| \leq |u_0 U_0 (\nabla Su - \nabla Sv) + u_0 (U_0 - V_0) \nabla Sv + (u_i - v_i) V_0 \nabla Sv| \leq a^{-1/2} |u_i| |\nabla Su - \nabla Sv| + |u_i - v_i| |\nabla Sv|| u_0 - v_0| k(a, b).$$

Then,

$$b_i \leq k_1^1 \left( \int_G |u_i| |\nabla Su - \nabla Sv| |\nabla w| \, dG \right.$$

$$\left. + \int_G |u_i| |\nabla Sv| |u_0 - v_0| |\nabla w| \, dG + \int_G |u_i - v_i| |\nabla Sv| |\nabla w| \, dG \right), \tag{1.7}$$

where $k_1$ is a positive constant not depending on $u$ and $v$. For $p > n$, as $W$ is imbedded into $C$, we get by the Hölder inequality, putting $q = p/(p - 1)$ and denoting the $L_p$-norm by $||\cdot||_p$, the $C$-norm by $||\cdot||_C$, \begin{align*}
 b_i \leq & k_1^1 |u_i|_C |\nabla Su - \nabla Sv|_q + |u_i|_C |u_0 - v_0|_C |\nabla Sv|_q \\
 & + |u_i - v_i|_C |\nabla Sv|_q |\nabla w|_p.
\end{align*}

Next, we estimate $||\nabla Su - \nabla Sv||_p$ and $||\nabla Sv||_q$. We have

$$\nabla \cdot (|\nabla Su|^{r-2} \nabla Su) = k(N + u_0) \quad \text{and} \quad \nabla \cdot (|\nabla Sv|^{r-2} \nabla Sv) = k(N + v_0).$$

From this we get by subtraction and partial integration for $w \in W_r(G)$

$$\int_G (|\nabla Su|^{r-2} \nabla Su - |\nabla Sv|^{r-2} \nabla Sv) \nabla w \, dG = -k \int_G (u_0 - v_0) w \, dG.$$

For $s, t \in R$ and $r \geq 2$ the Lemma gives $(s |s|^{r-2} - t |t|^{r-2}) (s - t) \geq \text{const } |s - t|^r$. Thus, if we take $w = Su - Sv \in W_r(G)$ then we get

$$\text{const } \int_G |\nabla Su - \nabla Sv|^r \, dG$$

$$\leq \int_G (|\nabla Su|^{r-2} \nabla Su - |\nabla Sv|^{r-2} \nabla Sv) (|\nabla Su - \nabla Sv|) \, dG$$

$$\leq \int_G |u_0 - v_0| |Su - Sv| \, dG \leq \text{const } |u_0 - v_0|_C |\nabla Su - \nabla Sv|_r,$$

as $W_r \subset W_r^{1} \subset L_r \subset L^1$ with continuous imbeddings. We get

$$||\nabla Su - \nabla Sv||_r \leq \text{const } |u_0 - v_0|_C ||\nabla Su - \nabla Sv||_r,$$

de $W_r \subset W_r^{1} \subset L_r \subset L^1$ with continuous imbeddings. We get

$$||\nabla Su - \nabla Sv||_r \leq \text{const } |u_0 - v_0|_C^{1/(r-1)}.$$

As $q \leq r$ we also have

$$||\nabla Su - \nabla Sv||_q \leq \text{const } |u_0 - v_0|_C^{1/(r-1)} \tag{1.8}$$
By multiplication with $Sv$ and partial integration the equation $\nabla \cdot (|\nabla Sv|^2 \nabla Sv) = k(N + v_0)$ gives
\[
\int_\Omega |\nabla Sv|^2 \nabla Sv \, dG = -k \int_\Omega (N + v_0) \, Sv \, dG,
\]
i.e. $|\nabla Sv| \leq k \|N + v_0\|_\Omega$. From $\|Sv\| \leq \text{const} \|\nabla Sv\|_\Omega$ we get as before
\[
\|\nabla Sv\|_\Omega \leq \text{const} \|N + v_0\|^{(r-1)}_\Omega.
\]
Now, taking into account that $\|\nabla w\|_p = \|w\|_{0,p}$, (1.7) and (1.8) yield
\[
\|B_i u - B_i v\|_{W^*} \leq \text{const} \left( |u_i|_C \|u_0 - v_0\|^{1/(r-1)}_C + |u_i|_C \|u_0 - v_0\|_C \right).
\]
In the same way as in [3] we are now able to show that $B_i$ is increasing continuous. Indeed, let $w^t \to u$ in $W^M$, i.e. $u_i^t \to u_i$ and $u_0^t \to u_0$ in $W$. Then, because of the compact imbedding $W \subset C$, $u_i^t \to u_i$ and $u_0^t \to u_0$ in $C$, and (1.10) gives us $\|B_i w^t - B_i u\|_{W^*} \to 0$ since $|u_i|_C$ is bounded.

Let us now define operators $A, B: W^M \to (W^M)^*$ as
\[
(Au, w)_M = \sum_{i=1}^{M} (A_i u, w_i) \quad \text{and} \quad (Bu, w)_M = \sum_{i=1}^{M} (B_i u, w_i),
\]
where $(\cdot, \cdot)_M$ is the dual pairing between $(W^M)^*$ and $W^M$. Then, if we define $\|u\|_{W^M} = \|u_1\|_{0,p} + \cdots + \|u_M\|_{0,p}$ from (1.6) we get
\[
(Au - Av, u - v)_M \geq \xi \sum_{i=1}^{M} \|u_i - v_i\|_{0,p} \geq \xi \|u - v\|_{W^M}.
\]
Furthermore, the operator $B$ is increasing continuous because the $B_i$ are. Indeed, we have for all $w \in W^M$
\[
|\langle Bu - Bv, w \rangle_M| \leq \sum_{i=1}^{M} |\langle B_i u - B_i v, w_i \rangle| \leq \sum_{i=1}^{M} \|B_i u - B_i v\|_{W^*} \|w_i\|_W
\]
\[
\leq \left( \sum_{i=1}^{M} \|B_i u - B_i v\|_{W^*}^2 \right)^{1/2} \|w\|_{W^M},
\]
i.e.
\[
\|Bu - Bv\|_{W^M} \leq \left( \sum_{i=1}^{M} \|B_i u - B_i v\|_{W^*}^2 \right)^{1/2}.
\]
Besides, we need an estimate for $|\langle B v, v \rangle_M|$. The Hölder inequality gives
\[
|\langle B v, v_i \rangle| = |k_i| \int_\Omega v_i(a + b v_0)\langle v v_i, v_i \rangle \, dG \leq \text{const} \|v_i\|_C \|\nabla Sv\|_\Omega \|v_i\|_W.
\]
From (1.9) and $|v_i|_C \leq \text{const} \|v_i\|_{0,p}$ we then obtain
\[
|\langle B v, v_i \rangle| \leq \text{const} \|N + v_0\|^{1/(r-1)} \|v_i\|_{0,p}^2.
\]
This gives
\[
|\langle B v, v \rangle_M| \leq \sum_{i=1}^{M} |\langle B v, v_i \rangle| \leq \text{const} \|N + v_0\|^{1/(r-1)} \|v\|_{W^M}^2.
\]
(1.12)
Furthermore, we can estimate
\[\|N + v_0\|^r \leq \text{const} \left( \|N\|^r + \|v_0\|^r \right)\]
\[(1.13)\]
since
\[\|v_0\|^r \leq \text{const} \|v_0\|_{0,p} \leq \text{const} \sum_{i=1}^{M} \|v_i\|_{0,p} \leq \text{const} \|v\|_{W^M}^r.
\]
Now we are ready to prove the coerciveness of \( T = A + B \). Because of (1.11), (1.12) and \( A_0 = 0 \) we have
\[\langle Tv, v \rangle_M = \langle Av, v \rangle_M + \langle Bv, v \rangle_M \geq \text{const} \|v\|_{W^M}^p - \text{const} \|N + v_0\|^r \|v\|_{W^M}^r,\]
and with (1.13)
\[\langle Tv, v \rangle_M \|v\|_M = \text{const} \|v\|_{W^M}^{p-1} - \text{const} \|v\|_M (\text{const} + \|v\|_M^{1/(r-1)}),\]
converging to \( \infty \) if \( \|v\|_M \to \infty \), provided \( 1 + 1/(r-1) < p - 1 \), i.e. \( r > (p - 1)/(p - 2) \).

This way, if \( p > n \) and \( r > (p - 1)/(p - 2) \), \( T \) is a coercive and pseudomonotone operator (\( T \) is the sum of a uniformly monotone, continuous operator and an increasing continuous one). Consequently, by a standard theorem of monotonicity theory the problem \( Tu = 0 \) has a solution \( u \in W^M \), taking into account that \( T \) is also bounded.

We have obtained

**Theorem 1:** Let be \( p, r \geq 2 \), \( p > n \), \( r > (p - 1)/(p - 2) \) and let our assumptions for \( D_i \), \( i = 1, ..., M \), and \( D_0 \) be fulfilled. Then the homogeneous boundary value problem (1), (3) has at least one solution.

2. The inhomogeneous problem

In this section we seek solutions \([v_1, ..., v_M, \Phi]\) of the inhomogeneous boundary value problem (1), (2). We suppose that there exist elements \( y_i \in W_p(G) \), \( i = 1, ..., M \), and \( y_0 \in W_1(G) \) such that
\[v_i|_{\partial \Omega} = y_i, \quad i = 1, ..., M, \quad \text{and} \quad v_0|_{\partial \Omega} = y_0.\]

(2.1)

Then, for \( u_i = v_i - y_i \) and \( \Psi = \Phi - y_0 \) we have the homogeneous conditions (3). Now, we can reduce the inhomogeneous problem to the homogeneous one considered in Section 1. To derive the corresponding operator equation we substitute \( v_i = u_i + y_i \), \( i = 1, ..., M \), and \( \Phi = \Psi + y_0 \) into the original differential equations (1), and we consider these equations in the variables \( u_i \) and \( \Psi \). Then we again generate operator equations in \( W_p(G) \) and \( W_1(G) \). Besides the homogeneous boundary conditions after the substitution we obtain the differential equations
\[\begin{align*}
\nabla \cdot (D_i(\nabla (u_i + y_i))) \nabla (u_i + y_i) + \nabla \cdot (k_i z_i (u_i + y_i) V_0 \nabla (\Psi + y_0)) &= f_i, \\
-\nabla \cdot (D_0(\nabla (\Psi + y_0))) \nabla (\Psi + y_0) &= (a + bv_0^2)^{-1/2} \nabla (N + v_0), \quad i = 1, ..., M
\end{align*}\)

(2.2)

where \( V_0 = (a + bv_0^2)^{-1/2} \) with \( v_0 = z_i (u_i + y_i) + \cdots + z_M (u_M + y_M) \).

Now, let us generate corresponding operators \( E_i \) and \( F_i \). We put
\[(F_i u, w) = I^F(u, w), \quad u \in W, \quad w \in W^M,\]
\[(E_i(u, \Psi), w) = I^E(u, \Psi, w), \quad \Psi \in W_1(G),\]
\[(F_0 \Psi, w_0) = I^F(\Psi, w_0), \quad \Psi, w_0 \in W_1(G)\]
where
\[ I^F_i(u_i, w) = \int_G D_i[(\nabla (u_i + y_i)) \nabla (u_i + y_i)] \nabla w \, dG, \]
\[ I^E_i(u_i, \Psi, w) = \int_G k_i \sigma(u_i + y_i) V_0 \nabla (\Psi + y_0) \nabla w \, dG, \]
\[ I^F(\Psi, w_0) = \int_G D_0[(\nabla (\Psi + y_0)) \nabla (\Psi + y_0)] \nabla w_0 \, dG. \]

We have \(|I^F_i(u_i, w)| \leq d_i \|\nabla (u_i + y_i)\|_p^{p-1} \|w\|_{0,p} \). Consequently, there exists an \( F_i \in (W \rightarrow (W^M)*) \) with
\[ (F_i u_i, w) = I^F_i(u_i, w) \quad \text{and} \quad \|F_i u_i\| \leq d_i 2^{p-2}(\|u_i\|_{0,p}^{p-1} + \|y_i\|_{1,p}^{p-1}). \]

Further, there exists an \( E_0 \in (W_p(G) \rightarrow (W_r(G))*) \) with
\[ (E_0 \Psi, w_0) = I^F(\Psi, w_0) \quad \text{and} \quad \|E_0 \Psi\| \leq 2^{p-2}(\|\Psi\|_{0,p}^{p-1} + \|y_0\|_{0,p}^{p-1}). \]

At last, we justify \((E_i(u_i, \Psi, w)) = I^E_i(u_i, \Psi, w)\). We obtain, with \( 1/p + 1/p' = 1 \),
\[ |I^E_i(u_i, \Psi, w)| \leq a_{1/2} |u_i + y_i| |\nabla (\Psi + y_0)| \|w\|_{0,p} \]
\[ \quad \leq a_{1/2} (|u_i| + |y_i|) (\|\Psi\|_{0,p} + \|y_0\|_{1,p}) \|w\|_{0,p}. \]

Next, we investigate the properties of the operators \( F_i, F_0 \) and \( E_i \). Under the assumptions (4) and (2.1) \( F_i \) is uniformly monotous with \( \delta_i(t) = g_i(t^p) \). This follows from the fact that \( F_i(u_i) \) can be understood as \( A_i(u_i + y_i) \), because \( A_i \) can be generated also on elements \( u_i + y_i \) as an operator in \( u_i \) on \( W_p(G) \). The uniform monotonicity of \( F_i \) then results from the uniform monotonicity of \( A_i \) (cf. (1.6)). Regarding \( F_0 \) we can proceed in an analogous way. Then for \( u \in W^M \) there exists a unique solution (again denoted by) \( S u \) of the problem (2.2) (now with \( v_0 \) instead of \( u_0 \)) under the boundary conditions (3) at \( \Psi \). Obviously, \( S \) is an increasing continuous mapping. We are going to show that, after substituting \( S u \), the operator \( E_i(u) = E_i(u_i, \nabla S u) \) is increasing continuous, too. The inequality
\[ |\langle E_i u^1 - E_i u^2, w \rangle| \]
\[ \leq k_i \int_G |u_i^1 + y_i| V_0^1 \nabla S u^1 \nabla S u^1 \nabla w \, dG \]
\[ \leq k_i \int_G |u_i^1 + y_i| \nabla (S u^1 - S u^2) \nabla w \, dG \]
\[ + \int_G |u_i^1 + y_i| \nabla S u^2 \nabla v_0^1 - v_0^2 \, dG + \int_G |u_i^1 - u_i^2| \nabla S u^2 \nabla w \, dG \]
holds true, where \( k_i \) is a constant and \( v_0^j = v_0(w^j), V_0^j = (a + b(v_0^j)^2)^{-1/2}, j = 1, 2. \)

The increasing continuity then follows in the same way as in Section 1. The solution of the inhomogeneous boundary value problem is equivalent to the solution of the operator equation \( F u + E u = f \), where \( F = [F_1, \ldots, F_M] \) is a uniformly monotous operator and \( E = [E_1, \ldots, E_M] \) is an increasing continuous one.

**Theorem 2:** Under the formulated assumptions, especially (4) on \( D_i \), (2.1) on \( y_i \) and \( r > 2, p > (2r - 1)/(r - 1) \), the boundary value problem (1), (2) has at least one solution. The solution set is strongly compact and weakly closed and its diameter can be estimated from above by a concrete finite number.
Proof: The operator $F + E$ is bounded, coercive and pseudomonotone. The coerciveness follows from the uniform monotonicity of $F$ with $\delta(t) = g^p, p > (2r - 1)/(r - 1)$, and the fact that $(Eu, u)$ must increase to infinity at least as fast as $\|u\|_{L_p^p}^{2r-1}/r-1$. The solution set is weakly compact and weakly closed (cf. e.g. [6]). $F + E$ satisfies the $S_\ast$-condition, i.e. especially that every weakly (to a solution) converging sequence of solutions also converges strongly. From this, the strong compactness of the solution set follows.

Now to the last assertion. Because of the coerciveness of the mapping $F + G$ and the resulting a priori boundedness we have for $v^1, v^2 \in W^M$ and $r > 2$:

$$(Fv^1 - Fv^2, v^1 - v^2) + (Ev^1 - Ev^2, v^1 - v^2) \geq g \|v^1 - v^2\|_{L_p^p}^2 - c_1 \|v^1 - v^2\|_{L_p^p}^2 \geq c_2 \|v^1 - v^2\|_{L_p^p}^{2r-1}(g\|v^1 - v^2\|_{L_p^p}^{p-2} - c_1) - c_2$$

if

$$\|v^1 - v^2\|_{L_p^p} \geq \max \{[(c_1 + \varepsilon_1)/g]^{1/(p-2)}, [(c_2 + \varepsilon_2)/\varepsilon_1]^{1/(r-2)}\} = K(\varepsilon_1, \varepsilon_2)$$

is assumed with arbitrary $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. Hence for solutions $v^1, v^2$ we can conclude $\|v^1 - v^2\|_{L_p^p} \leq K(\varepsilon_1, \varepsilon_2)$.

Remark: The constants $c_1$ and $c_2$ in the proof of Theorem 2 enclose e.g. the embedding constant of $W^M_p(G)$ into $C(G)$.

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