On Classes of Stieltjes Type Operator-Valued Functions with Gaps

V. E. TSEKANOVSKII

We introduce and investigate classes of operator-valued functions with gaps, which can be realized as fractional linear transformations of operator-valued transfer functions of conservative scattering systems.

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Classes of Stieltjes type operator-valued functions with gaps on the positive semi-axis (i.e., with intervals of holomorphy and definiteness) are considered. We prove criteria that a given function, whose values are operators in a finite-dimensional Hilbert space, belongs to these classes. Moreover, we investigate classes of Stieltjes type operator-valued functions which admit a realization, i.e., which can be represented as fractional linear transformations of operator-valued transfer functions of conservative scattering systems of the form

$$\Theta = (\mathcal{S}_+ \subset \mathcal{S} \subset \mathcal{S}_-, A_1, K, I, E)$$

where $A_1 \in [\mathcal{S}_+ \subset \mathcal{S}_-], \quad \text{Im}A_1 = KK^*, A_1 \supset T \supset A, A_1^* \supset T^* \supset A$, $A$ is a closed Hermitian operator in $\mathcal{S}_+$ and $T$ is closed with dense domain of definition in $\mathcal{S}$.

In the class of realizable Stieltjes type operator-valued functions the following subclasses are investigated:

1. the subclass, where $\mathcal{D}(A) = \mathcal{S}_+, \mathcal{D}(T) \supset \mathcal{D}(T^*)$
2. the subclass, where $\mathcal{D}(A) = \mathcal{S}_-, \mathcal{D}(T) \supset \mathcal{D}(T^*)$
3. the subclass, where $\mathcal{D}(A) = \mathcal{S}_+ \cap \mathcal{S}_-, \mathcal{D}(T) = \mathcal{D}(T^*)$.

We prove analytical criteria for a given operator-valued function to belong to the mentioned subclasses (with gaps). These criteria are analoga, supplements, and refinements of some of the results stated by M.G. Krein and A.A. Nudel'man [7].

§ 1 The classes $\mathcal{S}_\triangleleft \bigcup_{j=1}^m (\alpha_j, \beta_j)$ of operator-valued functions

According to M.G. Krein [8], a function $V(z)$, whose values are operators in a finite-dimensional Hilbert space $E$, will be called a Stieltjes type operator-valued function if the following conditions hold:

1. $V(z)$ is holomorphic on $\text{Ext}[0, \infty) := \{z: z \notin [0, \infty)\}$
2. $V(z) \geq 0$ for $z < 0$
3. $V(z)$ is an operator-valued R-function, i.e., $\text{Im}V(z) / \text{Im}z \geq 0$.

The class of Stieltjes type operator-valued functions will be denoted by $S$.

Let $\{(\alpha_j, \beta_j)\}_{j=1}^m$ be a system of mutually disjoint intervals on the positive semi-axis.
Definition: By $S_k[\bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$ we denote the class of functions $V(z)$, whose values are operators in a finite-dimensional Hilbert space $E$, such that the following two conditions hold:
1. $V(z) \in S$.
2. $V(z)$ is holomorphic and positive on all intervals $(\alpha_j, \beta_j)$, i.e., $(V(z)f, f) > 0$ for all $f \in E$, $f \neq 0$, and all $z \in \alpha_j, \beta_j$; $(V(z)f, f)$ is holomorphic and negative on the intervals $(\alpha_j, \beta_j)$, respectively, i.e., $(V(z)f, f) < 0$ for all $f \in E$, $f \neq 0$, and all $z \in (\alpha_j, \beta_j)$.

Theorem 1: A scalar function $V(z)$ belongs to the classes $S_k[\bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$ if and only if the following two conditions hold:
(i) $V(z) \in S$.
(ii) $\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z}V(z) \in S \left( \prod_{j=1}^{m} \frac{\alpha_j - z}{\beta_j - z}V(z) \in S, \text{ respectively} \right)$.

Proof: First we consider the class $S_k[\bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$. Let (i) and (ii) be fulfilled. Since (i), a well known theorem (see [7]) gives us
\[
V(z) = c \exp \left( \int_{-\infty}^{\infty} \left( \frac{1}{1 + t^2} - \frac{t}{1 + t^2} \right) f(t) \, dt \right),
\]
where $c > 0$, $f(t)$ is a summable function such that $0 \leq f(t) \leq 1$ a.e. and $\int_{-\infty}^{\infty} (1 + t^2)^{-1} f(t) \, dt < \infty$. Moreover, the representation (2) is unique. It is not hard to see that
\[
\frac{\beta_j - z}{\alpha_j - z} = \frac{c_j}{\alpha_j - z} \exp \left( \int_{-\infty}^{\infty} \left( \frac{1}{1 + t^2} - \frac{t}{1 + t^2} \right) f(t) \, dt \right) .
\]
Since (ii), we get
\[
\prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z}V(z) = c_1 \exp \left( \int_{-\infty}^{\infty} \left( \frac{1}{1 + t^2} - \frac{t}{1 + t^2} \right) f_1(t) \, dt \right) ,
\]
in an analogous way, where $c_1 > 0$ and the function $f_1(t)$ has the same properties as $f(t)$. Using (3) and (4), we obtain
\[
V(z) = c_2 \exp \left( \int_{-\infty}^{\infty} \left( \frac{1}{1 + t^2} - \frac{t}{1 + t^2} \right) f_2(t) \, dt \right) ,
\]
where
\[
f_2(t) = \begin{cases} f_1(t) & \text{for } t \in \mathbb{R} \setminus \bigcup_{j=1}^{m}(\alpha_j, \beta_j) \\ f_1(t) - 1 & \text{for } t \in \bigcup_{j=1}^{m}(\alpha_j, \beta_j) \end{cases} .
\]
Because of the uniqueness of the representation (2) it follows
\[
V(z) = c \exp \int_{\mathbb{R} \setminus \bigcup_{j=1}^{m}(\alpha_j, \beta_j)} \left( \frac{1}{1 + t^2} - \frac{t}{1 + t^2} \right) f(t) \, dt ,
\]
where $c > 0$, $0 \leq f(t) \leq 1$ a.e. on $\mathbb{R} \setminus \bigcup_{j=1}^{m}(\alpha_j, \beta_j)$. By a well known theorem (see [7]), the relation (5) implies that $V(z)$ is holomorphic and positive on all intervals $(\alpha_j, \beta_j)$.

Now assume that $V(z) \in S_k[\bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$. Then $V(z) \in S_k[\bigcup_{j=1}^{m}(\alpha_j, \beta_j)]$. We will show that the inclusion $((\beta_j - z)/(\alpha_j - z))V(z) \in S$ is true. In fact, setting $\xi = (\beta_j - z)/(\alpha_j - z)$ and $V_\xi(\xi) = V(z)$
we get
\[ \frac{\Im V_1(\xi)}{\Im \xi} = \frac{|\alpha_1 - z|^2}{\beta_1 - \alpha_1} \frac{\Im V(z)}{\Im z} \geq 0, \]
hence, \( V_1(\xi) \in R \), i.e., \( V_1(\xi) \) is an R-function. It is not hard to see that \( z \in (\alpha_1, \beta_1) \) implies \( \xi \in (-\infty, 0) \), and since \( V(z) \in S_{[\alpha_1, \beta_1]} \), it follows \( V_1(\xi) \in S \). Now a theorem of M.G. Krein (see [8]) gives us \( \xi V_1(\xi) \in R \). Thus \( ((\beta_1 - z)/(\alpha_1 - z))V(z) \in R \). Since \( ((\beta_1 - z)/(\alpha_1 - z))V(z) \geq 0 \) if \( z \in (-\infty, 0) \), we obtain \( ((\beta_1 - z)/(\alpha_1 - z))V(z) \in S \).

We will show that the implication
\[ \prod_{j=1}^{k} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S \quad (1 < k < m) \]
is true. In fact, it is not hard to see that
\[ \prod_{j=1}^{k} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S \left[ (\alpha_{k+1}, \beta_{k+1}) \right]. \]
Now by analogous arguments as above we get
\[ \prod_{j=1}^{k+1} \frac{\beta_j - z}{\alpha_j - z} V(z) = \prod_{j=1}^{k+1} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S. \]
Thus the first part of the theorem is proved.

Now let \( V(z) \in S \left[ \bigcup_{j=1}^{m}(\alpha_j, \beta_j) \right] \). We will show that the \( (\cdot) \)-part of (ii) is true. It is not hard to see (cf. [6]) that \( V(z) \in R \) if and only if \( -V(z)^{-1} \in R \). Thus, the relation \( V(z) \in S_{[\alpha_1, \beta_1]} \) implies \( -V(z)^{-1} \in R \) and \( V(z)^{-1} > 0 \), \( z \in (\alpha_1, \beta_1) \). Setting \( \xi = (\beta_1 - z)/(\alpha_1 - z) \) and \( V_1(\xi) = -V(z)^{-1} \), we get
\[ \frac{\alpha_1 - z}{\beta_1 - z} \left( -\frac{1}{V(z)} \right)^{-1} = \frac{\alpha_1 - z}{\beta_1 - z} V(z) \in R. \]
Since \( \frac{\alpha_1 - z}{\beta_1 - z} V(z) \geq 0 \) if \( z \in (-\infty, 0) \), it follows \( \frac{\alpha_1 - z}{\beta_1 - z} V(z) \in S \). Using an analogous induction method as in the first part of the proof we obtain the \( (\cdot) \)-part of (ii).

Now assume (i) and (ii)/(\cdot) -part. Consider the function \( -V(z)^{-1} \in R \) and use analogous arguments as in the proof of the sufficiency in the first part. This gives us that \( -V(z)^{-1} \) is holomorphic and positive on all intervals \((\alpha_j, \beta_j)\). Thus the theorem is proved.

**Theorem 2:** A function \( V(z) \), whose values are operators in a finite-dimensional Hilbert space \( E \), belongs to the classes \( S_{\left[ \bigcup_{j=1}^{m}(\alpha_j, \beta_j) \right]} \) if and only if the following two conditions hold:

(i) \( V(z) \in S \).

(ii) \( \prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S \left( \prod_{j=1}^{m} \frac{\alpha_j - z}{\beta_j - z} V(z) \in S \right. \), respectively \).

**Proof:** Let \( V(z) \in S_{\left[ \bigcup_{j=1}^{m}(\alpha_j, \beta_j) \right]} \). Considering the scalar function \( (V(z)f, f) \in S \) and using Theorem 1 we get
\[ \prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} (V(z)f, f) = \left( \prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z)f, f \right) \in S. \]
Hence, the operator-valued function \( \prod_{j=1}^{m} \frac{\beta_j - z}{\alpha_j - z} V(z) \) belongs to \( S \).
The sufficiency of conditions (i) and (ii) is trivial. The proof for the class $S_{\mathbb{U}(x,13)} \cap S_{\mathbb{U}(c,d)}$ is analogous. Thus the theorem is proved.

**Definition:** We will say that a function $V(z)$, whose values are operators in a finite-dimensional Hilbert space $E$, belongs to the class

$$S_{\mathbb{U}(x,13)} \cap S_{\mathbb{U}(c,d)}$$

if the following three conditions hold:
1. $V(z) \in S$.
2. $V(z)$ is holomorphic and positive on the intervals $(\alpha_j, \beta_j)$ ($j = 1, \ldots, m$).
3. $V(z)$ is holomorphic and negative on the intervals $(c_k, d_k)$ ($k = 1, \ldots, n$).

Theorem 2 immediately implies the following

**Theorem 3:** A function $V(z)$, whose values are operators in a finite-dimensional Hilbert space $E$, belongs to the class $S_{\mathbb{U}(x,13)} \cap S_{\mathbb{U}(c,d)}$ if and only if the following two conditions hold:
1. $V(z) \in S$.
2. $V(z) \in S_{\mathbb{U}(x,13)} \cap S_{\mathbb{U}(c,d)}$.

§ 2 Realizable operator-valued functions of the class $S_{\mathbb{U}(x,13)}$

Let $A$ be a closed Hermitian operator in a Hilbert space $\mathcal{H}$, whose defect numbers are finite and coincide. This operator can be considered as acting from $\mathcal{H}_0 = \mathcal{H}(A)$ into $\mathcal{H}$. Let $A^*$ be the adjoint operator. Clearly, $\mathcal{D}(A^*) = \mathcal{H}$ (where the closure is taken in $\mathcal{H}$). We set $\mathcal{H}_+ = \mathcal{D}(A^*)$ and introduce the scalar product $(f, g)_+ = (f, g) + (A^* f, A^* g)$ ($f, g \in \mathcal{H}_+$. We consider the rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_+$. (cf. [2]).

We will say that a closed and densely defined operator $T$ in $\mathcal{H}$ belongs to the class $\Omega_A$ if the following two conditions are fulfilled.
1. $T \supset A$, $T^* \supset A$ ($A$ is closed and Hermitian)
2. $-i$ is a regular point of $T$

are fulfilled.

A bounded operator $A_1 : \mathcal{H}_+ \rightarrow \mathcal{H}_-$ (i.e., $A_1 \in [\mathcal{H}_+, \mathcal{H}_-]$) is called a biextension of the Hermitian operator $A$ if $A_1 \supset A$ and $A_1^* \supset A$. Identifying the dual space of $\mathcal{H}_+$ with $\mathcal{H}_-$, we see that $A_1^* \in [\mathcal{H}_-, \mathcal{H}_+]$. If $A_1 = A_1^*$, then $A_1$ is called a selfadjoint biextension of $A$.

By $\hat{\mathcal{A}}$ we denote the restriction of $A_1$ to $\mathcal{D}(\hat{\mathcal{A}}) = \{ f \in \mathcal{H}_+ : A_1 f \in \mathcal{H} \}$. It is called a quasikernel of $A_1$ (cf. [10, 11]). A selfadjoint biextension is called a strong biextension if $\hat{\mathcal{A}} = \hat{\mathcal{A}}^*$ (cf. [10, 11]).

Let $T \in \Omega_A$. Then $A_1 \in [\mathcal{H}_+, \mathcal{H}_-]$ is called a $(*)$-extension of $T$ if

$$A_1 \supset T \supset A, A_1^* \supset T^* \supset A.$$ (6)

Moreover, if $A_1 = (A_1 + A_1^*)/2$ is a strong selfadjoint biextension, then $A_1$ is called a correct $(*)$-extension of $T$.

By definition, the class $\Lambda_A$ denotes the set of all operators $T \in \Omega_A$ such that $A$ coincides with the maximal common Hermitian part of $T$ and $T^*$. 
Definition: The operator colligation
\[ \Theta = \begin{pmatrix} A & K & J \\ \hat{S}_+ & \hat{S} & \hat{S}_- \\ E \end{pmatrix} \] (7)
is called rigged if the following four conditions hold:
1. \( J = J^* = J^{-1} \) (dim \( E < \infty \)).
2. \( K \) is a bounded linear operator from \( E \) into \( \hat{S}_- \).
3. \( A \) is a correct (*)-extension of \( T \in \Lambda_A \), and
   \[ \text{Im} A = (A^* - A)/2i = KJ^*K^*. \] (8)
4. The ranges of \( K \) and \( \text{Im} A \) coinside.

The operator-valued function
\[ W_\Theta(z) = I - 2iK(A - zI)^{-1}J \] (9)
is called a Livsic type characteristic function of the colligation \( \Theta \).

Furthermore, we introduce the function
\[ V_\Theta(z) = K^*(A - zI)^{-1}K. \] (10)
It is well known (cf. [3,9]) that the functions \( V_\Theta(z) \) and \( W_\Theta(z) \) are associated by the relations
\[ V_\Theta(z) = i(W_\Theta(z) + I)^{-1}(W_\Theta(z) - I)J \quad \text{and} \quad W_\Theta(z) = (I + iV_\Theta(z))^{-1}(I - iV_\Theta(z))J. \] (11)

We consider the conservative system (cf. [9])
\[ (A - zI)x = KJ\varphi_-, \] (12)
where \( x \in \hat{S}_-, \varphi_- \in E \), \( \varphi_- \) is the so-called input vector, \( \varphi_+ \) is the output vector, and \( x \) is the inner state. It is not hard to see that the transfer function \( \Pi(z) \) of such a system (i.e., \( \varphi_+ = \Pi(z)\varphi_- \)) coincides with the operator function \( W_\Theta(z) \). If \( J + I \), then the system is called a crossing system and if \( J = I \), it is called a scattering system (cf. [1]). In the following we will write a conservative system \( \Theta \) in the form of a rigged operator colligation.

Definition: A function \( V(z) \), whose values are operators in a finite-dimensional Hilbert space \( E \), is called realizable if it can be represented as
\[ V(z) = V_\Theta(z) = K^*(A - zI)^{-1}K = i(W_\Theta(z) + I)^{-1}(W_\Theta(z) - I), \] (13)
where \( \Theta \) is a conservative scattering system of the form (7).

Theorem 4: Let \( V(z) \) be a realizable function, whose values are operators in a finite-dimensional Hilbert space \( E \), i.e., \( V(z) = K^*(A - zI)^{-1}K \). Let \( A > 0 \) and let \((\alpha, \beta)\) be an arbitrary interval of the positive semi-axis. Then \( V(z) \) belongs to the class \( \mathcal{S}_\alpha(\alpha, \beta) \) if and only if the following two conditions hold:
1. \( A_{1R} \geq 0 \).
2. For an arbitrary set \( \{z_i\}_{i=1}^P \) of non-real complex numbers such that \( z_i \neq \bar{z}_i \) and for all \( \varphi_i \in N_{z_i} \) (where \( N_{z_i} \) is the deficiency space of \( A \)) it holds
\[ \sum_{i, j=1}^P (B(z_i, z_j)\varphi_i, \varphi_j) \geq 0, \] (15)
where
\[ B(\lambda, \mu) = \frac{\beta - \alpha}{(\alpha - \mu)(\alpha - \mu)} A_1^R + \frac{\alpha\beta - \beta(\lambda + \mu) + \lambda\mu}{(\alpha - \lambda)(\alpha - \mu)} I. \] (16)

**Proof:** Assume that the conditions (14) and (15) hold. We will show that \( V(z) \in S, [\alpha, \beta] \). Since \( V(z) \) is realizable, there exists a conservative scattering system
\[ \Theta = \begin{pmatrix} A_1 & K \\ S_+ \subset \mathcal{S}_- \subset S_- \end{pmatrix} \]
such that \( V(z) = V_0(z) = K^*(A^R_1 - zI)^{-1}K \). The operator \( A_1 \) is a \((\ast)\)-extension of some densely defined closed operator \( T \), i.e., condition (6) holds, where \( A \) is the common maximal Hermitian part of \( T \) and \( T^* \). Let \( N_z \) be the deficiency space of the Hermitian operator \( A \) and let \( \{z_i\}_{i=1}^p \) be an arbitrary set of non-real complex numbers such that \( z_i \neq \overline{z_i} \). Moreover, let \( \varphi_i \in N_{z_i} \). According to [11], there exists a vector \( h_i \in E \) such that
\[ \varphi_i = (A^R_1 - zI)^{-1}Kh_i \quad (i = 1, \ldots, p). \] (17)

Set \( w_i = (\beta - z_i)/(\alpha - z_i) \). We will prove the inequality
\[ \sum_{i, l=1}^p \left( \frac{w_i V(z_i) - \overline{w_l} V(z_l)}{z_i - z_l} \right) h_i, h_l \geq 0. \] (18)

In fact,
\[ \sum_{i, l=1}^p \left( \frac{w_i V(z_i) - \overline{w_l} V(z_l)}{z_i - z_l} \right) h_i, h_l \]
\[ = \sum_{i, l=1}^p \left( \frac{w_i (A^R_1 - z_i I)^{-1} - \overline{w_l} (A^R_1 - z_l I)^{-1}}{z_i - z_l} \right) Kh_i, Kh_l \]
\[ = \sum_{i, l=1}^p \left( (A^R_1 - z_l I)^{-1} (w_i (A^R_1 - z_l I) - \overline{w_l} (A^R_1 - z_l I)) (A^R_1 - z_i I)^{-1} \right) \varphi_i, \varphi_l \]
\[ = \sum_{i, l=1}^p \left( \frac{\beta - \alpha}{(\alpha - z_l)(\alpha - z_l)} A_1^R + \frac{\alpha\beta - \beta(\lambda + \mu) + \lambda\mu}{(\alpha - \lambda)(\alpha - \mu)} I \right) \varphi_i, \varphi_l \]
\[ = \sum_{i, l=1}^p \left( B(z_j, z_l) \varphi_i, \varphi_l \right) \geq 0. \]

Setting \( p = 1 \), \( z_i = z \), \( h_i = h \), we obtain from (17) and (18)
\[ \left( \frac{\beta - \alpha}{(\alpha - z) V(z) - \frac{\beta - \alpha}{\alpha - z} \overline{V(z)}}{z - \overline{z}} \right) h, h \geq 0 \quad \text{and hence} \quad \text{Im} \left( \frac{\beta - \alpha}{(\alpha - z) V(z)} \right) \geq 0, \]
\[ \text{i.e., the operator-valued function} \ ((\beta - \alpha)(\alpha - z) V(z)) \text{is an operator-valued R-function. Now condition (14) implies that} \ V(z) \in S, [\alpha, \beta] \]. Since for \( z < 0 \) we have \( (\beta - z)(\alpha - z) \geq 0 \) it holds \( (\beta - z)(\alpha - z) V(z) \in S \). By Theorem 2 we get \( V(z) \in S, [\alpha, \beta] \).
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Now assume \( V(z) \in S_\mathcal{C}[\alpha, \beta] \). Thus \( V(z) \in S \) and hence \( A_{IR} \geq 0 \) (cf. [4]). It remains to prove (15). In fact, by Theorem 2 the condition \( V(z) \in S_\mathcal{C}[\alpha, \beta] \) implies that \( (\beta - z)(\alpha - z)V(z) \in S \). Thus, the operator-valued function \( (\beta - z)(\alpha - z)V(z) \) has a representation of the form

\[
\frac{\beta - z}{\alpha - z} V(z) = \gamma + \int_0^\infty \frac{1}{t - z} \, d\sigma(t),
\]

(19)

where \( \gamma \geq 0 \) if \( \sigma(t) \) is a non-decreasing operator-valued function in \( E \) such that \( \int_0^\infty (1 + t)^{-1} \, d\sigma(t) < \infty \).

Let \( \{ z_i \}_{i=1}^P \) be an arbitrary set of non-real complex numbers such that \( z_i \neq \overline{z}_i \). Let \( \varphi_i \in \mathbb{N}_{z_i} \). Then according to [11] there exist vectors \( h_i \in E \) such that (17) holds. Setting \( w_i = (\beta - z_i)(\alpha - z_i) \) we obtain

\[
\sum_{i,j=1}^P \left( w_i V(z_i) - \overline{w}_j V(z_j) \right) h_i, h_j = \sum_{i,j=1}^P \left( \int_0^\infty \frac{1}{(t - z_i)(t - \overline{z}_j)} \, d\sigma(t) h_i, h_j \right) \geq 0.
\]

It is clear from the proof of the sufficiency part that the inequalities (15) and (18) are equivalent. Thus, the above inequality yields the needed result.

Remark: If there is no gap, i.e., if \( \alpha = \beta \), the inequality (15) holds trivially and we obtain the results of [4].

**Theorem 5:** Let \( V(z) \) be a realizable function, whose values are operators in a finite-dimensional Hilbert space \( E \), i.e., \( V(z) = K^*(A_{IR} - zI)^{-1}K \). Let \( (\alpha, \beta) \) be an arbitrary interval of the positive semi-axis. Then \( V(z) \) belongs to the class \( S_\mathcal{C}[\alpha, \beta] \) if and only if the following two conditions hold:

1. \( A_{IR} \geq 0 \).
2. For an arbitrary set \( \{ z_i \}_{i=1}^P \) of non-real complex numbers such that \( z_i \neq z_j \) and for all \( \varphi_i \in \mathbb{N}_{z_i} \) (where \( \mathbb{N}_{z_i} \) is the deficiency space of \( A \)) it holds

\[
\sum_{i=1}^P (B(z_i, z_i) \varphi_i, \varphi_i) \geq 0,
\]

where

\[
B(\lambda, \mu) = \frac{\alpha - \beta}{(\beta - \lambda)(\beta - \mu)} A_{IR} + \frac{\alpha \beta - \alpha(\lambda + \mu) + \lambda \mu}{(\beta - \lambda)(\beta - \mu)} I.
\]

(20)

**Proof:** This theorem can be proved in the same way as Theorem 4 (set \( w_i = (\alpha - z_i)(\beta - z_i) \)).

Note further that in the case \( \alpha = \beta \) (i.e., if there are no gaps) Theorem 5 is an extension of results of [4]. Moreover, it is not hard to see that the above used method allows us to obtain analogous results for the classes

\[
S_\mathcal{C}[\bigcup_{j=1}^m (\alpha_j, \beta_j)] \text{ and } S_\mathcal{C}[\bigcup_{j=1}^m (\alpha_j, \beta_j)] \cap S_\mathcal{C}[\bigcup_{k=1}^n (c_k, d_k)].
\]

In fact, we have the following

**Theorem 6:** Let \( V(z) \) be a realizable function, whose values are operators in a finite-dimensional Hilbert space \( E \), i.e., \( V(z) = K^*(A_{IR} - zI)^{-1}K \). Let \( (\alpha_j, \beta_j) \) \( (j = 1, \ldots, m) \) and \( (c_k, d_k) \) \( (k = 1, \ldots, n) \) be arbitrary mutually disjoint intervals of the positive semi-axis. Then \( V(z) \) belongs to the class \( S_\mathcal{C}[\bigcup_{j=1}^m (\alpha_j, \beta_j)] \cap S_\mathcal{C}[\bigcup_{k=1}^n (c_k, d_k)] \) if and only if the following two conditions hold:
1. $A|_{R} \geq 0$.
2. For an arbitrary set $\{z_i\}_{i=1}^{P}$ of non-real complex numbers such that $z_i \neq \overline{z}_i$ and for all $\varphi_i \in N_{z_i}$ (where $N_{z_i}$ is the deficiency space of $A$) it holds
\[
\sum_{i, l=1}^{P} (B(z_i, z_l) \varphi_i, \varphi_l) \geq 0,
\]
where
\[
B(\lambda, \mu) = \frac{w(\lambda) - w(\mu)}{\lambda - \mu} A|_{R} + \frac{\lambda w(\mu) - \mu w(\lambda)}{\lambda - \mu}.
\] (21)
and
\[
w(\lambda) = \prod_{j=1}^{m} \frac{\beta_j - \lambda}{\alpha_j - \lambda} \prod_{k=1}^{n} \frac{c_k - \lambda}{d_k - \lambda}.
\] (22)

§ 3 Some subclasses of realizable Stieltjes type operator-valued functions with gaps

By a result of M. G. Krein (see [8]) each Stieltjes type function $V(z)$, whose values are operators in a finite-dimensional Hilbert space $E$, can be represented in the form
\[
V(z) = \gamma + \int_{0}^{\infty} \frac{d\sigma(t)}{\sigma(t - z)}.
\] (23)
where $\gamma \geq 0$, $\sigma(t)$ is a non-decreasing operator-valued function in $E$ such that $\int_{0}^{\infty} (1 + t)^{-1} d\sigma(t) < \infty$. According to [5], we introduce the following notion.

**Definition:** We will say that a Stieltjes type function $V(z)$, whose values are operators in a finite-dimensional Hilbert space $E$, belongs to the class $S(R)$, if $\gamma f = 0$ for all $f$ of the subclass
\[
E_{\gamma} = \left\{ f \in E : \int_{0}^{\infty} (d\sigma(t) f, f)_{E} < \infty \right\}.
\] (24)
As it was proved in [5], each operator-valued function $V(z) \in S(R)$ can be realized by a conservative scattering system $\Theta$, i.e., it holds (13).

**Definition:** Following [5], we introduce the following subclasses of $S(R)$:
(i) The class $S^{0}(R)$ consisting of all $V(z) \in S(R)$ such that
\[
\int_{0}^{\infty} (d\sigma(t) f, f) = \infty \quad (f \in E, f \neq 0).
\] (25)
(ii) The class $S^{1}(R)$ consisting of all $V(z) \in S(R)$ such that $\gamma = 0$ and
\[
\int_{0}^{\infty} (d\sigma(t) f, f) < \infty \quad (f \in E)
\] (26)
in the representation (23).
(iii) The class $S^{01}(R)$ consisting of all $V(z) \in S(R)$ such that $E_{\gamma} = \{0\}$ and $E_{\gamma} = E$. 
It is not hard to see that
\[ S(R) = S^0(R) \cup S'(R) \cup S^{0\prime}(R). \] (27)

**Definition:** We introduce the following subclasses of \( S(R), S^0(R), S'(R) \) and \( S^{0\prime}(R) \).

(i) The class \( S_\pm[R, \cup_j \mu_j(a_j, \beta_j)] \) consisting of all \( V(z) \in S(R) \) such that \( V(z) \) is holomorphic and positive (negative) on all intervals \( (a_j, \beta_j) \).

(ii) The class \( S^0_\pm[R, \cup_j \mu_j(a_j, \beta_j)] \) consisting of all \( V(z) \in S^0(R) \) such that \( V(z) \) is holomorphic and positive (negative) on all intervals \( (a_j, \beta_j) \).

(iii) The class \( S^{0\prime}_\pm[R, \cup_j \mu_j(a_j, \beta_j)] \) consisting of all \( V(z) \in S^{0\prime}(R) \) such that \( V(z) \) is holomorphic and positive (negative) on all intervals \( (a_j, \beta_j) \).

(iv) The class \( S_{\pm}^{0\prime}[R, \cup_j \mu_j(a_j, \beta_j)] \) consisting of all \( V(z) \in S^{0\prime}(R) \) such that \( V(z) \) is holomorphic and positive (negative) on all intervals \( (a_j, \beta_j) \).

Let \( \Theta \) be a conservative scattering system of the form (7) such that \( V_\Theta(z) = V(z) \) and let \( A \) and \( T \) be the operators of (6). Then (cf. [5])

\[ \mathcal{D}(A) = S_\Theta, \quad \mathcal{D}(T) \ast \mathcal{D}(T^*) \text{ if } V(z) \in S^0_\pm[R, \cup_j \mu_j(a_j, \beta_j)], \]

\[ \mathcal{D}(A) = S_\Theta, \quad \mathcal{D}(T) = \mathcal{D}(T^*) \text{ if } V(z) \in S^{0\prime}_\pm[R, \cup_j \mu_j(a_j, \beta_j)], \]

\[ \mathcal{D}(A) = S_\Theta, \quad \mathcal{D}(T) = \mathcal{D}(T^*) \text{ if } V(z) \in S_{\pm}^{0\prime}[R, \cup_j \mu_j(a_j, \beta_j)]. \]

**Theorem 7:** A function \( V(z) \), whose values are operators in a finite-dimensional Hilbert space \( E \), belongs to the class \( S^0_\pm[R, (a, \alpha)] \) if and only if the following two conditions hold:

(i) \( V(z) \in S^0(R) \).

(ii) \( \frac{\beta - z}{\alpha - z} V(z) \in S^0(R) \left( \frac{\alpha - z}{\beta - z} V(z) \in S^0(R) \right. \) respectively \( \left. \right) \) (28)

**Proof:** Assume that the conditions (28) hold. Since \( S^0(R) \in S \), we have \( V(z) \in S^0_\pm[R, (a, \alpha)] \) by Theorem 2 and, hence, \( V(z) \in S^0_\pm[R, (a, \alpha)] \) because \( V(z) \in S^0(R) \). Conversely, assume that \( V(z) \in S^0_\pm[R, (a, \alpha)] \). Then, clearly, \( V(z) \in S^0(R) \). It remains to show the first inclusion of (ii). It is well known that

\[ \int_0^\infty (d \sigma(t), f, f) = \lim_{\eta \to \infty} (\eta \text{Im} V(\eta \text{Im}) f, f), \] (29)

where \( \sigma(t) \) is the operator-valued measure of the representation (23) (cf. [6]). The definitive of \( S^0(R) \) and (29) imply

\[ \lim_{\eta \to \infty} (\eta \text{Im} V(\eta \text{Im}) f, f) = \infty. \] (30)

Since \( V(z) = K^*\left(\mathcal{A}_R - z I\right)^{-1}K \), we obtain

\[ \text{Im} V(z) = \text{Im} z K^*\left(\mathcal{A}_R - z I\right)^{-1}\left(\mathcal{A}_R - z I\right)^{-1}K. \] (31)

Set

\[ f_\eta = \left(\mathcal{A}_R - \eta I\right)^{-1}K f. \] (32)

From (30) - (32) it follows

\[ \lim_{\eta \to \infty} \eta^2(f_\eta, f_\eta) = \lim_{\eta \to \infty} \eta^2(\left(\mathcal{A}_R - \eta I\right)^{-1}K f, (\mathcal{A}_R - \eta I)^{-1}K f) \] (33)
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\[ \lim_{\eta \to \infty} \eta^2 (K^* (A_1 R + i \eta I)^{-1} (A_1 R - i \eta I)^{-1} K f, f) = \lim_{\eta \to \infty} (\eta \text{Im} V(\eta) f, f) = \infty. \]

We will show that

\[ \lim_{\eta \to \infty} \left( \eta \text{Im} \frac{\beta - \eta}{\alpha - \eta} V(\eta) f, f \right) = \infty. \]  

(34)

In fact, setting \( z_1 = z_2 = \eta \) in the inequality (18) and regarding the considerations of the proof of this inequality we obtain

\[ \lim_{\eta \to \infty} \left( \eta \text{Im} \frac{\beta - \eta}{\alpha - \eta} V(\eta) f, f \right) = \lim_{\eta \to \infty} \left( \eta \frac{\beta - \alpha}{2 \eta} (A_1 R f_{\eta}, f_{\eta}) + \frac{\eta \alpha \beta}{\alpha^2 + \eta^2} (f_{\eta}, f_{\eta}) + \frac{\eta^2}{\alpha^2 + \eta^2} (f_{\eta}, f_{\eta}) \right) \]

\[ \geq \lim_{\eta \to \infty} \frac{\eta^2}{\alpha^2 + \eta^2} (f_{\eta}, f_{\eta}) = \lim_{\eta \to \infty} \frac{\eta^2}{\alpha^2 + \eta^2} \eta^2 (f_{\eta}, f_{\eta}) = \infty. \]

We mention that we have also used Theorem 4 \((A_1 R \geq 0)\) and (33).

Finally, assume that \( V(z) \in S^0(\mathbb{R}) \). We will show that \((\alpha - z)/(\beta - z) V(z) \in S^0(\mathbb{R})\).

Setting \( w = (\alpha - \eta \beta - \eta) \) and using (13) we get

\[ (\text{Im} w V(\eta) f, f) = \frac{1}{2} \left( (w K^* (A_1 R - \eta I)^{-1} \eta - w K^* (A_1 R + \eta I)^{-1} \eta) f, f \right) \]

\[ = \frac{1}{2i} \left( (w (A_1 R - \eta I)^{-1} \eta - w (A_1 R + \eta I)^{-1} \eta) K f, K f \right) \]

\[ = \left( \frac{w - \eta}{2i} A_1 R f_{\eta}, f_{\eta} \right) + \frac{w - \eta}{\beta - \eta} (f_{\eta}, f_{\eta}) \]

\[ = \eta (\alpha - \beta) (A_1 R f_{\eta}, f_{\eta}) + \frac{\eta (\alpha \beta + \eta^2)}{\beta - \eta} (f_{\eta}, f_{\eta}), \]

where \( f_{\eta} \) has the form (32). But \((A_1 R f_{\eta}, f_{\eta}) = (\text{Im} \text{Im} V(\eta) f, f)\). In fact,

\[ \left( \text{Im} \text{Im} V(\eta) f, f \right) = \left( \text{Im} V(\eta) + \text{Im} V(-\eta) f, f \right) \]

\[ = \left( \frac{\text{Im} K^* (A_1 R - \eta I)^{-1} \eta + \text{Im} K^* (A_1 R + \eta I)^{-1} \eta} {2i \eta} f, f \right) \]

\[ = \left( \frac{(A_1 R + \eta I)^{-1} \eta (A_1 R + \eta I) + \eta (A_1 R - \eta I) (A_1 R - \eta I)^{-1} \eta}{2i \eta} f, f \right) \]

\[ = (A_1 R f_{\eta}, f_{\eta}). \]

Regarding (23) we obtain

\[ \left( \text{Im} \text{Im} V(\eta) f, f \right) = (\gamma f, f) + \frac{\infty}{\beta} \int_{t^2 + \eta^2} (d \sigma(t) f, f). \]

Using Lebesgue's Dominated Convergence Theorem (cf. [6]), it follows

\[ \lim_{\eta \to \infty} (A_1 R f_{\eta}, f_{\eta}) = \lim_{\eta \to \infty} \left( \text{Im} \text{Im} V(\eta) f, f \right) = (\gamma f, f) < \infty. \]  

(36)

Now (35) and (36) imply

\[ \lim_{\eta \to \infty} \left( \eta \text{Im} \frac{\beta - \eta}{\alpha - \eta} V(\eta) f, f \right) = \lim_{\eta \to \infty} \left( \eta^2 (\alpha - \beta) (A_1 R f_{\eta}, f_{\eta}) + \frac{\eta^2 \alpha \beta}{\beta^2 + \eta^2} (f_{\eta}, f_{\eta}) + \frac{\eta^2}{\beta^2 + \eta^2} \eta^2 (f_{\eta}, f_{\eta}) \right) \]
Thus the theorem is proved.

**Theorem 8:** A function $V(z)$, whose values are operators in a finite-dimensional Hilbert space $E$, belongs to the class $S^1_1[R, (\alpha, \beta)]$ if and only if the following two conditions hold:

(i) $V(z) \in S^1_1(R)$.

(ii) $\frac{\beta - z}{\alpha - z} V(z) \in S^1_1(R)$, respectively.

**Proof:** The sufficiency is obvious (compare the proof of Theorem 7). Now assume that $V(z) \in S^1_1[R, (\alpha, \beta)]$. Clearly, $V(z) \in S^1_1(R)$. We will show that $(\frac{\beta - z}{\alpha - z})V(z) \in S^1_1(R)$. Since $V(z)$ is realizable, the relation (33) holds. In this relation, the operator $A_{1R}$ is a bounded linear operator from $S^1_1$ into $S^1_1$. Let $R$ be the (isometric) Riesz-Berezanskii operator, which arises in a natural way in the theory of nested Hilbert spaces (cf. [2]). The operator $R$ has the properties $(f, g)_R = (Rf, Rg)_R = (f, Rg)_R$ $(f, g) \in S^1_1$. Thus

$$[(A_{1R}f, \eta)] \leq \|R A_{1R}f, \eta\|^2 = \|R A_{1R}\|(\|f\|^2 + \|A^* f\|^2)$$

where $P$ is the orthoprojector of $S^1_1$ onto $D(A)$ and the operator $A$ is the maximal common Hermitean part of the operators $T$ and $T^*$ that arise realizing the operator-valued function $V(z)$ as a transfer function of the conservative scattering system (7). Furthermore, as it was noted in the proof of Theorem 7, we have

$$\lim_{\eta \to \infty} \frac{\beta - \eta}{\alpha - \eta} V(\eta)f, f = \lim_{\eta \to \infty} \frac{\eta^2(\beta - \alpha)}{\alpha^2 + \eta^2} (A_{1R}f, \eta) + \frac{\eta^2}{\alpha^2 + \eta^2} (f, \eta) + \frac{\eta^2}{\alpha^2 + \eta^2} (f, \eta)$$

Since $\int_0^\infty \|d\eta(t)\| < \infty$, the realization (38) implies $\lim_{\eta \to \infty} \frac{\beta - \eta}{\alpha - \eta} V(\eta)f, f < \infty$. Now assume $V(z) \in S^1_1[R, (\alpha, \beta)]$. We will show that $(\frac{\alpha - z}{\beta - z})V(z) \in S^1_1(R)$ Using (33) we get

$$\lim_{\eta \to \infty} \frac{\beta - \eta}{\alpha - \eta} V(\eta)f, f \leq \lim_{\eta \to \infty} \frac{\eta^2}{\beta^2 + \eta^2} (f, \eta) + \frac{\eta^2}{\beta^2 + \eta^2} (f, \eta) < \infty.$$
Theorem 10: A function \( V(z) \), whose values are operators in a finite-dimensional Hilbert space \( E \), belongs to the class \( S_{\alpha,\beta}^n[R, \cup_{j=1}^m (\alpha_j, \beta_j)] \) if and only if the following two conditions hold:

(i) \( V(z) \in S(R) \).

(ii) \( \frac{\alpha - z}{\alpha - z} V(z) \in S(R) \), \( \frac{\beta - z}{\beta - z} V(z) \in S(R) \), respectively.

Theorem 11: A function \( V(z) \), whose values are operators in a finite-dimensional Hilbert space \( E \), belongs to the class \( S_{\alpha,\beta}^0[R, \cup_{j=1}^m (\alpha_j, \beta_j)] \) if and only if the following two conditions hold:

(i) \( V(z) \in S_0(R) \).

(ii) \( \prod_{j=1}^m \frac{\beta_j - z}{\alpha_j - z} V(z) \in S_0(R) \) \( \prod_{j=1}^m \frac{\beta_j - z}{\alpha_j - z} V(z) \in S_0(R) \), respectively.

Proof: The sufficiency of the conditions is easy to prove. Since \( S_0(R) \subset S \), we have \( V(z) \in S_{\alpha,\beta}^n[R, \cup_{j=1}^m (\alpha_j, \beta_j)] \) because of Theorem 2. But since \( V(z) \in S_0(R) \), we obtain \( V(z) \in S_0^0[R, \cup_{j=1}^m (\alpha_j, \beta_j)] \).

The necessity is proved with aid of mathematical induction. For \( n = 1 \) the result was proved in Theorem 7. Now assume that for \( m = p \) from \( V(z) \in S_0^0[R, \cup_{j=1}^p (\alpha_j, \beta_j)] \) it follows that

(i) \( V(z) \in S_0^0(R) \).

(ii) \( \prod_{j=1}^p \frac{\beta_j - z}{\alpha_j - z} V(z) \in S_0^0(R) \).

We will show that this fact remains true for \( n = p + 1 \). Assume that \( V(z) \in S_0^0[R, \cup_{j=1}^p (\alpha_j, \beta_j)] \). Then, clearly, \( V(z) \in S_0^0[R, \cup_{j=1}^p (\alpha_j, \beta_j)] \) and hence

(i) \( V(z) \in S_0^0(R) \).

(ii) \( \prod_{j=1}^p \frac{\beta_j - z}{\alpha_j - z} V(z) \in S_0^0(R) \).

Since \( V(z) \) is holomorphic and positive on the interval \( (\alpha_{p+1}, \beta_{p+1}) \), we obtain

\[
\prod_{j=1}^p \frac{\beta_j - z}{\alpha_j - z} V(z) \in S_0^0[R, (\alpha_{p+1}, \beta_{p+1})].
\]

Hence by Theorem 7,

\[
\frac{\beta_{p+1} - z}{\alpha_{p+1} - z} \prod_{j=1}^p \frac{\beta_j - z}{\alpha_j - z} V(z) = \prod_{j=1}^{p+1} \frac{\beta_j - z}{\alpha_j - z} V(z) \in S_0^0(R).
\]

An analogous proof works in the case \( V(z) \in S_0^0[R, \cup_{j=1}^p (\alpha_j, \beta_j)] \).

It is not hard to see that for the classes

\[ S_{\alpha,\beta}^n[R, \cup_{j=1}^m (\alpha_j, \beta_j)], S_{\alpha,\beta}^0[R, \cup_{j=1}^m (\alpha_j, \beta_j)] \text{ and } S_{\alpha,\beta}^n[R, \cup_{j=1}^m (\alpha_j, \beta_j)] \]

analogous results hold. Combining the above stated theorems we get the following

Theorem 12: A function \( V(z) \), whose values are operators in a finite-dimensional Hilbert space \( E \), belongs to the class \( S_{\alpha,\beta}^n[R, \cup_{j=1}^m (\alpha_j, \beta_j)] \cap S. \cup_{k=1}^p (c_k, d_k) \) if and only if the following two conditions hold:

(i) \( V(z) \in S(R) \).
(ii) \[ \sum_{j=1}^{m} \frac{\beta_j}{\alpha_j - z} - \sum_{k=1}^{n} \frac{c_k}{\delta_k - z} V(z) \in S(R). \]

Note that analogous results can be formulated for the classes

\[ S^o[R, \cup_{j=1}^{m}(\alpha_j, \beta_j)] \cap S^o[\cup_{k=1}^{n}(c_k, \delta_k)], \]
\[ S^e[R, \cup_{j=1}^{m}(\alpha_j, \beta_j)] \cap S^e[\cup_{k=1}^{n}(c_k, \delta_k)], \]
\[ S^{oo}[R, \cup_{j=1}^{m}(\alpha_j, \beta_j)] \cap S^{oo}[\cup_{k=1}^{n}(c_k, \delta_k)]. \]

**Definition:** Let \( A \) be a symmetric operator in a Hilbert space \( \mathcal{H} \). The interval \((\alpha, \beta)\) is called a gap of the operator \( A \) if

\[
\left\| Af - \frac{\alpha + \beta}{2} f \right\| \leq \frac{\beta - \alpha}{2} \| f \| \quad \text{for all} \quad f \in \mathcal{D}(A).
\]

**Theorem 13:** Let \( V(z) \) be a realizable operator-valued function in a finite-dimensional Hilbert space \( E \), i.e., \( V(z) = K^*(A_{IR} - zI)^{-1}K \), where (6) holds. Let \( \mathcal{D}(A) = \mathcal{D} \) and \( A \geq 0 \). Let \((\alpha, \beta)\) be an arbitrary interval of the positive semi-axis. Then \( V(z) \in S_e[(\alpha, \beta)] \) and \((\alpha, \beta)\) is a gap of the operator \( A \) if and only if the following two conditions hold:

(i) \( A_{IR} \geq 0 \).

(ii) \( (A_{IR} \varphi_i, \varphi_i) + \frac{\alpha \beta}{\beta - \alpha}(\varphi_i, \varphi_i) - \frac{\beta}{\beta - \alpha}(A^* \varphi_i, \varphi_i) - \frac{\beta}{\beta - \alpha}(\varphi_i, A^* \varphi_i) + \frac{1}{\beta - \alpha}(A^* \varphi_i, A^* \varphi_i) \geq 0 \quad \forall \varphi_i \in \mathcal{D}_+(A). \)

**Proof:** Assume that (40) holds. Let \( \{z_i\}_{i=1}^{P} \) be an arbitrary set of non-real complex numbers such that \( z_i \neq z_j \). Let \( N_{z_i} \) be the deficiency space of the operator \( A \) and \( \varphi_i \in N_{z_i} \). Set \( \varphi = \sum_{i=1}^{P} \frac{A^* \varphi_i}{(\alpha - z_i)}, \) Since \( A^* \varphi_i = z_i \varphi_i \), we obtain from (40)

\[
(A_{IR} \varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha}(\varphi, \varphi) - \frac{\beta}{\beta - \alpha}(A^* \varphi, \varphi) - \frac{\beta}{\beta - \alpha}(\varphi, A^* \varphi) + \frac{1}{\beta - \alpha}(A^* \varphi, A^* \varphi) \geq 0 \forall \varphi \in \mathcal{D}_+(A).
\]

where \( B(\lambda, \mu) \) has the form (16). Thus Theorem 4 yields \( V(z) \in S_e[(\alpha, \beta)] \).

Now we will show that \((\alpha, \beta)\) is a gap of \( A \). In fact, if the vector \( \varphi \) of the inequality (40) belongs to \( \mathcal{D}(A) \), then with regard to the inclusions \( A^* \supset A \) and \( A_{IR} \supset A \) we obtain

\[
(A_{IR} \varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha}(\varphi, \varphi) - \frac{\beta}{\beta - \alpha}(A^* \varphi, \varphi) - \frac{\beta}{\beta - \alpha}(\varphi, A^* \varphi) + \frac{1}{\beta - \alpha}(A^* \varphi, A^* \varphi) \geq 0.
\]
This implies
\[(\alpha + \beta)(A\varphi, \varphi) \leq \alpha \beta (\varphi, \varphi) + (A\varphi, A\varphi).\] (41)

But conditions (39) and (41) are equivalent. In fact, if (39) holds, we get

\[\left((A - \alpha^2 - \beta f)\varphi, (A - \alpha^2 - \beta f)\varphi\right) \leq \left(\frac{\beta - \alpha}{2}\right)^2 (\varphi, \varphi),\]

hence

\[\left((A\varphi, A\varphi) - (\alpha + \beta)(A\varphi, \varphi) + \left(\frac{\alpha + \beta}{2}\right)^2 (\varphi, \varphi) \leq \left(\frac{\beta - \alpha}{2}\right)^2 (\varphi, \varphi)\]

and \((\alpha + \beta)(A\varphi, \varphi) \leq \alpha \beta (\varphi, \varphi) + (A\varphi, A\varphi), i.e., (41).\] It is not hard to see that the converse conclusion is also true. Thus, the interval \((\alpha, \beta)\) is a gap of the operator \(A\).

Now let \(V(z) \in \mathcal{S}_1[\alpha, \beta]\) and the interval \((\alpha, \beta)\) be a gap of the operator \(A\). Then by Theorem 4 it holds (15). As it was proved above, this yields the inequality (40) for all vectors \(\varphi\) of the form

\[\varphi = \sum_{i=1}^{n} (\alpha - z_i)^{-1} \varphi_i,\] (42)

where \(\{z_i\}_{i=1}^{n}\) be an arbitrary set of non-real complex numbers such that \(z_i \neq z_j\) and \(\varphi_i\) is an arbitrary vector of the deficiency space \(N_{z_i}\). Let \(S_1 = \bigvee_{z \in \mathbb{Z}} N_z\), where the closure is taken with respect to the metric of the space \(S_1\). Then, clearly, \(S_1 = S_1 \oplus S_2\), where the subspaces \(S_1\) and \(S_2\) are invariant subspaces of the operator \(A\) and the operator \(A_2 = A|S_2\) is selfadjoint. Thus, \(A = A_1 \oplus A_2\), where \(A_1 = A|S_1\). It is easy to see that \(A^* = A_1^* \oplus A_2\). It follows that each vector \(\varphi \in S_1\) can be represented in the form \(\varphi = \varphi_1 + \varphi_2\), where \(\varphi_1 \in \mathcal{D}(A_1^*)\) and \(\varphi_2 \in \mathcal{D}(A_2)\). Since the operators \(A_1^*\) and \(A^*\) are continuous operators from \(S_1\) into \(S_1\), we can extend the inequality (40) from all vectors of the form (42) to all vectors \(\varphi \in \mathcal{D}(A^*_1)\). It is easy to see that an arbitrary selfadjoint extension \(\hat{A}\) of the operator \(A\) has the form \(\hat{A} = \hat{A}_1 \oplus \hat{A}_2\), where \(\hat{A}_1\) is a selfadjoint extension of the operator \(A_1\) in the space \(S_1\). Since by assumption the interval \((\alpha, \beta)\) is a gap of the operator \(A\), it is also a gap of the operator \(A_2\). Thus, for the operator \(A_2\) it holds (39) and hence (41), as it was shown above. Setting \(\varphi = \varphi_1 + \varphi_2\) in (40) we obtain

\[(A_{R\varphi}, \varphi) + \alpha \beta (\varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (A^* \varphi, \varphi) - \frac{\beta}{\beta - \alpha} (\varphi, A^* \varphi) + \frac{1}{\beta - \alpha} (A^* \varphi, A^* \varphi) = \]

\[= (A_{R\varphi_1}, \varphi_1) + \alpha \beta (\varphi_1, \varphi_1) - \frac{\beta}{\beta - \alpha} (A^* \varphi_1, \varphi_1) - \frac{\alpha}{\beta - \alpha} (\varphi_1, A^* \varphi_1) + \frac{1}{\beta - \alpha} (A^* \varphi_1, A^* \varphi_1)\]

\[+ (A_{R\varphi_2}, \varphi_2) + \frac{\alpha \beta}{\beta - \alpha} (\varphi_2, \varphi_2) - \frac{\beta}{\beta - \alpha} (A^* \varphi_2, \varphi_2) - \frac{\alpha}{\beta - \alpha} (\varphi_2, A^* \varphi_2) + \frac{1}{\beta - \alpha} (A^* \varphi_2, A^* \varphi_2) \]

\[= (A_{R\varphi_1}, \varphi_1) + \alpha \beta (\varphi_1, \varphi_1) - \frac{\beta}{\beta - \alpha} (A^* \varphi_1, \varphi_1) - \frac{\alpha}{\beta - \alpha} (\varphi_1, A^* \varphi_1) + \frac{1}{\beta - \alpha} (A^* \varphi_1, A^* \varphi_1)\]

\[+ \frac{\alpha \beta}{\beta - \alpha} (\varphi_2, \varphi_2) - \frac{\beta}{\beta - \alpha} (A_2 \varphi_2, A_2 \varphi_2) - \frac{\alpha}{\beta - \alpha} (A_2 \varphi_2, A_2 \varphi_2) \geq 0.\]

We note that the last inequality holds since the terms in the big brackets are non-negative.
**Theorem 14:** Let $V(z)$ be a realizable operator-valued function in a finite-dimensional Hilbert space $E$, i.e., $V(z) = K^\ast (A_R - zI)^{-1}K$, where (6) holds. Let $\mathcal{D}(A) = S_\uparrow$ and $A \preceq 0$. Let $(\alpha, \beta)$ be an arbitrary interval of the positive semi-axis. Then $V(z) \in S_\uparrow [\alpha, \beta]$ and $(\alpha, \beta)$ is a gap of the operator $A$ if and only if the following two conditions hold:

(i) $\alpha \preceq 0$.

(ii) $-(A_R \varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (A^\ast \varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (\varphi, A^\ast \varphi) + \frac{1}{\beta - \alpha} (A^\ast \varphi, A^\ast \varphi) \preceq 0 \forall \varphi \in S_\uparrow$ \quad (43)

**Proof:** Assume that (43) holds. Let $\{z_i\}_{i=1}^p$ be an arbitrary set of non-real complex numbers such that $z_i \neq z_k$. Let $N_{z_i}$ be the efficiency space of the operator $A$ and $\varphi_i \in N_{z_i}$. Setting $\varphi = \sum_{i=1}^p (\alpha - z_i)^{-1} \varphi_i$ in (43), we obtain with regard to $A^\ast \varphi_i = z_i \varphi_i$

$$-(A_R \varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (A^\ast \varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (\varphi, A^\ast \varphi) + \frac{1}{\beta - \alpha} (A^\ast \varphi, A^\ast \varphi)$$

$$= -\sum_{i=1}^p \left( \frac{1}{(\beta - z_i)(\beta - z_i)} \right) (A_R \varphi_i, \varphi_i) + \frac{\alpha \beta}{\beta - \alpha} \sum_{i=1}^p \left( \frac{1}{(\beta - z_i)(\beta - z_i)} \right) (\varphi_i, \varphi_i)$$

$$= \frac{\alpha}{\beta - \alpha} \sum_{i=1}^p \left( \frac{1}{(\beta - z_i)(\beta - z_i)} \right) (A^\ast \varphi_i, A^\ast \varphi_i)$$

where $B(\lambda, \mu)$ has the form (20). This implies the inclusion $V(z) \in S_\uparrow [\alpha, \beta]$ by Theorem 5.

We will now show that $(\alpha, \beta)$ is a gap of the operator $A$. In fact, if the vector $\varphi$ in the inequality (43) belongs to $\mathcal{D}(A)$, then with regard to the inclusions $A^\ast \supset A$ and $A_R \supset A$ we obtain

$$-(A_R \varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (A^\ast \varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (\varphi, A^\ast \varphi) + \frac{1}{\beta - \alpha} (A^\ast \varphi, A^\ast \varphi)$$

$$= -(A \varphi, \varphi) + \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) - \frac{2\alpha}{\beta - \alpha} (A \varphi, \varphi) + \frac{1}{\beta - \alpha} (A^\ast \varphi, A^\ast \varphi)$$

$$= \frac{\alpha \beta}{\beta - \alpha} (\varphi, \varphi) - \frac{1}{\beta - \alpha} (A \varphi, \varphi) - \frac{\alpha}{\beta - \alpha} (A \varphi, A \varphi) \preceq 0.$$ 

This yields $(\alpha + \beta)(A \varphi, \varphi) \preceq \alpha \beta (\varphi, \varphi) + (A \varphi, A \varphi).$ Therefore, the relation (41) is true for the operator $A$. As it was shown above, this implies that the interval $(\alpha, \beta)$ is a gap of $A$. The necessity part can be proved in an analogous way as the necessity part of Theorem 13.

As a corollary of Theorems 13 and 14 we obtain the following general

**Theorem 15:** Let $V(z)$ be a realizable operator-valued function in a finite-dimensional Hilbert space $E$, i.e., $V(z) = K^\ast (A_R - zI)^{-1}K$, where (6) holds. Let $\mathcal{D}(A) = S_\uparrow$ and $A \preceq 0$. Let $(\alpha_j, \beta_j)$ $(j = 1, \ldots, m)$ and $(c_k, d_k) (k = 1, \ldots, n)$ two arbitrary sets of mutually disjoint intervals of the po-
sitive semi-axis. Then \( V(z) \in S_c \left( \bigcup_{j=1}^{m} (\alpha_j, \beta_j) \right) \cap S_c \left( \bigcup_{k=1}^{n} (c_k, d_k) \right) \) and all intervals \((\alpha_j, \beta_j)\) and \((c_k, d_k)\) are gaps of the operator \( A \) if and only if the following three conditions hold:

(i) \( A \mid \neq 0 \).

(ii) \( (A^\dagger \varphi, \varphi) + \frac{\alpha_j \beta_j}{\beta_j - \alpha_j} (\varphi, \varphi) - \frac{\beta_j}{\beta_j - \alpha_j} (A^* \varphi, \varphi) + \frac{1}{\beta_j - \alpha_j} (A^* \varphi, A^* \varphi) \geq 0 \)

for each \( \varphi \in \Delta_+ \) and all \( j = 1, \ldots, m \).

(iii) \( -(A^\dagger \varphi, \varphi) + \frac{c_k d_k}{d_k - c_k} (\varphi, \varphi) - \frac{c_k}{d_k - c_k} (A^* \varphi, \varphi) + \frac{1}{d_k - c_k} (A^* \varphi, A^* \varphi) \geq 0 \)

for each \( \varphi \in \Delta_+ \) and all \( k = 1, \ldots, n \).

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Dr. Vladislav Eduardovic Tsekanovskii
Artema 84 kw. 65
340055 - Donetsk, Ukraine