Regularity and Derivative Bounds
for a Convection-Diffusion Problem
with Neumann Boundary Conditions
on Characteristic Boundaries

Aidan Naughton and Martin Stynes

Abstract. A convection-diffusion problem is considered on the unit square, with
convection parallel to two of the square’s sides. Dirichlet conditions are imposed on
the inflow and outflow boundaries, with Neumann conditions on the other two sides.
No assumption is made regarding the corner compatibility of the data. The regularity
of the solution is expressed precisely in terms of the regularity and compatibility of
the data. Pointwise bounds on all derivatives of the solution are derived and their
dependence on the data regularity, its corner compatibility, and on the small diffusion
parameter is made explicit. These results extend previous bounds of Jung and Temam

Keywords. Singular perturbation, convection-diffusion, characteristic boundary layer,
regularity, a priori bound

Mathematics Subject Classification (2000). Primary 35B25, secondary 35B45,
35B65, 35E20

1. Introduction

Bounds on derivatives of solutions to singularly perturbed convection-diffusion
boundary value problems are of importance for two main reasons: they reveal
the fine structure of the solution, and they are also needed in the analysis of
numerical methods for such problems. While many papers on this topic address
ordinary differential equations, progress for problems posed in two dimensions
has been much slower – relatively few papers, such as [1, 2, 6, 7, 9, 11, 12], rig-
orously prove pointwise derivative bounds for singularly perturbed problems
posed in two-dimensional domains.

A. Naughton: School of Mathematics and Statistics, University of St. Andrews, North
Haugh, St. Andrews, Fife KY16 9SS, Scotland; naughton_aidan@hotmail.com
M. Stynes: Department of Mathematics, National University of Ireland, Cork, Ire-
land; m.stynes@ucc.ie
The problem studied in this paper is posed on the unit square with Dirichlet conditions along \( x = 0 \) and \( x = 1 \) and Neumann conditions along \( y = 0 \) and \( y = 1 \). Related problems on the unit square but with different boundary conditions were considered in [6, 7, 11]. Some of this earlier work coincides with what is here and we shall exploit this overlap fully; nevertheless the alteration of the boundary conditions changes significantly the nature of the solution.

Problems similar to ours are considered by Clavero et al. [1] and by Jung and Temam [5], but in both these papers the analysis is simplified by an assumption that the data satisfies certain corner compatibility conditions that exclude corner singularities. We make no such assumption. Furthermore, unlike these earlier papers, we make precise the relationship between the given data and the regularity of the solution.

In [1] Robin conditions are imposed along \( y = 0 \) and \( y = 1 \) and pointwise bounds are proved under strong compatibility assumptions at the corners of the domain, but the arguments are not written down in full. Our pointwise bounds on derivatives of the solution agree with those of [1] when their corner compatibility conditions are satisfied. In [5] only \( L_2 \) and \( H^1 \)-type bounds on derivatives are obtained, and our pointwise bounds imply these weaker estimates.

Shishkin [13] considers a convection-diffusion problem on a two-dimensional rectangle with Dirichlet boundary data, where the differential operator has variable coefficients, with small parameters \( \varepsilon_1 \) multiplying the second-order derivatives and \( \varepsilon_2 \) multiplying one of the first-order derivatives. When \( \varepsilon_2 = 0 \) this problem is related to our problem (1). The techniques of [13] are suitable for studying problems with stronger parabolic layers than those considered here.

The problem we shall consider is as follows. Let \( u(x, y) \) be the solution to the boundary value problem

\[
Lu(x, y) := -\varepsilon \Delta u(x, y) + pu_x(x, y) + qu(x, y) = f(x, y) \quad \forall (x, y) \in Q = (0, 1)^2
\]

\[
u_y(x, 0) = h_s(x), \quad u_y(x, 1) = h_n(x) \quad \text{for } 0 < x < 1 \tag{1b}
\]

\[
u(0, y) = g_w(y), \quad u(1, y) = g_e(y) \quad \text{for } 0 < y < 1. \tag{1c}
\]

The constants \( p \) and \( q \) satisfy \( p > 0, q > 0 \). As we are interested in the singularly perturbed case, without loss of generality the diffusion parameter \( \varepsilon \) satisfies \( 0 < \varepsilon \leq \min \{1, 12 \frac{p^2}{q}\} \). Assume that \( f(x, y) \) and the boundary data lie in certain Hölder spaces:

\[
f \in C^{2\ell, \alpha}(\bar{Q}), \quad g_w, g_e \in C^{2\ell, \alpha}[0, 1] \tag{2a}
\]

\[
\int_0^x h_s(t) \, dt \in C^{2\ell, \alpha}[0, 1], \quad \int_0^x h_n(t) \, dt \in C^{2\ell, \alpha}[0, 1], \tag{2b}
\]

for some non-negative integer \( \ell \) and \( \alpha \in (0, 1) \). If \( \ell > 0 \), the condition on \( h_s \) and \( h_n \) is equivalent to requiring \( h_s, h_n \in C^{2\ell - 1, \alpha}[0, 1] \).
By a solution of (1) we mean a function \( u \in C^{2,\alpha}(Q) \) that satisfies (1a) and can be extended up to the boundary to satisfy (1b)–(1c). Existence and uniqueness of this solution can be shown by combining techniques from [3, 14].

The purpose of this paper is to derive pointwise derivative bounds for the solution of (1), while making explicit their dependence on the parameter \( \varepsilon \) and on the corner compatibility and regularity of the data.

A typical solution to (1) will have an exponential boundary layer along \( x = 1 \) and weaker parabolic boundary layers along \( y = 0 \) and \( y = 1 \), with weak corner layers at the outflow corners \((1,0)\) and \((1,1)\). Furthermore, depending on the compatibility of the data at corners, the solution may contain corner singularities. We decompose the solution to (1) as

\[
 u = S + E + w_{00} + w_{01} + w_{10} + w_{11} + ˘u \quad \text{in } Q;
\]

here each function (except the remainder \( ˘u \)) is the solution of a simpler half-plane or quarter-plane problem. The definitions of these functions will be given later, but for convenience it is summarized in Table 1, where the column labelled “\( L \)” gives the output when \( L \) is applied to each function in the first column and the other columns show the boundary conditions for each problem.

<table>
<thead>
<tr>
<th></th>
<th>( L )</th>
<th>\begin{tabular}{c} \text{Dirichlet} \text{ cond.}\end{tabular} \text{ at } x = 0</th>
<th>\begin{tabular}{c} \text{Dirichlet} \text{ condition}\end{tabular} \text{ at } x = 1</th>
<th>\begin{tabular}{c} \text{Neumann} \text{ condition}\end{tabular} \text{ at } y = 0</th>
<th>\begin{tabular}{c} \text{Neumann} \text{ condition}\end{tabular} \text{ at } y = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td>( f^*(x, y) )</td>
<td>( g_w^*(y) )</td>
<td>( g_e^*(y) - S(1, y) )</td>
<td>( h^*_s(x) - S_y(x, 0) )</td>
<td>( h^*_n(x) - S_y(x, 1) )</td>
</tr>
<tr>
<td>( E )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( w_{00} )</td>
<td>0</td>
<td>0</td>
<td>( -\chi(1-y)w_{00}(1, y) )</td>
<td>( -\chi(x)E_y(x, 0) )</td>
<td>( -\chi(x)E_y(x, 1) )</td>
</tr>
<tr>
<td>( w_{01} )</td>
<td>0</td>
<td>0</td>
<td>( -\chi(y)w_{01}(1, y) )</td>
<td>( -\chi(x)E_y(x, 0) )</td>
<td>( -\chi(x)E_y(x, 1) )</td>
</tr>
<tr>
<td>( w_{10} )</td>
<td>0</td>
<td>0</td>
<td>( -\chi(1-y)w_{10}(1, y) )</td>
<td>( -\chi(x)E_y(x, 0) )</td>
<td>( -\chi(x)E_y(x, 1) )</td>
</tr>
<tr>
<td>( w_{11} )</td>
<td>0</td>
<td>0</td>
<td>( \tilde{g}_w(y) )</td>
<td>( \tilde{g}_e(y) )</td>
<td>( \tilde{h}_s(x) )</td>
</tr>
</tbody>
</table>

where

\[
 \begin{align*}
 \tilde{g}_w(y) &= -E(0, y) - w_{10}(0, y) - w_{11}(0, y) \\
 \tilde{g}_e(y) &= -[1 - \chi(1-y)]w_{00}(1, y) - [1 - \chi(y)]w_{01}(1, y) \\
 \tilde{h}_s(x) &= -w_{01,y}(x, 0) - w_{11,y}(x, 0) - [1 - \chi(x)]E_y(x, 0) \\
 \tilde{h}_n(x) &= -w_{00,y}(x, 1) - w_{10,y}(x, 1) - [1 - \chi(x)]E_y(x, 1). 
\end{align*}
\]

Table 1: The decomposition used for \( u \).

The functions \( f^*, g_w^*, g_e^*, h^*_s \) and \( h^*_n \) are smooth extensions of \( f, g_w, g_e, h_s \) and \( h_n \), respectively, that vanish outside some bounded set. In our notation the letter \( h \) is in general reserved for Neumann boundary conditions, while \( g \) is used for Dirichlet boundary conditions.
To ensure existence and uniqueness of the functions defined in (3) as solutions of problems on unbounded domains, one must impose certain growth restrictions at infinity. We do not state these explicitly in this current paper as their derivation is routine but tedious; see [6, Section 3.2] for an example of this. All the barrier functions that we shall employ on unbounded domains satisfy the requisite growth conditions. Some of these make use of the assumption that \( q > 0 \); cf. [6, Lemma 3.5].

To bound the derivatives of \( u \), we shall derive bounds separately for each function in the decomposition (3). One of our estimates (Lemma 4.2) sharpens a similar bound obtained in [6].

The function \( S \) is the principal component in the solution of \( u \). It provides a good approximation of \( u \) on \( Q \) except near the ordinary and parabolic layers and the corner singularities. The other terms in the decomposition handle these more difficult regions, as we now describe. The function \( E \) is a correction to \( S \) along \( x = 1 \) that yields the correct boundary condition there for \( u \). The boundary data for \( u \) along \( y = 0 \) is provided by \( w_{00} \); any corner singularity in \( u \) at \((0,0)\) is also contained in \( w_{00} \). The function \( w_{01} \) performs a role analogous to \( w_{00} \) along \( y = 1 \) and at \((0,1)\). Any corner singularities at \((1,0)\) and \((1,1)\) are contained in \( w_{10} \) and \( w_{11} \); these two functions also correct some of the boundary data of the earlier functions. Any boundary data not accounted for at this juncture is corrected by the remainder function \( \tilde{u} \).

1.1. Notation. Set \( \Pi_x = \{ (x,y) \in \mathbb{R}^2 : x > 0 \} \), \( \Pi_y = \{ (x,y) \in \mathbb{R}^2 : y > 0 \} \) and \( Q = \{ (x,y) \in \mathbb{R}^2 : x > 0, y > 0 \} \). For various measurable sets \( \Omega \), with integers \( k \geq 0 \) and \( p \geq 1 \), and \( 0 < \alpha < 1 \), let \( W^{k,p}(\Omega) \) denote the usual Sobolev space of functions on \( \Omega \) whose weak derivatives of order at most \( k \) are in \( L^p(\Omega) \), while \( C^{k,\alpha}(\Omega) \) denotes the space of Hölder-continuous functions on \( \Omega \). Set \( H^k(\Omega) = W^{k,2}(\Omega) \) and write \( \| \cdot \|_{k,p,\Omega} \) for the norm in \( W^{k,p}(\Omega) \).

Finally, we use \( C \) to denote a generic constant that is independent of the parameter \( \varepsilon \) but may depend on the remaining data of \( (1) \); note that \( C \) can take different values in different places during our analysis.

2. Compatibility conditions

To discuss the regularity of \( u \) on the closed domain \( \bar{Q} \), one must consider the compatibility of the data of \( (1) \) at the corners of \( \bar{Q} \). Number the corners \((0,1), (0,0), (1,0) \) and \((1,1)\) as 1, 2, 3 and 4, respectively. The data for the problem \( (1) \) is given by the 5-tuple \( X = (g_w, g_e, h_s, h_n, f) \). Let \( \ell \geq 1 \) be an integer and let \( 0 < \alpha < 1 \). For each integer \( k \geq 2 \), define the Banach space

\[
\mathcal{D}_{k,\alpha} = \left( C^{k-1,\alpha}[0,1] \right)^2 \times \left( C^{k-2,\alpha}[0,1] \right)^2 \times C^{k-3,\alpha}(\bar{Q}).
\]
For the Poisson equation $\Delta w = f$ with boundary conditions (1b) and (1c), necessary and sufficient conditions for the solution to belong to the space $C^{k,\alpha}(\overline{Q})$ were established by Volkov [14]. Corresponding conditions for the problem (1) are given in Theorem 2.1, which can be derived similarly to [11, Theorem 2.1]; see [10] for details.

**Theorem 2.1.** Let $\ell \geq 1$ and $\nu$ be integers. Let $X \in D_{2\ell,\alpha}$. Let $u$ be the solution of (1) with data $X$. Then there are numbers $a_{\mu,\nu}^{(i)}, i = 1, \ldots, 4, \mu \geq 0$ and $b_{\mu_1,\mu_2,\nu}^{(i)}, i = 1, \ldots, 4, \mu_1 \geq 0, \mu_2 \geq 0,$ which depend only on $\varepsilon$, $p$ and $q$, such that when one sets (where each sum is interpreted as 0 if the upper limit is less than the lower limit)

\[
\begin{align*}
\Lambda_\nu^{(1)}(X) &= g_w^{(2\nu+1)}(1) + \sum_{\mu=0}^{2\nu} a_{\mu,\nu}^{(1)} h_n^{(\mu)}(0) + \sum_{\mu_1+\mu_2\leq 2\nu-1} b_{\mu_1,\mu_2,\nu}^{(1)} D_x^{\mu_1} D_y^{\mu_2} f(0,1), \\
\Lambda_\nu^{(2)}(X) &= g_w^{(2\nu+1)}(0) + \sum_{\mu=0}^{2\nu} a_{\mu,\nu}^{(2)} h_n^{(\mu)}(0) + \sum_{\mu_1+\mu_2\leq 2\nu-1} b_{\mu_1,\mu_2,\nu}^{(2)} D_x^{\mu_1} D_y^{\mu_2} f(0,0), \\
\Lambda_\nu^{(3)}(X) &= g_e^{(2\nu+1)}(0) + \sum_{\mu=0}^{2\nu} a_{\mu,\nu}^{(3)} h_n^{(\mu)}(1) + \sum_{\mu_1+\mu_2\leq 2\nu-1} b_{\mu_1,\mu_2,\nu}^{(3)} D_x^{\mu_1} D_y^{\mu_2} f(1,0), \\
\Lambda_\nu^{(4)}(X) &= g_e^{(2\nu+1)}(1) + \sum_{\mu=0}^{2\nu} a_{\mu,\nu}^{(4)} h_n^{(\mu)}(1) + \sum_{\mu_1+\mu_2\leq 2\nu-1} b_{\mu_1,\mu_2,\nu}^{(4)} D_x^{\mu_1} D_y^{\mu_2} f(1,1),
\end{align*}
\]

then $u \in C^{2\ell-1,\alpha}(\overline{Q})$ if and only if

\[
\Lambda_\nu^{(i)}(X) = 0 \text{ for } i = 1, 2, 3, 4, \text{ and } \nu = 0, 1, \ldots, \ell - 1. \tag{4}
\]

Furthermore, if (4) holds and $X \in D_{2\ell+1,\alpha}$, then $u \in C^{2\ell,\alpha}(\overline{Q})$. If $\nu \leq \ell - 1$, the expressions $\Lambda_\nu^{(i)}$ for $i = 1, 2, 3, 4$ define bounded linear functionals on $D_{2\ell,\alpha}$.

By assumption (2) the data $X \in D_{2\ell+1,\alpha}$ (in fact for the proof of Lemma 3.1 the assumption on $f$ in (2) is stronger than this; see also the comment preceding Lemma 4.2).

### 2.1. Definition

Given a set of data $X \in D_{2\ell+1,\alpha}$, define a compatibility index $\nu$ at each vertex as follows: set $jk = 10, 00, 01, 11$ if $i = 1, 2, 3, 4$ respectively, then for each couple $jk$ set $\nu_{jk}(X) = m$ if $\Lambda_{\nu_{jk}}^{(i)}(X) = 0$ for $\nu = 0, \ldots, m$ and $\Lambda_{m+1}^{(i)}(X) \neq 0$. If $\Lambda_{0}^{(i)}(X) \neq 0$, set $\nu_{jk}(X) = -1$.

### 3. Smooth component $S$

The first component in the decomposition (3) of $u$ is the function $S$. Let $f^*$ and $g_w^*$ be smooth extensions of $f$ and $g_w$ to $\Pi_x$ and $(-\infty, \infty)$ respectively
that vanish outside some bounded neighbourhoods of $\bar{Q}$ and $[0,1]$ respectively. Similar extensions $g^*_e, h^*_s$ and $h^*_n$ of $g_e, h_s$ and $h_n$ are used later.

Define $S$ to be the solution of the half-plane problem

$$LS = f^* \text{ for } (x,y) \in \Pi_x, \quad S(0,y) = g_w^*(y) \text{ for } -\infty < y < \infty.$$  

Then $S \in C^{2\ell,\alpha}(\bar{\Pi}_x)$. The same function appears in [11]; from there one has

**Lemma 3.1 ([11, Theorem 3.2]).** There exists a constant $C$ such that

$$\|S\|_{m+n,\infty,\Pi_x} \leq C \left( \|f^*\|_{m+n,\infty,\Pi_x} + \|g_w^*\|_{C^{m+n,\alpha}(\mathbb{R})} \right) \text{ for } m + n \leq 2\ell. \quad (5)$$

Recalling that the parameter $\varepsilon$ may be close to zero and the constant $C$ in (5) is independent of $\varepsilon$, we surmise that the regularity demanded of the data $f$ in Lemma 3.1 is optimal while that demanded of $g_w^*$ is slightly suboptimal.

### 4. Exponential layer component $E$

Let $E$ be the solution to the half-plane problem

$$LE = 0 \quad \text{for } x < 1, -\infty < y < \infty \quad (6a)$$

$$E(1,y) = -S(1,y) + g^*_e(y) \quad \text{for } -\infty < y < \infty. \quad (6b)$$

This function $E$ is the same as that defined in [6, Section 5]. Here we use Fourier transforms to bound the derivatives of $E$ as this requires less regularity than the approach followed in [6].

For convenience set $W(x,y) = E(1-x,y)$. Then

$$L^*W := -\varepsilon W_{xx} - \varepsilon W_{yy} - pW_x + qW = 0 \quad \text{on } \Pi_x \quad (7a)$$

$$W(0,y) = g(y) \quad \text{for } -\infty < y < \infty, \quad (7b)$$

where $g(y) := -S(1,y) + g^*_e(y)$.

We shall need the following Mikhlin multiplier result.

**Theorem 4.1 ([4, Theorem 6.2.3]).** Let $M \in C^1(\mathbb{R})$. Let $K$ be a constant (independent of $\eta$) such that

$$|M^{(j)}(\eta)| \leq K(1 + |\eta|)^{-j} \quad \text{for } j = 0, 1 \quad \text{and } \eta \in \mathbb{R}. \quad (8)$$

Let $s \in C^{k,\alpha}(\mathbb{R})$ for some non-negative integer $k$. Define $h \in L_2(\mathbb{R})$ implicitly from its Fourier transform $\hat{h}$ by setting $\hat{h}(\eta) = M(\eta)\hat{s}(\eta)$. Then $h \in C^{k,\alpha}(\mathbb{R})$ and there exists a constant $C$ such that

$$\|h\|_{C^{k,\alpha}(\mathbb{R})} \leq CK\|s\|_{C^{k,\alpha}(\mathbb{R})}.$$
Theorem 4.1 is now used to establish bounds for derivatives of $E$. The bounds in the next lemma require more regularity of the data than one would expect from Theorem 2.1 because we need the constant $C$ to be independent of $\varepsilon$. Recall that $f^* \in C^{2d,\alpha}(\Pi_x)$ and $g_u^*, g_v^* \in C^{2d,\alpha}(\mathbb{R})$.

**Lemma 4.2.** Let $m$ and $n$ be non-negative integers and let $\bar{p} \in (0, p)$. Then for $m + n \leq 2\ell - 1$, one has

$$|D_x^m D_y^n E(x, y)| \leq C \left[ \|f^*\|_{m+n+1,\infty, \Pi_x} + \|g_u^*\|_{C^{m+n+1}, \mathbb{R}} + \|g_v^*\|_{C^{m+n+1}, \mathbb{R}} \right] \varepsilon^{-m} e^{-\frac{2(1-x)}{x^2}} \quad \text{for} \quad (x, y) \in \Pi_x. \quad (9a)$$

**Proof.** Define the Fourier transform of $W$ with respect to $y$ by

$$\hat{W}(x, \eta) = (\mathcal{F}W)(x, \eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} W(x, y) e^{-i\eta y} dy. \quad (9b)$$

Similarly define the Fourier transform $\hat{g}$ of $g$. The Fourier transform of the problem satisfied by $W$ is

$$\varepsilon \eta^2 \hat{W} - \varepsilon \hat{W}_{xx} - p \hat{W}_x + q \hat{W} = 0 \quad \text{for} \quad x > 0, \quad -\infty < \eta < \infty \quad (9a)$$

$$\hat{W}(0, \eta) = \hat{g}(\eta) \quad \text{for} \quad -\infty < \eta < \infty. \quad (9b)$$

Setting $r(\eta) = \frac{1}{2\varepsilon} \left(p + \sqrt{p^2 + 4\varepsilon^2 \eta^2 + 4\varepsilon q}\right)$, one can verify that the solution to (9) is $\hat{W}(x, \eta) = e^{-r(\eta)x} \hat{g}(\eta)$. Hence

$$\mathcal{F}(D_x^m W)(x, \eta) = D_x^m \hat{W}(x, \eta) = (-r(\eta))^m e^{-r(\eta)x} \hat{g}(\eta). \quad (10)$$

Set $\hat{g}_m(\eta) = (1 + \eta)^m \hat{g}(\eta)$. Then (10) can be rewritten as

$$D_x^m \hat{W}(x, \eta) = (-r(\eta))^m (1 + \eta)^{-m} e^{-r(\eta)x} \hat{g}_m(\eta). \quad (11)$$

Since $g_m(y) = (1 + \frac{d}{dy})^m g(y)$ is a linear combination of derivatives of $g$ of order at most $m$, it is clear that

$$\|g_m\|_{C^{0,\alpha}(\mathbb{R})} \leq C \|g\|_{C^{m,\alpha}(\mathbb{R})} \quad \text{provided} \quad g \in C^{m,\alpha}(\mathbb{R}). \quad (12)$$

We shall use $M_m(\eta) = (-r(\eta))^m (1 + \eta)^{-m} e^{-r(\eta)x}$ as the Mikhlin multiplier. In [11, Lemma 4.1] a related multiplier $M(\eta)$ is used; in fact $|M_m(\eta)| = |M(\eta)r(\eta)(1 + \eta)^{-1}|$ and $M(\eta)$ satisfies (8) with $K = C\varepsilon^{1-m} e^{-\frac{2\varepsilon}{x^2}}$. Combining these facts with the inequalities $|r(\eta)| \leq C(|\eta| + \varepsilon^{-1})$ and $|r'(\eta)| \leq C$, one sees that $M_m(\eta)$ satisfies (8) with $K = C\varepsilon^{-m} e^{-\frac{2\varepsilon}{x^2}}$. Applying Theorem 4.1 with the multiplier $M_m(\eta)$ to (11) yields

$$\|D_x^m W(x, \cdot)\|_{C^{0,\alpha}(\mathbb{R})} \leq C\varepsilon^{-m} e^{-\frac{2\varepsilon}{x^2}} \|g_m\|_{C^{0,\alpha}(\mathbb{R})} \leq C\varepsilon^{-m} e^{-\frac{2\varepsilon}{x^2}} \|g\|_{C^{m,\alpha}(\mathbb{R})},$$

where we used (12). On recalling the definitions of $W$ and $g$ and invoking (5), the lemma is proved in the case $n = 0$.

Now suppose that $n \geq 1$. Set $v(x, y) = D_y^n W(x, y)$, apply $D_y^n$ to (7), and apply the case $n = 0$ result to the function $v$. \qed
Remark 4.3. Lemma 4.2 requires less data regularity than the bound
\[ |D_x^m D_y^n E(x, y)| \leq C \|g_e\|_{2\ell, \infty, (0,1)} + \|S(1, \cdot)\|_{2\ell, \infty, \mathbb{R}} \varepsilon^{-\frac{(1-x)}{\ell}} \]
of [6, p.119].

The following result will be needed in the proof of Lemma 6.2.

Lemma 4.4. For \( m \leq 2\ell - 2 \) there is a constant \( C \) such that
\[ |D_x^m \left( e^{\frac{x}{\varepsilon}} E_{y} (1 - x, y) \right) | \leq C \left( \|f^*\|_{m+1, \infty, \Pi_x} + \|g_{e, e}^*\|_{m+2, \infty, \mathbb{R}} + \|g_{e, e}^*\|_{m+1, \infty, \mathbb{R}} \right) \]
on \( \Pi_x \).

**Proof.** Apply \( D_y \) to (6) to get \( LE_y = 0 \) for \( x < 1 \), \( E_y (1, y) = (g_e^*)'(y) - S_y (1, y) \). This is a problem similar to (6) but with one degree less regularity. Set \( W_1 (x, y) = e^{\frac{x}{\varepsilon}} E_y (1 - x, y) \); then \( LW_1 = 0 \) on \( \Pi_x \) and \( W_1 (0, y) = -S_y (1, y) + (g_e^*)'(y) \). Now invoking Lemma 3.1 yields the desired result. \( \square \)

5. Incoming corner functions

At this stage of our construction the function \( S + E \) essentially matches the boundary data for \( u \) in \( Q \) along the inflow \((x = 0, \) where \( E \) is exponentially small\) and outflow \((x = 1) \) boundaries but may not agree with \( \frac{\partial u}{\partial y} \) on the sides \( y = 0, 1 \). The incoming corner function \( w_{00} \) will handle the Neumann boundary data along the side \( y = 0 \) and any corner singularity at the point \((0, 0) \). A related problem in [6] defines an incoming corner function \( z_{00} \) with Dirichlet boundary conditions; our analysis is based on [6, Section 2] but has many differences, especially in the construction of the function \( \zeta \) below.

Define \( w_{00} \in L_2 (\mathbb{Q}) \) to be the solution of the quarter-plane problem
\[ \begin{align*}
Lw_{00} &= 0 & \text{on } \mathbb{Q} \quad (13a) \\
w_{00, y} (x, 0) &= h_{e}^* (x) - S_y (x, 0) & \text{for } x > 0 \quad (13b) \\
w_{00} (0, y) &= 0 & \text{for } y > 0. \quad (13c)
\end{align*} \]

By the Lax-Milgram theorem, this problem is well-posed in \( H^1 (\mathbb{Q}) \). Then [8, Sections 10 & 12] imply that \( w_{00} \in C^{2\ell, \alpha} (\mathbb{Q}) \). The well-posedness of later boundary value problems can be handled similarly.

We shall present our arguments in the setting of a general quarter-plane problem so that the results can be applied to the later problem (32) as well as to (13). Set \( \beta = \min \left\{ \frac{p}{12}, \frac{q}{2p}, \sqrt{q} \right\} \). Assume that \( g \) and \( h \) are functions with
Let $r$ transform $(15)$ yields problems. Extend $h$ is the solution of $(16)$. 

Lemma 5.1. For $\ell$ derivatives of $w$ will be decomposed into a sum of solutions of half-plane problems. Extend $h(x)$ to a function $h_1(x)$ that vanishes for $x \leq -1$ with $\int_{-\infty}^{x} h_1(t) \, dt \in C^{2\ell,\alpha}(\mathbb{R})$. Let $w_1 \in C^{2\ell}(\bar{\Pi}_y)$ satisfy the grazing half-plane problem

$\begin{align*}
Lw_1 &= 0 \quad \text{on } \bar{\Pi}_y \\
w_{1,y}(x,0) &= h_1(x) \quad \text{for } x \in \mathbb{R}.
\end{align*}$

(15a) \hspace{1cm} (15b)

Write $\hat{w}_1(\xi, y)$ for the partial Fourier transform of $w_1$ with respect to $x$. Then transforming $(15)$ yields

$\varepsilon^2 \hat{w}_1 - \varepsilon \hat{w}_{1,yy} + \upsilon \varepsilon \hat{w}_1 + q \hat{w}_1 = 0 \quad \text{for } -\infty < \xi < \infty, y > 0$

$\hat{w}_{1,y}(\xi,0) = \hat{h}_1(\xi) \quad \text{for } \xi \in \mathbb{R}.$

(16)

Let $r(\xi) = \sqrt{\xi^2 + q\varepsilon^{-1} + \upsilon \varepsilon^{-1}}$. One can verify that $r(\xi) = s + it$ where $s = \frac{1}{\sqrt{2}} \left[ \xi^2 + q\varepsilon^{-1} + \sqrt{(\xi^2 + q\varepsilon^{-1})^2 + \upsilon^2 \xi^2 \varepsilon^{-2}} \right]^{\frac{1}{2}} > 0$ and $t = \frac{\upsilon \xi \varepsilon^{-1}}{2s}$. Thus

$\hat{w}_1(\xi, y) = -\frac{\hat{h}_1(\xi)}{\sqrt{\xi^2 + q\varepsilon^{-1} + \upsilon \varepsilon^{-1}}} e^{-r(\xi)y}$

(17)

is the solution of $(16)$.

We bound the pure $y$-derivatives of $w_1$ in Lemma 5.1 and the remaining derivatives of $w_1$ in Lemma 5.2.

Lemma 5.1. For $n \leq 2\ell$ there is a constant $C$ such that

$\|w_1(\cdot, y)\|_{C^{n,\alpha}(\mathbb{R})} \leq C e^{-\frac{n}{2\sqrt{2}} \|h_1\|_{C^{n,\alpha}(\mathbb{R})}}.$

Proof. Assume that $n \geq 1$ as the case $n = 0$ can be proved by a similar argument. From $(17)$, using the same idea that took us from $(10)$ to $(11)$, for $n \geq 1$ we write

$F(D^n \hat{w}_1)(\xi, y) = D^n_y \hat{w}_1(\xi, y) = e^{-r(\xi)y}[r(\xi)]^{n-1}(1 + i\xi)^{1-n} \hat{h}_{1,n-1}(\xi),$

(18)
where \( \hat{h}_{1,n-1}(\xi) = (1 + i\xi)^{-1}h_1(\xi) \). Thus for \( n = 1, \ldots, 2\ell \) one has \( h_{1,n-1}(x) = (1 + \frac{d}{d\xi})^{n-1}h_1(x) \) and \( \|h_{1,n-1}\|_{C^{n,\alpha}(\mathbb{R})} \leq C\|h_1\|_{C^{n-1,\alpha}(\mathbb{R})} \). We shall apply Theorem 4.1 to (18) with the multiplier

\[
M(\xi) = [-r(\xi)]^{n-1}(1 + i\xi)^{1-n}e^{-r(\xi)y} \quad \text{and} \quad K = C\varepsilon \frac{1+n}{2} e^{-\frac{\sqrt{n}y}{\sqrt{2}}}. 
\]

Note that \( |\Re(r(\xi))| = s \geq \sqrt{\xi^2 + q \varepsilon - 1} \geq \frac{|\xi| + \sqrt{q^2 - \frac{1}{2}}}{\sqrt{2}} \). Also \( \frac{r'(\xi)}{1 + |\xi|} \leq C\varepsilon^{-\frac{1}{2}} \). Hence \( |M(\xi)| \leq K \). Now \( r'(\xi) = \frac{2\xi + ip}{2\xi \sqrt{\xi^2 + q \varepsilon - 1 + ip \varepsilon}} \) so

\[
M'(\xi) = (-1)^{n+1}e^{-r(\xi)y} \left\{ (n-1)[r(\xi)]^{n-2}r'(\xi)(1 + i\xi)^{-n} - r'(\xi)y[r(\xi)]^{n-1}(1 + i\xi)^{-n} + i[r(\xi)]^{n-1}(1 - n)(1 + i\xi)^{-n} \right\}.
\]

It follows that

\[
|M'(\xi)| \leq C\left\{ |r(\xi)|^{n-1}(1 + i\xi)^{1-n}e^{-r(\xi)y}| \left\{ |[r(\xi)]^{-1}r'(\xi)| + |r'(\xi)|y + |(1 + i\xi)^{-1}| \right\}. 
\]

Recall that \( r = s + it \); now \( |t| = \frac{q \varepsilon + 1}{2\pi} \leq s \) from the formula for \( s \). Hence

\[
|r(\xi)y e^{-r(\xi)y}| \leq \sqrt{2}s e^{-s} \leq Ce^{-\sqrt{n}y} \leq Ce^{-\frac{\sqrt{n}y}{\sqrt{2}}}. 
\]

Using this inequality to bound the second term in \( \{ \cdots \} \) above, we get

\[
|M'(\xi)| \leq C\varepsilon \frac{1+n}{2} e^{-\frac{\sqrt{n}y}{\sqrt{2}}} \left\{ |[r(\xi)]^{-1}r'(\xi)| + |(1 + i\xi)^{-1}| \right\} 
\]

\[
\leq C\varepsilon \frac{1+n}{2} e^{-\frac{\sqrt{n}y}{\sqrt{2}}} \left[ \frac{2\xi + ip}{2\xi \sqrt{\xi^2 + q \varepsilon - 1 + ip \varepsilon}} + \frac{1}{1 + |\xi|} \right] 
\]

\[
\leq \frac{K}{1 + |\xi|},
\]

on considering separately the cases \( |\xi| \leq 1 \) and \( |\xi| > 1 \). Thus \( M(\xi) \) is seen to satisfy (8) with \( j = 1 \). The lemma follows from Theorem 4.1.

We now proceed to bound all derivatives of \( w_1 \) in terms of the norm \( \| \cdot \|_{\infty, \Pi_y} \).

**Lemma 5.2.** For \( m + n \leq 2\ell - 1 \) there is a constant \( C \) such that

\[
\|D_x^m D_y^n w_1\|_{\infty, \Pi_y} \leq Ce^{-\frac{\sqrt{n}y}{2\sqrt{2}}} \varepsilon \frac{1+n}{2} \|h_1\|_{m+n+1, \infty, \mathbb{R}} \quad (19)
\]

**Proof.** Let \( v(x, y) = D_x^m w_1(x, y) \). Applying \( D_x^m \) to (15) yields \( Lv = 0 \) on \( \Pi_y \) with \( v(x, 0) = D_x^m h_1(x) \) for \( x \in \mathbb{R} \). Then an invocation of Lemma 5.1 gives

\[
\|w_1(\cdot, y)\|_{C^{m+n, \alpha}(\mathbb{R})} \leq Ce^{-\frac{\sqrt{n}y}{2\sqrt{2}}} \varepsilon \frac{1+n}{2} \|h_1\|_{C^{m+n, \alpha}(\mathbb{R})}. 
\]

The inequality (19) follows. \[\square\]
Continuing with the decomposition, set \( w_2(x, y) = w(x, y) - w_1(x, y) \) and \( g_2(y) = g(y) - w_1(0, y) \). Then \( w_2 \) satisfies the quarter-plane problem

\[
Lw_2 = 0 \quad \text{on } \Omega \\
w_2(0, y) = g_2(y) \quad \text{for } y > 0 \\
w_{2,y}(x, 0) = 0 \quad \text{for } x > 0.
\]

Like \( w \), the function \( w_2 \) will have compatibility index \( \nu \) at \((0, 0)\), which implies \( g_2^{(2k+1)}(+0) = 0 \) for \( k = 0, \ldots, \nu \). Let \( w_3(x, y) \) and \( g_3(y) \) be even extensions of \( w_2 \) and \( g_2 \) for \( y < 0 \). The functions \( w_{3,x}, w_{3,xx} \) and \( w_{3,yy} \) are even functions of \( y \) and hence continuous across \( y = 0 \). Furthermore, \( w_{3,y} \) is an odd function of \( y \) and \( w_{3,y}(x, 0) = 0 \) so \( w_{3,y}(x, y) \) is continuous across \( y = 0 \). Thus \( w_3(x, y) \) is a classical solution of the boundary value problem \( Lw_3 = 0 \) on \( \Pi_x \), \( w_3(0, y) = g_3(y) \) for \( y \in \mathbb{R} \).

Next, we deal with discontinuities in derivatives of \( g_3(y) \). As \( g_3 \) is even, its even-order derivatives are automatically continuous on \( \mathbb{R} \). Define the numbers \( d_0, d_1, \ldots, d_{\nu+1} \) by setting

\[
\sum_{\mu=0}^{\nu+1} d_\mu 2^{2k\mu} = \begin{cases} 
1 & \text{if } k = 0 \\
0 & \text{if } k = 1, \ldots, \nu + 1.
\end{cases}
\]  

(If \( \nu = -1 \), set \( d_0 = 1 \).) This Vandermonde system of linear equations has a unique solution \( d_0, \ldots, d_{\nu+1} \). Define the even function \( \zeta_j(y) = \sum_{\nu+1} c_j \zeta_j(y) \) for all \( y \in \mathbb{R} \), where

\[
\zeta_j(y) = \sqrt{\varepsilon} \sum_{\mu=0}^{\nu+1} d_\mu (\sqrt{q} + j)^{2\mu} \exp \left\{ -\frac{(\sqrt{q} + j)^{2\mu} |y|}{\sqrt{\varepsilon}} \right\} \quad \text{for } j = \nu + 1, \ldots, \ell
\]

and the \( \ell - \nu \) numbers \( c_j \) will be chosen shortly. For all \( k \), clearly

\[
\zeta_j^{(2k+1)}(\pm 0) = \mp \varepsilon^{-k} (\sqrt{q} + j)^{2k+2} \sum_{\mu=0}^{\nu+1} d_\mu 2^{(2k+2)\mu}.
\]  

From (20) and (22) we see that \( \zeta_j^{(2k+1)}(y) \) is continuous at \( y = 0 \) for \( k = 0, \ldots, \nu \). All even-order derivatives of \( \zeta_j \) are automatically continuous at \( y = 0 \). Thus \( \zeta_j \in C^{2k+2}(\mathbb{R}) \). To specify the \( \ell - \nu \) numbers \( c_j \), we first impose the \( \ell - \nu - 1 \) conditions

\[
\zeta_j^{(2k+1)}(+0) = g_3^{(2k+1)}(+0) \quad \text{for } k = \nu + 1, \ldots, \ell - 1.
\]

Equivalently, using (22) and \( \zeta = \sum_{j=\nu+1}^{\ell} c_j \zeta_j \), we require

\[
\varepsilon^k g_3^{(2k+1)}(+0) = -\left( \sum_{\mu=0}^{\nu+1} d_\mu 2^{(2k+2)\mu} \right) \sum_{j=\nu+1}^{\ell} c_j (\sqrt{q} + j)^{2k}
\]

for \( k = \nu + 1, \ldots, \ell - 1 \).
Recall that \( g_3(y) \) is an even extension of \( g_2(y) \) and \( g_2(y) = g(y) - w_1(0, y) \), so \( g_2 \in C^{2\ell,\alpha}(\mathbb{R}) \). From (14) and Lemma 5.2 one sees that
\[
\varepsilon^k |g_3^{(2k+1)}(y)| \leq C \quad \text{for } k = 0, \ldots, \ell.
\] (24)

Now
\[
\left| \int_{0}^{\infty} g_3(y) \, dy \right| = \left| \int_{0}^{\infty} g_2(y) \, dy \right| = \left| \int_{0}^{\infty} [g(y) - w_1(0, y)] \, dy \right| \leq C\varepsilon
\]
by (14) and (19), while (21) and (20) yield
\[
\int_{0}^{\infty} \zeta(y) \, dy = \varepsilon \sum_{j=\nu+1}^{l} c_j \sum_{\mu=0}^{\nu+1} d_\mu = \varepsilon \sum_{j=\nu+1}^{l} c_j.
\]
We now impose the further condition
\[
\int_{0}^{\infty} g_3(y) \, dy = \int_{0}^{\infty} \zeta(y) \, dy; \quad (25)
\]
our calculations above show that this is equivalent to setting
\[
\sum_{j=\nu+1}^{\ell} c_j = \phi, \quad \text{where } |\phi| \leq C. \quad (26)
\]
As the \( d_\mu \) are already determined, (23) and (26) together form a Vandermonde system for the \( c_j \). Furthermore, (24) and \(|\phi| \leq C\) imply that the \( c_j \) are bounded independently of \( \varepsilon \). Finally, the construction of \( \zeta \) ensures that \( g_3 - \zeta \) is in \( C^{2\ell,\alpha}(\mathbb{R}) \).

To shorten the analysis we now aim to appeal to a result in [6], but this cannot be done directly because we have even boundary data for our half-plane problems while [6] has odd boundary data. This motivates the main idea in the proof of Lemma 5.3, which reveals the purpose of the condition (25).

Define the function \( \Phi(x, y) \) by
\[
L \Phi = 0 \quad \text{on } \Pi_x, \quad \Phi(0, y) = \zeta(y) \quad \text{for } y \in \mathbb{R}.
\]
Set \( w_4(x, y) = w_3(x, y) - \Phi(x, y) \) and \( g_4(y) = g_3(y) - \zeta(y) \). Then \( w_4 \) is the solution of the half-plane problem
\[
Lw_4 = 0 \quad \text{on } \Pi_x, \quad w_4(0, y) = g_4(y) \quad \text{for } y \in \mathbb{R}.
\]

**Lemma 5.3.** Let \( \varepsilon < \frac{p^2}{q} \). Then for non-negative integers \( m \) and \( n \) satisfying \( 2m + n \leq 2\ell - 1 \) there exists a constant \( C \) such that for all \((x, y) \in \bar{\Pi}_x \) one has
\[
|D_x^m D_y^n w_4(x, y)| \leq C\varepsilon^{\frac{4m}{p^2}} e^{-\frac{\varepsilon}{p^2}} e^{-\frac{\sqrt{q} \varepsilon |y|}{2\varepsilon}}.
\]
Proof. Our construction puts \( g_4 \) in \( C^{2\ell,\alpha}(\mathbb{R}) \). By (14), (21) and Lemma 5.2, the even function \( g_4 \) satisfies
\[
|g_4^{(k)}(y)| \leq C \varepsilon^{1-k} e^{-\frac{\sqrt{q}y}{2\sqrt{\varepsilon}}} \quad \text{for } y \in \mathbb{R} \text{ and } k = 0, 1, \ldots, 2\ell - 1. \tag{27}
\]
Define the function \( G_4 \) by
\[
G_4(y) = \begin{cases} 
\int_{-\infty}^{y} g_4(t) \, dt & \text{if } y \geq 0 \\
\int_{y}^{\infty} g_4(t) \, dt & \text{if } y < 0.
\end{cases}
\]
Then \( G'_4(y) = g_4(y) \) for \( y \neq 0 \) and \( G_4 \) is an odd function. Furthermore, \( G_4(0) = 0 \) because (25) holds. Hence \( G_4 \in C^{2\ell+1,\alpha}(\mathbb{R}) \), and from (27) and the definition of \( G_4 \) we infer that
\[
|G_4^{(k)}(y)| \leq C \varepsilon^{1-k} e^{-\frac{\sqrt{q}y}{2\sqrt{\varepsilon}}} \quad \text{for } y \in \mathbb{R} \text{ and } k = 0, 1, \ldots, 2\ell. \tag{28}
\]
Define the function \( W_4 \) by
\[
LW_4 = 0 \quad \text{on } \Pi_x, \quad W_4(0, y) = G_4(y) \quad \text{for } y \in \mathbb{R}. \tag{29}
\]
Bearing (28) in mind, we see that the problem (29) is identical to [6, (3.12)] except that the data has been multiplied by \( \varepsilon \). Thus one can multiply the bound of [6, Theorem 3.2] by \( \varepsilon \) to get
\[
|D^m_x D^n_y W_4(x, y)| \leq C \varepsilon^{\frac{2-n}{2}} e^{-\frac{\alpha}{2p} e^{-\frac{\sqrt{q}y}{2\sqrt{\varepsilon}}}} \quad \text{on } \Pi_x \text{ for } 2m + n \leq 2\ell;
\]
this result is valid for \( \varepsilon < \frac{p^2}{q} \). But \( w_4 = W_{4,y} \), so we are done.

Set \( r = \sqrt{x^2 + y^2} \); this is the distance from \((x, y)\) to, the corner \((0, 0)\). The following two definitions will recur frequently in subsequent bounds. Set
\[
\phi_2(x, y) = \exp\left(-\frac{qx}{2p}\right) \exp\left(-\frac{\beta y}{2\sqrt{\varepsilon}}\right) \tag{30}
\]
and
\[
\psi(\nu, m, n, r) = \begin{cases} 
 r^{2\nu+3-m-n} |\ln r| & \text{if } m + n \leq 2\nu + 3 \\
 r^{2\nu+3-m-n} & \text{if } m + n > 2\nu + 3.
\end{cases} \tag{31}
\]
For \( \mu = 0, \ldots, \nu+1 \) and \( j = \nu+1, \ldots, \ell \), let \( \Phi_{\mu,j} \) be the solution to the half-plane problem
\[
L\Phi_{\mu,j} = 0 \quad \text{on } \Pi_x, \quad \Phi_{\mu,j}(0, y) = \sqrt{\varepsilon} e^{-(\sqrt{q}+j)2\mu |y|/\sqrt{\varepsilon}} \quad \text{for } y \in \mathbb{R}.
\]
Thus \( \Phi(x, y) = \sum_{j=\nu+1}^{\ell} c_j \sum_{\mu=0}^{\nu+1} d_{\mu}(\sqrt{q}+j)2^{\mu}\Phi_{\mu,j}(x, y) \) on \( \Pi_x \).
Lemma 5.4. Let $r^* \geq \varepsilon$ be given and $n$ be a non-negative integer. Then there exists a constant $C$ which depends on $r^*$, $n$ and $\nu$ such that
\[
|D_y^n \Phi(x, y)| \leq C \left[ \varepsilon^{\frac{1-n}{2}} + \varepsilon^{-\nu-1} \varepsilon^{2\nu+3-n} \right] \quad \text{for } r < \varepsilon
\]
\[
|D_y^n \Phi(x, y)| \leq C \varepsilon^{\frac{1-n}{2}} \left[ 1 + r^{\nu+\frac{1-n}{2}} \right] \phi_2(x, |y|) \quad \text{for } \varepsilon \leq r \leq r^*.
\]
Proof. First consider the case $n = 0$. Use the barrier functions $W_1(x, y) = C\sqrt{\varepsilon} e^{-\frac{y^2}{2\varepsilon} e^{-\frac{x^2}{2\varepsilon}}}$ and $W_2(x, y) = C\sqrt{\varepsilon} e^{-\frac{y^2}{2\varepsilon} e^{-\frac{x^2}{2\varepsilon}}}$. Then for $j = \nu + 1, \ldots, \ell$ and $i = 1, 2$ one has $LW_i \geq 0 = L\Phi_{\mu,j}$ in $\Pi_x$ and $W_i(0, y) \geq |\Phi_{\mu,j}(0, y)|$. The growth conditions derived in [6, Section 3] show that the use of a maximum principle is justified; this establishes the bounds on $\Phi_{\mu,j}$. By linear superposition the same bounds hold for $\Phi$.

Assume that $n \geq 1$. For $j = \nu + 1, \ldots, \ell$, let $\theta_j$ be the solution of the half-plane problem $L\theta_j = 0$ on $\Pi_x$, $\theta_j(0, y) = \zeta_j(y)$ for $y \in \mathbb{R}$. Applying $D_y$ to $\theta_j$ yields $L(D_y \theta_j) = 0$ on $\Pi_x$ with
\[
D_y \theta_j(0, y) = \zeta_j'(y) = -\sum_{\mu=0}^{\nu+1} d_{\mu}(\sqrt{q} + j)^{2\mu} (\text{sgn} y) \exp \left\{ -\frac{(\sqrt{q} + j)^{2\mu} |y|}{\sqrt{\varepsilon}} \right\}
\]
for $y \in \mathbb{R}$. Equations (20) and (22) give $\zeta_j^{(2k+1)}(+0) = 0$ for $k = 0, \ldots, \nu$. Thus $D_y \theta_j$ has the same properties as $z(x, y)$ in [6, §4]. Consequently [6, Corollary 4.1, Lemmas 4.6 and 4.8] provide bounds for $D_y^n \theta_j(x, y)$ and by linearity for $D_y^n \Phi$. \hfill $\square$

Bounds for all derivatives of $\Phi(x, y)$ are given in Lemma 5.5.

Lemma 5.5. Let $r^* \geq \varepsilon$ be given. Let $m$ and $n$ be non-negative integers. Then there exists a constant $C$ which depends on $r^*$, $m$, $n$ and $\nu$ such that
\[
|D_x^n D_y^m \Phi(x, y)| \leq C \left[ \varepsilon^{\frac{1-n}{2}} + \varepsilon^{\nu+2-m-n} + \varepsilon^{-\nu-1}\psi(\nu, m, n, r) \right] \quad \text{for } r < \varepsilon
\]
\[
|D_x^n D_y^m \Phi(x, y)| \leq C \varepsilon^{\frac{1-n}{2}} \left[ 1 + r^{\nu-m+\frac{1-n}{2}} \right] \phi_2(x, |y|) \quad \text{for } \varepsilon \leq r \leq r^*.
\]
Proof. The case $m = 0$ is covered in Lemma 5.4. The inductive argument on $m$ of [6, Theorem 4.1] gives the bounds for the higher-order $x$-derivatives. \hfill $\square$

Recall that $w = w_1 + w_4 + \Phi$. Combining Lemmas 5.2, 5.3 and 5.5 yields

Lemma 5.6. Let $m$ and $n$ be non-negative integers satisfying $2m + n \leq 2\ell - 1$. Fix $r^* \geq \varepsilon$. Then there is a constant $C$ which depends on $r^*$, $\ell$ and $\nu$, such that
\[
|D_x^n D_y^m w(x, y)| \leq C \left[ \varepsilon^{\frac{1-n}{2}} + \varepsilon^{\nu+2-m-n} + \varepsilon^{-\nu-1}\psi(\nu, m, n, r) \right] \quad \text{for } r < \varepsilon
\]
\[
|D_x^n D_y^m w(x, y)| \leq C \varepsilon^{\frac{1-n}{2}} \left[ 1 + r^{\nu-m+\frac{1-n}{2}} \right] \phi_2(x, |y|) \quad \text{for } \varepsilon \leq r \leq r^*.
\]
where the functions $\phi_2$ and $\psi$ were defined in (30) and (31).
To close this section, we define the incoming corner function \( w_{01} \), which handles the boundary data \( h_n^* \) on the side \( y = 1 \) of \( Q \) and any corner singularity at \((0, 1)\). It is the solution of the quarter-plane problem

\[
Lw_{01}(x, y) = 0 \quad \text{for } x > 0, \; y < 1 \\
w_{01, y}(x, 1) = h_n^*(x) - S_y(x, 1) \quad \text{for } x > 0 \\
w_{01}(0, y) = 0 \quad \text{for } y < 1.
\]

6. Outgoing corner functions

The next term in our decomposition of \( u \) is the outgoing corner function \( w_{10} \) which deals with the Dirichlet boundary data along the side \( x = 1 \) of \( Q \) as well as corner singularities that (1) may have at \((1,0)\). The inadvertent introduction of any incompatibility at other corners of \( Q \) is avoided through a \( C^\infty \) cut-off function \( \chi : \mathbb{R} \to [0, 1] \) that satisfies

\[
\chi(t) = \begin{cases} 
0 & \text{for } t \leq \frac{1}{3} \\
1 & \text{for } t \geq \frac{2}{3}.
\end{cases}
\]

Define \( w_{10} \) to be the solution of the quarter-plane problem

\[
Lw_{10}(x, y) = 0 \quad \text{for } x < 1, \; y > 0 \quad (32a) \\
w_{10}(1, y) = -\chi(1 - y)w_{00}(1, y) \quad \text{for } y > 0 \quad (32b) \\
w_{10, y}(x, 0) = -\chi(x)E_y(x, 0) \quad \text{for } x < 1. \quad (32c)
\]

The functions \( S, E, w_{01}, w_{10} \) and \( w_{11} \) are all smooth at \((0,0)\) and – as we shall see in Section 7 – the function \( \tilde{u} \) is compatible to arbitrary order at \((0,0)\). Thus, recalling the decomposition (3), it follows that \( w_{00} \) enjoys the same degree of compatibility as \( u \) at the corner \((0,0)\) as the function \( u \). A similar argument demonstrates that each of our four corner functions has the same degree of compatibility at its “home” corner as \( u \) has there.

Lemma 6.1. Let \( m \) and \( n \) be nonnegative integers satisfying \( 2m + n \leq 2\ell - 1 \). Fix \( r^* \geq \varepsilon \). Then there is a constant \( C \), which depends on \( r^* \), \( \ell \) and \( \nu_{00} \), such that

\[
\left| D_x^m D_y^n w_{00}(x, y) \right| \leq C \left( \varepsilon^{\frac{1-n}{2} + \varepsilon^{\nu_{00} + 2 - m - n}} \right) \quad \text{for } m + n < 2\nu_{00} + 3, \; r < \varepsilon \\
\left| D_x^m D_y^n w_{00}(x, y) \right| \leq C \left( \varepsilon^{\frac{1-n}{2} + \varepsilon^{-\nu_{00} - 1} \ln |r|} \right) \quad \text{for } m + n = 2\nu_{00} + 3, \; r < \varepsilon \\
\left| D_x^m D_y^n w_{00}(x, y) \right| \leq C \left( \varepsilon^{\frac{1-n}{2} + \varepsilon^{-\nu_{00} - 1} r^{2\nu_{00} + 3 - m - n}} \right) \quad \text{for } m + n > 2\nu_{00} + 3, \; r < \varepsilon \\
\left| D_x^m D_y^n w_{00}(x, y) \right| \leq C \varepsilon^{\frac{1-n}{2} \left[ 1 + r^{\nu_{00} + \frac{3-n}{2}} \right]} e^{-\frac{\delta y}{2}} \quad \text{for } \varepsilon \leq r \leq r^*.
\]
From Lemmas 4.4 and 6.1 the boundary conditions for \( (0,0) \). Furthermore, (2) and Lemma 3.1 imply that the boundary conditions (13b), (13c) satisfy (14). We can therefore invoke Lemma 5.6 to obtain the desired result follows; note that in the case \( m < 2 \nu_{10} + 3 \), \( r \) replaced by \( \nu_{10} \) and \( r_{01} \).

The next result resembles [7, Lemma 2].

**Lemma 6.2.** Let \( m \) and \( n \) be nonnegative integers satisfying \( 2m + n \leq 2\ell - 2 \). Fix \( r^* \geq \varepsilon \). Then there is a constant \( C \), which depends on \( r^* \), \( \ell \) and \( \nu_{10} \), such that

\[
\begin{align*}
|D_x^m D_y^n w_{10}(x,y)| &\leq C \varepsilon^{-m+1-\frac{m}{2}n} \quad \text{for } m+n < 2\nu_{10}+3, \quad r < \varepsilon \\
|D_x^m D_y^n w_{10}(x,y)| &\leq C \left[ \varepsilon^{-m+1-\frac{m}{2}n} + \varepsilon^{-r_{10}-1} \ln |r_{10}| \right] \quad \text{for } m+n = 2\nu_{10}+3, \quad r < \varepsilon \\
|D_x^m D_y^n w_{10}(x,y)| &\leq C \left[ \varepsilon^{-m+1-\frac{m}{2}n} + \varepsilon^{-r_{10}-1} \nu_{10}^{2\nu_{10}+3-m-n} \right] \quad \text{for } m+n > 2\nu_{10}+3, \quad r < \varepsilon \\
|D_x^m D_y^n w_{10}(x,y)| &\leq C \varepsilon^{-m+1-\frac{m}{2}n} \left[ 1 + \nu_{10} \nu_{10}^{r_{10}+\frac{m}{2}n} \right] e^{-\frac{r(1-r)}{2}} e^{-\frac{\partial y}{2\varepsilon}} \quad \text{for } \varepsilon \leq r_{10} \leq r^*.
\end{align*}
\]

**Proof.** Let \( v(x,y) = e^{\frac{\nu_{10}}{r}} w_{10}(1-x,y) \). Then

\[
\begin{align*}
Lv(x,y) &= 0 \quad \text{for } x > 0, \quad y > 0 \\
v(0,y) &= -\chi(1-y) w_{10}(1,y) \quad \text{for } y > 0 \\
v_y(x,0) &= -\varepsilon \frac{\nu_{10}}{r} \chi(1-x) E_y(1-x,0) \quad \text{for } x > 0.
\end{align*}
\]

From Lemmas 4.4 and 6.1 the boundary conditions for \( v \) satisfy (14) with \( 2\ell \) changed to \( 2\ell - 1 \). Also the function \( v \) has the same compatibility \( \nu_{10} \) at \( (0,0) \) as the function \( w_{10} \) had at \((1,0)\). We can therefore apply Lemma 5.6 to \( v \). Now

\[
\begin{align*}
|D_x^m D_y^n w_{10}(x,y)| &\leq C \sum_{i+j=m} |D_x^i (e^{-\frac{r(1-r)}{2}}) D_y^j D_y^n v(1-x,y)| \\
&\leq C \varepsilon^{-\frac{r(1-r)}{2}} \sum_{i+j=m} \varepsilon^{-i} |D_x^i D_y^j v(1-x,y)|
\end{align*}
\]

and the desired result follows; note that in the case \( m+n < 2\nu_{10} + 3 \) and \( r < \varepsilon \) we get the bound

\[
|D_x^m D_y^n w_{10}(x,y)| \leq C \left[ \varepsilon^{-m+\frac{1-n}{2}} + \varepsilon^{\nu_{10}-m-n+2} \right] = C \varepsilon^{-\frac{m+1-n}{2}} \left[ 1 + \varepsilon^{\nu_{10}+(3-n)} \right],
\]

but \( m+n < 2\nu_{10} + 3 \) implies that \( \varepsilon^{\nu_{10}+(3-n)} \leq C \).
The outgoing corner function $w_{11}$ is introduced to deal with the boundary conditions at the corner $(1,1)$. It is the solution of the quarter-plane problem

$$Lw_{11}(x,y) = 0 \quad \text{for } x < 1, y < 1$$
$$w_{11}(1,y) = -\chi(y)w_{01}(1,y) \quad \text{for } y < 1$$
$$w_{11,y}(x,1) = -\chi(x)E_y(x,1) \quad \text{for } x < 1.$$

Bounds on the derivatives of $w_{11}$ follow from Lemma 6.2 on making the change of variable $y \mapsto 1 - y$ with $\nu_{10}$ and $r_{10}$ replaced respectively by $\nu_{11}$ and $r_{11}$.

7. The remainder term $\hat{u}$

Finally we come to the remainder term $\hat{u}$. This function satisfies

$$L\hat{u} = 0 \quad \text{on } Q$$
$$\hat{u}(0,y) = \hat{g}_w(y) := -E(0,y) - w_{10}(0,y) - w_{11}(0,y)$$
$$\hat{u}(1,y) = \hat{g}_n(y) := [\chi(1-y) - 1]w_{00}(1,y) + [\chi(y) - 1]w_{01}(1,y)$$
$$\hat{u}_y(x,0) = \hat{h}_s(x) := [\chi(x) - 1]E_y(x,0) - w_{01,y}(x,0) - w_{11,y}(x,0)$$
$$\hat{u}_y(x,1) = \hat{h}_n(x) := [\chi(x) - 1]E_y(x,1) - w_{00,y}(x,1) - w_{10,y}(x,1).$$

We check compatibility at the origin; the other vertices are similar. Near $(0,0)$ the boundary data for $\hat{u}$ can be written as

$$\hat{g}_w(y) = -E(0,y) - w_{10}(0,y) - w_{11}(0,y) - w_{01}(0,y)$$
$$\hat{h}_n(x) = -E_y(x,0) - w_{01,y}(x,0) - w_{11,y}(x,0) - w_{10,y}(x,0).$$

Thus the boundary data for $\hat{u}$ is compatible to arbitrary order at $(0,0)$ since $E$, $w_{10}$, $w_{11}$ and $w_{01}$ are all $C^\infty$ functions in a neighbourhood of $(0,0)$. As in [6, Theorem 5.1], an energy argument and Sobolev imbedding now show that $\|\hat{u}\|_{2\sigma-2,\infty,Q} \leq C$.

8. Bound on the derivatives of $u$

In Section 1 we outlined a decomposition for $u$ into half-plane and quarter-plane problems. Subsequent sections proved derivative bounds for the solutions to each of these problems. In Theorem 8.1 the derivative bounds for the half-plane and quarter-plane problems are brought together to give derivative bounds for $u$. 
Theorem 8.1. Let m, n be non-negative integers satisfying 2m + n ≤ 2ℓ − 2. Let \( r_{ij} \) be defined by (33). Let \( \beta = \min \{ \frac{p}{12}, \frac{2}{2p}, \sqrt{q} \} \). Let \( \rho \in (0, p) \). Let \( \nu_{ij} \) give the compatibility of the data at the corner \((i, j)\). Then for \((x, y) \in Q\), the solution \( u \) of (1) satisfies

\[
|D_x^m D_y^n u(x, y)| \leq C(1 + T_0 + T_{01} + T_{10} + T_{11} + T_E)
\]

with \( T_E = \varepsilon^{-m} e^{-\frac{\beta(1-x)}{2\sqrt{x}}} \), where for \( \mu = 0, 1 \), one has

\[
T_{0\mu} = \varepsilon^{-\frac{1}{2}} + \varepsilon^{\nu_{0\mu} - m - n + 2} \quad \text{for } m + n < 2\nu_{0\mu} + 3, r_{0\mu} < \varepsilon
\]
\[
T_{0\mu} = \varepsilon^{-\frac{1}{2}} + \varepsilon^{-\nu_{0\mu} - 1} |\ln r_{0\mu}| \quad \text{for } m + n = 2\nu_{0\mu} + 3, r_{0\mu} < \varepsilon
\]
\[
T_{0\mu} = \varepsilon^{-\frac{1}{2}} + \varepsilon^{-\nu_{0\mu} - 1} r_{0\mu}^{2\nu_{0\mu} + 3 - m - n} \quad \text{for } m + n > 2\nu_{0\mu} + 3, r_{0\mu} < \varepsilon
\]
\[
T_{0\mu} = \varepsilon^{-\frac{1}{2}} \left[ 1 + r_{0\mu}^{\nu_{0\mu} - m + \frac{3 - n}{2}} \right] \varepsilon^{-\nu_{0\mu} - 1} \mu \frac{(p-\rho)}{2\sqrt{p}} \quad \text{for } r_{0\mu} \geq \varepsilon
\]

and

\[
T_{1\mu} = \varepsilon^{-m - \frac{1}{2}} \quad \text{for } m + n < 2\nu_{1\mu} + 3, r_{1\mu} < \varepsilon
\]
\[
T_{1\mu} = \varepsilon^{-m - \frac{1}{2}} + \varepsilon^{-\nu_{1\mu} - 1} |\ln r_{1\mu}| \quad \text{for } m + n = 2\nu_{1\mu} + 3, r_{1\mu} < \varepsilon
\]
\[
T_{1\mu} = \varepsilon^{-m - \frac{1}{2}} + \varepsilon^{-\nu_{1\mu} - 1} r_{1\mu}^{2\nu_{1\mu} + 3 - m - n} \quad \text{for } m + n > 2\nu_{1\mu} + 3, r_{1\mu} < \varepsilon
\]
\[
T_{1\mu} = \varepsilon^{-m - \frac{1}{2}} \left[ 1 + r_{1\mu}^{\nu_{1\mu} + \frac{3 - n}{2}} \right] \varepsilon^{-\nu_{1\mu} - 1} \mu \frac{(p-\rho)}{2\sqrt{p}} \quad \text{for } r_{1\mu} \geq \varepsilon
\]

Proof. Use the decomposition (3) and add the bounds that we have proved for each of its terms. \( \square \)

References


Received August 5, 2008; revised March 19, 2009