Perron's Method and Barrier Functions for the Viscosity Solutions of the Dirichlet Problem for some Non-Linear Partial Differential Equations

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Abstract. The Dirichlet problem for some non-linear partial differential equations via Perron's method is studied in the viscosity set up, by considering two families of functions, instead of one, as considered by others before. The notion of barrier at a boundary point is introduced to study the regularity of boundary points. Barriers for some non-linear operators are also constructed.

Keywords: Viscosity super- and subsolutions, Perron families, resolutive functions, barriers

AMS subject classification: 35J25, 35J67, 35G30

1. Introduction

In this paper, we try to adopt the classical Perron method of solutions of the Dirichlet problem for the Laplacian to the case of some non-linear partial differential equations and discuss the regularity of the boundary points using barrier functions.

Perron's approach in the viscosity set up was studied earlier by Ishii [2] and several others (see the references in [5]). In these works, only one family of functions was used to get a solution. We follow here fully Perron's idea of introducing two families of functions to study the existence problem and the barrier functions for the boundary behaviour of the solutions.

2. Basic definitions

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ and $F: \Omega \times \mathbb{R} \times \mathbb{R}^n \times M_n \rightarrow \mathbb{R}$ a map, where $M_n$ is the space of all $n \times n$ real symmetric matrices. We study the solutions of the non-linear partial differential equation

$$F (x, u(x), Du(x), D^2 u(x)) = 0 \quad (2.1)$$

in the viscosity sense.
Definition 1: An extended real-valued function \( u \) on \( \Omega \) is said to be a \textit{viscosity supersolution} (respectively, a \textit{viscosity subsolution}) of equation \( F = 0 \) if, for all \( \varphi \in C^2(\Omega) \), \( u - \varphi \) has a local minimum (maximum) at a point \( x_0 \in \Omega \) implies

\[
F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0
\]

respectively

\[
F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0.
\]

If the function \( u \) is both a super- and subsolution of equation \( F = 0 \), then it is said to be a \textit{viscosity solution}. All the solutions, super- and subsolutions considered here are only in the viscosity sense.

Let \( f \) be a real-valued function defined on \( \partial \Omega \). We define the Perron families of functions \( \Phi_f \) and \( \Psi_f \) as

\[
\Phi_f = \left\{ v \mid \begin{array}{c}
v \text{ bounded below lower semi-continuous supersolution such that } \\
\liminf_{y \to X, y \in \Omega} v(y) \geq f(X) \forall X \in \partial \Omega
\end{array} \right\}
\]

and

\[
\Psi_f = \left\{ u \mid \begin{array}{c}
u \text{ bounded above upper semi-continuous subsolution such that } \\
\limsup_{y \to X, y \in \Omega} u(y) \leq f(X) \forall X \in \partial \Omega
\end{array} \right\}
\]

Define

\[
\overline{H}_f = \inf_{v \in \Phi_f} v \quad \text{and} \quad H_f = \sup_{u \in \Psi_f} u
\]

with the convention that the infimum and supremum over a void family is \( +\infty \) and \( -\infty \), respectively.

3. Existence of solutions

Let \( F \) be degenerate elliptic:

\[
F(x, r, p, X + Y) \leq F(x, r, p, X)
\]

(3.1)

for all \( (x, r, p, X) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times M_n \) and \( Y \geq 0 \), symmetric. Let us assume further that

\[
F(x, 0, 0, 0) = 0
\]

(3.2)

for all \( x \in \Omega \) and \( F \) is monotone non-decreasing in the sense that

\[
F(x, t, p, A) \geq F(x, s, p, A)
\]

(3.3)

for all \( (x, p, A) \in \Omega \times \mathbb{R}^n \times M_n \) and \( t, s \in \mathbb{R} \) such that \( t \geq s \).
Remark 1: Conditions (3.2) and (3.3) together imply that any positive real number is a positive supersolution and any negative real number is a negative subsolution of equation $F = 0$. Also, it can be easily proved using condition (3.3) that if $v$ is a supersolution and $u$ a subsolution of equation $F = 0$, then so is $v + \varepsilon$ and $u - \varepsilon$, respectively, for all $\varepsilon > 0$.

We shall assume further that the map $F$ has properties such that the following comparison theorem holds:

If $u$ is an upper semi-continuous subsolution bounded from above and $v$ a lower semi-continuous supersolution bounded from below of equation (2.1) such that

$$\limsup_n (u(x_n) - v(y_n)) \leq M$$

where $x_n, y_n \in \Omega, |x_n - y_n| \to 0$ and $\text{dist}(x_n, \partial \Omega) \to 0$, then

$$u(x) - v(x) \leq M$$

for all $x \in \Omega$.

Several sets of conditions on the map $F$ under which the above comparison property holds, appear in [4].

Let us assume that $f$ is a bounded function defined on $\partial \Omega$. Then, by using similar arguments as in [2], it can be proved that both $H_f$ and $\overline{H}_f$ are solutions of the equation $F = 0$. But, they need not be continuous. By the comparison theorem, it follows that every $u \in \Psi_f$ is less than or equal to every $v \in \Phi_f$. Hence $H_f \leq \overline{H}_f$ on $\Omega$.

Definition 2 (see [6]): The function $f$ is said to be semi-resolutive if $H_f = \overline{H}_f$. If $f$ is semi-resolutive, then the above common value is denoted by $H_f$.

Definition 3: The function $f$ is said to be resolutive if $f$ is semi-resolutive and $H_f$ is continuous.

Theorem 1: If $\{f_n\}$ is a sequence of semi-resolutive (resp. resolutive) functions converging to $f$ uniformly on $\partial \Omega$, then this limit function $f$ is also semi-resolutive (resp. resolutive) and $H_{f_n} \to H_f$ uniformly on $\Omega$.

For the proof of this theorem, we need the following lemma which can be easily proved.

Lemma 1: For all $\varepsilon > 0$, the inequalities $H_f + \varepsilon \geq H_f$ and $H_f - \varepsilon \leq H_f - \varepsilon$ are true.

Proof of Theorem 1: Since $f_n$ converges to $f$ uniformly on $\partial \Omega$, we have, for any $\varepsilon > 0$, the existence of an $N(\varepsilon) \in N$ such that

$$f_n(X) - \varepsilon \leq f(X) \leq f_n(X) + \varepsilon$$

for $X \in \partial \Omega$ and $n \geq N(\varepsilon)$.

Thus $H_{f_n - \varepsilon} \leq H_f \leq H_f \leq H_{f_n + \varepsilon}$. Using Lemma 1, it follows that

$$H_{f_n} - \varepsilon \leq H_f \leq H_f \leq H_{f_n} + \varepsilon.$$  \hspace{1cm} (3.4)

Since $f_n$ are semi-resolutive for all $n$, it follows that $|H_f - H_f| \leq 2\varepsilon$ for all $\varepsilon > 0$. This implies that $H_f = H_f$, proving that $f$ is semi-resolutive \(\blacksquare\)
From inequalities (3.4) we also get the estimate \(|H_{f_n} - H_f| \leq \varepsilon\) on \(\Omega\) for all \(n \geq N(\varepsilon)\). This implies that the sequence \(\{H_{f_n}\}\) converges to \(H_f\) uniformly on \(\Omega\). Thus, if the functions \(f_n\) are resolutive for all \(n\), so is also the function \(f\).

**Theorem 2:** Let \(f \in C(\partial \Omega)\). Suppose \(f\) admits a continuous extension \(f_1 : \Omega \to \mathbb{R}\) being a supersolution (resp. subsolution) of equation \(F = 0\). Then \(f\) is semi-resolutive and \(H_f\) is upper semi-continuous (resp. lower semi-continuous).

**Proof:** We shall prove the theorem assuming that the map \(f_1\) is a supersolution. The proof when \(f_1\) is a subsolution is analogous.

Obviously, \(f_1 \in \Phi_f\) and \(H_f \leq f_1\). As \(f_1\) is continuous, \((H_f)^* \leq f_1\). Thus, for \(X \in \partial \Omega\),

\[
\limsup_{x \to X, z \in \Omega} (H_f)^*(z) = \limsup_{x \to X, z \in \Omega} f_1(z) = f(X).
\] (3.5)

This implies that \((H_f)^* \in \Psi_f\). Hence, \(H_f \geq (H_f)^*\). Thus, \((H_f)^* \geq H_f \geq (H_f)^*\), implying that \(H_f = H_f = (H_f)^*\). Hence the function \(f\) is semi-resolutive and \(H_f\) is upper semi-continuous.

**Corollary** (see [3: Theorem 3.2]): Let the function \(f \in C(\partial \Omega)\) admit two continuous extensions \(f_1\) and \(f_2\) such that \(f_1\) is a subsolution and \(f_2\) is a supersolution of equation \(F = 0\). Then \(f\) is resolutive and \(H_f(x) \to f(x)\) for all \(X \in \partial \Omega\) as \(x \to X, x \in \Omega\).

**Proof:** It is immediate from Theorem 2 and the inequalities \(\limsup_{y \to X, y \in \Omega} H_f(y) \leq f(X) \leq \liminf_{y \to X, y \in \Omega} H_f(y)\).

**Remark 2:** When the function \(f\) admits a continuous extension \(f_1 : \overline{\Omega} \to \mathbb{R}\) being a supersolution (resp. subsolution) of equation \(F = 0\), then we get an upper semi-continuous (resp. lower semi-continuous) solution in \(\overline{\Omega}\) of the Dirichlet problem

\[
F(x, u, Du, D^2u) = 0 \quad \text{in} \quad \Omega, \quad u = f \quad \text{on} \quad \partial \Omega
\] (3.6)

by defining \(H_f(X) = f(X)\) on \(\partial \Omega\). The upper semi-continuity of this solution at the boundary points follows from (3.5) and (3.6).

**Application:** Consider a degenerate elliptic operator \(F : \Omega \times M_n \to \mathbb{R}\) satisfying conditions (3.1) and (3.2). Then, if \(A\) is positive-definite,

\[
F(x, A) \leq 0 \quad \text{for all} \quad x \in \Omega
\]

as \(F(x, 0) = 0\). Thus, if \(\Psi\) is a \(C^2\) convex function on \(\overline{\Omega}\), where \(\Omega\) is convex, then \(F(x, D^2 \Psi(x)) \leq 0\), implying that \(\Psi\) is a subsolution of equation \(F = 0\). If \(f = \Psi\) on \(\partial \Omega\), then by Theorem 2 \(f\) is semi-resolutive and \(H_f\) is lower semi-continuous.

As every convex function \(\Psi\) is a uniform limit of \(C^2\) convex functions, it follows from Theorem 1 that if \(f = \Psi\) on \(\partial \Omega\), then \(f\) is semi-resolutive and \(H_f\) is lower semi-continuous.
4. Regular points and barrier functions

Let $F$ be as in (2.1) and the function $f$ be semi-resolutive. Now, we seek conditions on $\Omega$ so that if $f$ is continuous on $\partial\Omega$, then $H_f(y) \rightarrow f(X)$ as $y \in \Omega \rightarrow X$ for all $X \in \partial\Omega$. To this end, we define the notion of a barrier at a boundary point.

**Definition 4:** Let $X_0 \in \partial\Omega$. A pair of functions $(v, u)$ is called a barrier at $X_0$ if the following conditions are satisfied:

(i) $v \geq 0$ and $\lambda v$ is a lower semi-continuous supersolution of $F = 0$ for all $\lambda \geq 0$.
(ii) $u \leq 0$ and $\lambda u$ is an upper semi-continuous subsolution of $F = 0$ for all $\lambda \geq 0$.
(iii) $\lim_{y \in \Omega, y \rightarrow X_0} v(y) = 0 = \lim_{y \in \Omega, y \rightarrow X_0} u(y)$
(iv) $\liminf_{y \in \Omega, y \rightarrow X} v(y) > 0 > \limsup_{y \in \Omega, y \rightarrow X} u(y)$ for all $X \in \partial\Omega, X \neq X_0$.

**Definition 5:** A point $X_0 \in \partial\Omega$ is said to be regular if there is a barrier at $X_0$.

From now onwards, we shall assume that $F$ is a mapping from $\Omega \times \mathbb{R}^n \times M_n$ to $\mathbb{R}$. This will ensure that if $v$ is a super- or subsolution of equation $F = 0$, then so is $v + c$ for every real number $c$.

**Theorem 3:** Let a barrier exist at $X_0 \in \partial\Omega$ and let $f$ be a bounded function on $\partial\Omega$ which is continuous at $X_0$. If $f$ is semi-resolutive, then $H_f(y) \rightarrow f(X_0)$ as $y \rightarrow X_0, y \in \Omega$.

**Proof:** The idea is to construct, for all $\varepsilon > 0$, a supersolution $\omega^\varepsilon \in \Phi_f$ and a subsolution $\omega_\varepsilon \in \Psi_f$ such that

\[ \limsup_{y \rightarrow X_0, y \in \Omega} \omega^\varepsilon(y) \leq f(X_0) + \varepsilon \quad \text{and} \quad \liminf_{y \rightarrow X_0, y \in \Omega} \omega_\varepsilon(y) \geq f(X_0) - \varepsilon. \]

Since $f$ is continuous at $X_0$, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(X) - f(X_0)| < \varepsilon$ for all $X$ with $|X - X_0| \leq \delta$. Let $(v, u)$ be a barrier at $X_0$. The function $g : \partial\Omega \rightarrow \mathbb{R}$ defined as

\[ g(X) = \liminf_{y \rightarrow X, y \in \Omega} v(y) \]

is lower semi-continuous and therefore attains its minimum $m_1$ on the compact set

\[ K = \partial\Omega \cap \{X : |X - X_0| \geq \delta\}. \]

As $g(X) > 0$ for $X \in K, m_1 > 0$. Let $M > \sup_{X \in \partial\Omega} |f(X)|$. Define

\[ \omega^\varepsilon = f(X_0) + \varepsilon + \frac{v}{m_1}(M - f(X_0)). \]

Then $\omega^\varepsilon$ is a supersolution.

Let us prove that $\omega^\varepsilon \in \Phi_f$. Let $X \in \partial\Omega$ be such that $|X - X_0| \leq \delta$. Then

\[ \liminf_{y \rightarrow X, y \in \Omega} \omega^\varepsilon(y) \geq f(X_0) + \varepsilon \geq f(X). \]

Let $X \in \partial\Omega$ be such that $|X - X_0| \geq \delta$. Then

\[ \liminf_{y \rightarrow X, y \in \Omega} \omega^\varepsilon(y) \geq f(X_0) + \varepsilon + (M - f(X_0)). \]
Thus, if \(|X - X_0| \geq \delta\), then

\[
\liminf_{y \to X, y \in \Omega} \omega^\varepsilon(y) \geq M + \varepsilon > f(X).
\]

Hence \(\omega^\varepsilon \in \Phi_f\). Therefore, \(H_f \leq \omega^\varepsilon\) and

\[
\limsup_{y \to X_0, y \in \Omega} H_f(y) \leq \limsup_{y \to X_0, y \in \Omega} \omega^\varepsilon(y) = f(X_0) + \varepsilon.
\]

As \(\varepsilon > 0\) is arbitrary,

\[
\limsup_{y \to X_0, y \in \Omega} H_f(y) \leq f(X_0).
\]

Let \(m < \inf_{x \in \partial \Omega} f(X)\) and \(m_2 < 0\) be such that

\[
\limsup_{y \to X, y \in \Omega} u(y) \leq m_2
\]

for all \(X\) with \(|X - X_0| \geq \delta\). Define

\[
\omega^\varepsilon = f(X_0) - \varepsilon - \frac{u}{m_2}(f(X_0) - m).
\]

Then \(\omega^\varepsilon\) is a subsolution and let us prove that \(\omega^\varepsilon \in \Psi_f\).

Let \(X \in \partial \Omega\) be such that \(|X - X_0| < \delta\). By the choice of \(m\) and \(m_2\),

\[
\omega^\varepsilon \leq f(X_0) - \varepsilon \quad \text{and} \quad \limsup_{y \to X, y \in \Omega} \omega^\varepsilon(y) \leq f(X_0) - \varepsilon \leq f(X).
\]

Let \(X \in \partial \Omega\) be such that \(|X - X_0| \geq \delta\). Let \(\lambda = -1/m_2\). Then \(\lambda > 0\) and

\[
\limsup_{y \to X, y \in \Omega} \omega^\varepsilon(y) = f(X_0) - \varepsilon + \lambda f(X_0) - m)
\]

\[
\leq f(X_0) - \varepsilon + \lambda (f(X_0) - m) m_2
\]

\[
\leq f(X_0) - \varepsilon + (-1/m_2)(f(X_0) - m) m_2
\]

\[
= f(X_0) - \varepsilon - f(X_0) + m
\]

\[
= m - \varepsilon
\]

\[
< f(X).
\]

Hence \(\omega^\varepsilon \in \Psi_f\) and \(H_f \geq \omega^\varepsilon\). Therefore

\[
\liminf_{y \to X_0, y \in \Omega} H_f(y) \geq \liminf_{y \to X_0, y \in \Omega} \omega^\varepsilon(y) = f(X_0) - \varepsilon.
\]

As \(\varepsilon > 0\) is arbitrary, \(\liminf_{y \to X_0, y \in \Omega} H_f(y) \geq f(X_0)\). Thus,

\[
f(X_0) \leq \liminf_{y \to X_0, y \in \Omega} H_f(y) \leq \limsup_{y \to X_0, y \in \Omega} H_f(y) \leq f(X_0),
\]

implying \(\lim_{y \to X_0, y \in \Omega} H_f(y) = f(X_0)\).
5. Barrier functions for some $\Omega$ and some operators

In this section, we shall construct barriers for some operators $F$ and some domains $\Omega$.

**Example 1:** Consider the quasilinear degenerate elliptic equations of the type

$$F(x, Du, D^2 u) = -\text{Tr}(a(Du)D^2 u) = 0 \quad \text{in } \Omega$$

(5.1)

where $a \in C(\mathbb{R}^n, M_n)$. This type is considered in [4]. We shall construct a barrier at every boundary point of a ball $B(y; r) \subset \Omega$.

Let $\Psi(x) = |x - y|^2 - r^2$ and $d(x) = |x - X_0|^2$ where $X_0 \in \partial B(y; r)$. Let

$$v = d(x) - \Psi(x).$$

Then $D^2(\lambda v) = 0$ for all $\lambda \geq 0$. Hence $\lambda v$ is a solution in $B(y; r)$ of equation (5.1). Further, $v \geq 0$ and $v(X_0) = 0, v(X) > 0$ for points $X \in \partial B(y; r)$ with $X \neq X_0$.

Consider $u = \psi(x) - d(x) = -v$. Then $\lambda u$ is a solution in $B(y; r)$ of equation (5.1). Further $u \leq 0, u(X_0) = 0$ and $u(X) < 0$ for points $X \in \partial B(y; r)$ with $X \neq X_0$. Hence $(v, u)$ is a barrier at $X_0$. Thus, every point on $\partial B(y; r)$ is regular.

**Example 2:** Let $\Omega$ be an open set in $\mathbb{R}^n$ ($n \geq 2$). The set $\Omega$ is said to satisfy an exterior sphere condition at $X_0 \in \partial \Omega$ if there exists an $R > 0$ such that $\overline{B(y; R)} \cap \Omega = \{X_0\}$. We shall construct a barrier at such a point $X_0$ for equations of type (5.1) for some suitable function $a$.

Without loss of generality, let us assume that $y = 0$. We define on $\Omega$, for all $\sigma > 0$, the function $v_{\sigma}(x) = 1/R^\sigma - 1/|x|^\sigma$. Then we have, for all $x \in \Omega$ and $\lambda \in \mathbb{R}$,

$$D(\lambda v_{\sigma})(x) = \frac{\lambda \sigma}{|x|^{\sigma+2}} x \quad \text{and} \quad D^2(\lambda v_{\sigma})(x) = \frac{\lambda \sigma}{|x|^{\sigma+2}} I_n - \frac{\lambda \sigma(\sigma + 2)}{|x|^{\sigma+4}} B(x)$$

where $I_n$ is the $n \times n$ identity matrix and $B(x)$ is the positive definite matrix $[b_{ij}]$ with $b_{ij} = x_i x_j$. Let us assume that, for some $l \in \mathbb{N}$,

$$a(k\xi) = k^{2l}a(\xi) \quad \text{for all } k > 0.$$  

(5.2)

Then it is seen easily that

$$F(x, D(\lambda v_{\sigma}), D^2(\lambda v_{\sigma})) = \frac{(\lambda \sigma)^{2l+1}}{|x|^{2l(\sigma+2)+\sigma+4}} \{(\sigma + 2)((x)x, x) - |x|^2\text{Tr}(a(x))\}.$$  

Let us assume further that

$$\inf_{x \in \Omega} \{(a(x)x, x)\} = m > 0.$$  

(5.3)

This is always verified if $a(x)$ is a positive definite matrix, as $d(0, \Omega) > 0$.

Let $M = \sup_{x \in \Omega} |x|^2\text{Tr}(a(x))$. Then, if $\lambda \geq 0$,

$$F(x, D(\lambda v_{\sigma})(x), D^2(\lambda v_{\sigma})(x)) \geq \frac{(\lambda \sigma)^{2l+1}}{|x|^{2l(\sigma+2)+\sigma+4}} \{(\sigma + 2)m - M\}.$$
Hence, if \( \sigma \) is such that
\[
(\sigma + 2)m - M \geq 0,
\]
then \( \lambda v_\sigma \) is a supersolution of equation (5.1) for all \( \lambda \geq 0 \). Further \( v_\sigma(X_0) = 0, v \geq 0 \) on \( \partial \Omega \setminus \{X_0\} \). Similarly, \( -\lambda v_\sigma \) is a subsolution of equation (5.1) for all \( \lambda \geq 0, -v_\sigma(X_0) = 0, -v_\sigma \leq 0 \) and \( -v_\sigma < 0 \) on \( \partial \Omega \setminus \{X_0\} \). Thus, \((v_\sigma, -v_\sigma)\) is a barrier at \( X_0 \).

Now, we shall give a particular \( \alpha \in C(\mathbb{R}^n, M) \), satisfying (5.2) and (5.3). For example, let \( a_{i j}(\xi) = \xi_i \xi_j \) as considered in [1]. This equation arises as the limiting equation for the \( p \)-Laplacian as \( p \to \infty \) and is important in the study of plastic torsion. Then, relation (5.2) holds with \( \lambda = 1 \). Further,
\[
(a(x)x, x) = \sum_i \sum_j x_i x_j = |x|^4 \geq R^4 \quad \text{on } \overline{\Omega}.
\]
Thus, \( m = R^4 \) in condition (5.3) and
\[
F(x, \lambda v_\sigma(x), \lambda^2 v_\sigma(x)) = \frac{(\lambda \sigma)^3}{|x|^{3\sigma+8}} \{(\sigma + 2)|x|^4 - |x|^4\} = \frac{(\lambda \sigma)^3}{|x|^{3\sigma+4}}(\sigma + 1).
\]
Hence, \((v_\sigma - v_\sigma)\) is a barrier for any \( \sigma > 0 \).

**Concluding Remark:** Depending on the function \( F \) and the nature of comparison theorems available, one can modify the families \( \Phi_f \) and \( \Psi_f \). For example one can consider
\[
(\Phi_f^{Lip}, \Psi_f) \quad \text{or} \quad (\Phi_f, \Psi_f^{Lip})
\]
where
\[
\Phi_f^{Lip} = \left\{ u \middle| \text{a locally Lipschitz supersolution such that } \liminf_{y \to X, y \in \Omega} u(y) \geq f(X) \forall X \in \partial \Omega \right\}
\]
and
\[
\Psi_f^{Lip} = \left\{ u \middle| \text{a locally Lipschitz subsolution such that } \limsup_{y \to X, y \in \Omega} u(y) \leq f(X) \forall X \in \partial \Omega \right\}
\]
Then, under some conditions listed in [4], the comparison result holds. Then again, we can prove similar results. For example, if we consider \((\Phi_f^{Lip}, \Psi_f)\), then one can prove that if \( f \) admits a continuous extension as a supersolution, then \( f \) is semi-resolutive, \( H_f \) is upper semi-continuous and
\[
\limsup_{y \to X, y \in \Omega} H_f(y) \leq f(X)
\]
for all \( X \in \partial \Omega \).
References


Received 19.07.1993