The Classical and the Modified Neumann Problems for the Inhomogeneous Pluriholomorphic System in Polydiscs

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Abstract. The classical Neumann problem for the inhomogeneous pluriholomorphic system in a polydisc is considered. Its solvability conditions and its solution are given. It is shown that the problem is not well-posed. To fix the solution the boundary condition is modified. For the modified problem the solvability conditions and the solution which is unique up to an arbitrary constant are explicitly given.

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1. Introduction

The Neumann problem for the inhomogeneous pluriholomorphic system in the unit ball was studied in [1]. However, about the Neumann problem even for the homogeneous pluriholomorphic system in the unit polydisc nothing can be found in the literature, although a great deal of research has been done about the $\bar{\partial}$-Neumann problem in polydiscs (see, e.g., [2, 3, 6]).

Let

$$\mathbb{D}^n = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_k| < 1 \ (1 \leq k \leq n) \right\}$$

be the unit polydisc, $f_{k\ell}$ and $\gamma$ be given functions with $f_{k\ell} \in L_1(\mathbb{D}^n) \cap C(\mathbb{D}^n)$ and $\gamma \in C(\partial_0 \mathbb{D}^n)$. Consider the inhomogeneous system of $\frac{n(n+1)}{2}$ independent equations

$$\frac{\partial^2 u}{\partial z_k \partial z_\ell} = f_{k\ell}(z) \quad (1 \leq k, \ell \leq n) \quad (1)$$

with given right-hand sides satisfying the conditions

$$f_{k\ell}(z) = f_{\ell k}(z) \quad \text{and} \quad \frac{\partial f_{k\ell}}{\partial z_s} - \frac{\partial f_{ks}}{\partial z_\ell} = 0 \quad (1 \leq s \leq n).$$
**Problem (N_2).** Find a $C^1(D^n)$-solution of system (1) satisfying the Neumann condition
\[
\frac{\partial u}{\partial \nu} = \gamma_0(\zeta) \quad (\zeta \in \partial D^n)
\] (2)
where $\frac{\partial u}{\partial \nu} \zeta$ denotes the outward normal derivative of $u$ at the point $\zeta \in \partial D^n$.

By definition it is known (see [4]) that the Neumann condition (2) for the unit polydisc turns out to be
\[
\sum_{j=1}^{n} \left( z_j \frac{\partial u}{\partial z_j} + \bar{z}_j \frac{\partial u}{\partial \bar{z}_j} \right) \bigg|_{\zeta} = \gamma(\zeta) \quad (\zeta \in \partial_0 D^n)
\] (3)
with $\gamma(\zeta) = \gamma_0(\zeta) \sqrt{n}$. It is known that the general solution to system (1) is representable as
\[
u(z) = \phi_0(z) + \langle \phi(z), z \rangle + u_0(z)
\] (4)
where $\phi(z) = (\phi_1(z), \ldots, \phi_n(z))$, every $\phi_k$ ($k = 0, \ldots, n$) being an arbitrary function analytic in $D^n$, and $u_0$ is a special solution to system (1) given by
\[
u_0 = \sum_{\mu=1}^{n} (-1)^{\mu+1} \sum_{1 \leq \ell_1 \leq n}^{\mu \leq n} f^{\ell_1 \ell_2 \cdots \ell_{\mu}} \int_{\partial_0 D^n} \gamma(\zeta_1, \zeta_2) \zeta_1^\ell_1 \zeta_2^\ell_2 \cdots \zeta_\ell_\mu \zeta_1 d\zeta_1 \zeta_2 \cdots d\zeta_\ell_\mu
\]
(see [5]).

It is well known that for any given real-valued continuous function $\gamma$ on $\partial D$ there exists an analytic function $w$ in $D$, the real part of which has the boundary values $\gamma$ on $\partial D$, $\text{Re} w = \gamma$. A solution can be given by the Schwarz integral $S\gamma$ which is the complex counterpart of the Poisson integral $P\gamma$. Hence $\gamma$ turns out to be the boundary values of a harmonic function in $D$. For two complex variables in order that a given real-valued function on the distinguished boundary $\partial_0 D^2$ of the unit bidisc $D^2$ is the boundary value function of the real part of an analytic function in $D^2$ it has to belong to the space $\partial P h_{D^2}$ of boundary values of pluriharmonic functions in $D^2$. It is known that not any function defined on $\partial_0 D^2$ is in $\partial P h_{D^2}$ (see [1]). However, for our discussion we need to look at the problem a little bit further.

Let the real-valued function $\gamma$ on $\partial_0 D^2$ be representable by a Fourier series
\[
\gamma(z_1, z_2) = \sum_{i,k=-\infty}^{+\infty} a_{ik} z_1^i \bar{z}_2^k \quad ((z_1, z_2) \in \partial_0 D^2)
\]
\[
a_{ik} = \frac{1}{(2\pi i)^2} \int_{\partial_0 D^2} \gamma(\zeta_1, \zeta_2) \zeta_1^{-i} \zeta_2^{-k} \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} \quad (a_{-i,-k} = \bar{a}_{ik}).
\]
Thus for the given $\gamma$ we have two real pluriharmonic functions in $\mathbb{C}^2$: one in $\mathbb{D}^{++} = \mathbb{D}^2 - \{z = (z_1, z_2) : |z_1| > 1 \text{ and } |z_2| > 1\}$, i.e.,

$$\sum_{i,k=0}^{+\infty} \{a_{i,k} z_1^i z_2^k + a_{-i,-k} \bar{z}_1^i \bar{z}_2^k\} - a_{0,0},$$

and one in $\mathbb{D}^{+-} = \{z = (z_1, z_2) : |z_1| < 1 \text{ and } |z_2| > 1\}$ (D$^{+-}$), i.e.

$$\sum_{i,k=1}^{+\infty} \{a_{i,-k} z_1^i z_2^{-k} + a_{-i,k} \bar{z}_1^{-i} \bar{z}_2^k\}.$$

Clearly, if $\gamma \in \partial Ph_{\mathbb{D}^2}$, then obviously $a_{-i,k} = a_{i,-k} = 0$ for $i,k \in \mathbb{N}$, i.e.

$$a_{i,-k} = -\frac{1}{(2\pi i)^2} \int_{\partial \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \frac{\bar{\zeta}_1^i}{1 - \zeta_1} \frac{\bar{\zeta}_2^{-k}}{1 - \zeta_2} \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \quad (i,k \in \mathbb{N})$$

or, equivalently,

$$\frac{1}{(2\pi i)^2} \int_{\partial \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \frac{z_1^i}{1 - \zeta_1} \frac{\bar{z}_2^{-k}}{1 - \zeta_2} \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \quad ((z_1, z_2) \in \mathbb{D}^2). \quad (5)$$

If $\gamma \in \partial Ph_{\mathbb{D}^{+-}}$, then $a_{i,k} = a_{-i,-k} = 0$ for $i,k \in \{0\} \cup \mathbb{N}$. This means $\gamma$ satisfies

$$a_{i,k} = \frac{1}{(2\pi i)^2} \int_{\partial \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \frac{\gamma^{i,k}}{\zeta_1^i \zeta_2^{-k}} \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \quad (i,k \in \{0\} \cup \mathbb{N})$$

or, equivalently,

$$\frac{1}{(2\pi i)^2} \int_{\partial \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \frac{1}{1 - z_1 \zeta_1} \frac{1}{1 - z_2 \zeta_2} \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \quad ((z_1, z_2) \in \mathbb{D}^2).$$

Evidently, it is easy to see that $\partial Ph_{\mathbb{D}^2} = \partial Ph_{\mathbb{D}^{--}}$ and $\partial Ph_{\mathbb{D}^{+-}} = \partial Ph_{\mathbb{D}^{+-}}$. Further, if $\gamma$ belongs to $\partial H_{\mathbb{D}^2}$ (the space of boundary values of functions, holomorphic in $\mathbb{D}^2$), then $\gamma$ satisfies condition (5) and

$$a_{-i,-k} = \frac{1}{(2\pi i)^2} \int_{\partial \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \frac{1}{1 - \bar{z}_1 \zeta_1} \frac{1}{1 - \bar{z}_2 \zeta_2} \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \quad (i,k \in \{0\} \cup \mathbb{N}; i + k \neq 0)$$

as well, i.e.

$$\frac{1}{(2\pi i)^2} \int_{\partial \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \left(\frac{1}{1 - \bar{z}_1 \zeta_1} - 1\right) \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0 \quad ((z_1, z_2) \in \mathbb{D}^2)$$

or, equivalently,

$$\frac{1}{(2\pi i)^2} \int_{\partial \mathbb{D}^2} \gamma(\zeta_1, \zeta_2) \left(\frac{\bar{z}_1 \zeta_1}{1 - \bar{z}_1 \zeta_1} + \frac{\bar{z}_2 \zeta_2}{1 - \bar{z}_2 \zeta_2} - \frac{\bar{z}_1 \zeta_1}{1 - \bar{z}_1 \zeta_1} - \frac{\bar{z}_2 \zeta_2}{1 - \bar{z}_2 \zeta_2}\right) \frac{d\zeta_1}{\zeta_1} \frac{d\zeta_2}{\zeta_2} = 0.$$

On the basis of [1: Theorem 5.1] and from our discussion above we can get the following conclusion about the boundary values of holomorphic functions in polydiscs.
Lemma 1. Let $\gamma$ be a real-valued continuous function on $\partial_0 \mathbb{D}^n$ satisfying $\gamma \in \partial H_{\mathbb{D}^n}$:

$$
\sum_{\nu=1}^{n} \sum_{\lambda=0}^{\nu-1} \sum_{1 \leq k_1 < \ldots < k_{\lambda} \leq n}^{1 \leq k_{\lambda+1} < \ldots < k_{\nu} \leq n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \prod_{\tau=1}^{\lambda} \frac{z_{k_{\tau}}}{\zeta_{k_{\tau}} - z_{k_{\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_{\tau}}}{\zeta_{k_{\tau}} - z_{k_{\tau}}} d\zeta = 0.
$$

Then

$$
\phi(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \gamma(\zeta) \frac{d\zeta}{\zeta - z}
$$

is analytic in $\mathbb{D}^n$ satisfying $\phi(\zeta) = \gamma(\zeta)$ on $\partial_0 \mathbb{D}^n$.

2. The classical problem

From (4) it follows that

$$
u k = \phi_k(z) + u_{0\nu k}, \quad u_{zk} = \frac{\partial \phi_0}{\partial z_k} + \sum_{\mu=1}^{n} z_{\mu} \frac{\partial \phi_\mu}{\partial z_k} + \frac{\partial u_0}{\partial z_k}
$$

where

$$
u 0 \nu k = \sum_{\nu=1}^{n} (-1)^{\nu+1} \sum_{1 \leq k_1 < \ldots < k_{\nu} \leq n}^{1 \leq k_{\nu+1} < \ldots < k_{\nu} \leq n} T_{k_{\nu}} \ldots T_{k_1} f_{k_1 k_{\nu}} \ldots T_{k_{\nu}} (1 \leq k \leq n).
$$

Substituting these expressions into (3), we obtain an equality for $\zeta \in \partial_0 \mathbb{D}^n$:

$$
\sum_{k=1}^{n} \zeta_k \left( \phi_k(\zeta) + \sum_{j=1}^{n} \zeta_j \frac{\partial \phi_k}{\partial \zeta_j} + \frac{\zeta_k}{n} \sum_{j=1}^{n} \zeta_j \frac{\partial \phi_0}{\partial \zeta_j} \right) = \sum_{k=1}^{n} \zeta_k \left( \frac{\zeta_k}{n} \gamma(\zeta) - \frac{\partial u_0}{\partial \zeta_k} - \frac{\zeta_k}{n} \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right).
$$

Evidently, this equality is satisfied if

$$
\phi_k(\zeta) + \sum_{j=1}^{n} \zeta_j \frac{\partial \phi_k}{\partial \zeta_j} + \zeta_k \sum_{j=1}^{n} \zeta_j \frac{\partial \phi_0}{\partial \zeta_j} = \frac{\zeta_k}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] - \frac{\partial u_0}{\partial \zeta_k}
$$

holds for any $\zeta \in \partial_0 \mathbb{D}^n$ and $1 \leq k \leq n$. Since the left-hand side represents the boundary values of a holomorphic function in $\mathbb{D}^n$, the right-hand side does too. Thus according to Lemma 1, the problem is solvable if and only if the conditions

$$
\sum_{\nu=1}^{n} \sum_{\lambda=0}^{\nu-1} \sum_{1 \leq k_1 < \ldots < k_{\lambda} \leq n}^{1 \leq k_{\lambda+1} < \ldots < k_{\nu} \leq n} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \left\{ \frac{\langle \zeta, z \rangle}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] - \langle \text{grad} \zeta u_0, z \rangle \right\}
$$

$$
\times \prod_{\tau=1}^{\lambda} \frac{z_{k_{\tau}}}{\zeta_{k_{\tau}} - z_{k_{\tau}}} \prod_{\tau=\lambda+1}^{\nu} \frac{z_{k_{\tau}}}{\zeta_{k_{\tau}} - z_{k_{\tau}}} \frac{d\zeta}{\zeta} = 0 \quad (z \in \mathbb{D}^n)
$$

(7)
are satisfied. Then
\[
\phi_k(z) + \sum_{j=1}^{n} z_j \frac{\partial \phi_k}{\partial z_j} + \frac{z_k}{n} \sum_{j=1}^{n} z_j \frac{\partial \phi_0}{\partial z_j} = \frac{1}{(2\pi i)^n} \int_{\partial \Omega_D^n} \left\{ \frac{\zeta_k}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] - \frac{\partial u_0}{\partial \zeta_k} \right\} \frac{d\zeta}{\zeta - z} \quad (z \in \mathbb{D}^n)
\]

is analytic in \(\mathbb{D}^n\) and satisfies condition (7).

To derive the solution of problem \((N_2)\) we apply the Cauchy formula to (6), and by taking into account
\[
\frac{1}{2\pi i} \int_{\partial \mathbb{D}} T f(\zeta) \frac{d\zeta}{\zeta - z} = 0, \quad \text{i.e.} \quad \frac{1}{(2\pi i)^n} \int_{\partial \Omega_D^n} u_0 \frac{d\zeta}{\zeta - z} = 0
\]
we get the partial differential equations for \(z \in \mathbb{D}^n\)

\[
\phi_k(z) + \sum_{j=1}^{n} z_j \frac{\partial \phi_k}{\partial z_j} = -\frac{z_k}{n} \sum_{j=1}^{n} z_j \frac{\partial \phi_0}{\partial z_j} + \frac{1}{(2\pi i)^n} \int_{\partial \Omega_D^n} \left\{ \frac{\zeta_k}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] \right\} \frac{d\zeta}{\zeta - z}. \tag{8}
\]

By the transformation
\[
\begin{align*}
\omega_1 &= z_1 \\
\omega_2 &= \frac{z_1}{z_2} \\
& \vdots \\
\omega_n &= \frac{z_1}{z_n}
\end{align*}
\]
we obtain for (8) the equations
\[
\omega_1 \frac{\partial \phi_1}{\partial \omega_1} + \phi_1 = -\frac{\omega_1^2}{n} \frac{\partial \phi_0}{\partial \omega_1} + \frac{1}{(2\pi i)^n} \int_{\partial \Omega_D^n} \frac{\zeta_1}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] \times \frac{d\zeta_1}{\zeta_1 - \omega_1} \frac{d\zeta_2}{\zeta_2 - \frac{\omega_1}{\omega_2}} \cdots \frac{d\zeta_n}{\zeta_n - \frac{\omega_1}{\omega_n}}
\]
\[
\omega_1 \frac{\partial \phi_k}{\partial \omega_1} + \phi_k = -\frac{\omega_1^2}{n \omega_k} \frac{\partial \phi_0}{\partial \omega_1} + \frac{1}{(2\pi i)^n} \int_{\partial \Omega_D^n} \frac{\zeta_k}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] \times \frac{d\zeta_1}{\zeta_1 - \omega_1} \frac{d\zeta_2}{\zeta_2 - \frac{\omega_1}{\omega_2}} \cdots \frac{d\zeta_n}{\zeta_n - \frac{\omega_1}{\omega_n}} \quad (k = 2, \ldots, n).
\]
Integrating these equations we get

\[
\omega_1 \phi_1 = -\int_0^{\omega_1} \frac{t^2}{n} \frac{\partial \phi_0}{\partial t} \, dt + \int_0^{\omega_1} \frac{1}{(2\pi i)^n} \int_{\partial \mathbb{D}^n} \frac{\zeta_1}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] \frac{d\zeta}{\zeta - t \zeta_2 - \frac{t}{\omega_2} \cdots \frac{d\zeta_n}{\zeta - \frac{t}{\omega_n}} \, dt + C_1
\]

\[
\omega_k \phi_k = -\int_0^{\omega_1} \frac{t^2}{n \omega_k} \frac{\partial \phi_0}{\partial t} \, dt + \int_0^{\omega_1} \frac{1}{(2\pi i)^n} \int_{\partial \mathbb{D}^n} \frac{\zeta_k}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] \frac{d\zeta}{\zeta - t \zeta_2 - \frac{t}{\omega_2} \cdots \frac{d\zeta_n}{\zeta - \frac{t}{\omega_n}} \, dt + C_k \quad (k = 2, \ldots, n).
\]

Substituting \( \omega_1 = 0 \) on both sides we see that \( C_k = 0 \) \( (1 \leq k \leq n) \). Returning to the original variables we have

\[
z_1 \phi_1(z) = \int_0^1 \frac{z_1}{(2\pi i)^n} \int_{\partial \mathbb{D}^n} \frac{\zeta_1}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] \frac{d\zeta}{\zeta - sz} \, ds
\]

\[- z_1 \int_0^1 \frac{sz_1}{n} \sum_{j=1}^{n} (sz_j) \frac{\partial \phi_0(sz)}{\partial (sz_j)} \, ds
\]

\[
z_1 \phi_k(z) = \int_0^1 \frac{z_1}{(2\pi i)^n} \int_{\partial \mathbb{D}^n} \frac{\zeta_k}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] \frac{d\zeta}{\zeta - sz} \, ds
\]

\[- z_1 \int_0^1 \frac{sz_k}{n} \sum_{j=1}^{n} (sz_j) \frac{\partial \phi_0(sz)}{\partial (sz_j)} \, ds \quad (k = 2, \ldots, n),
\]

i.e. for \( k = 1, \ldots, n \) we have

\[
\phi_k(z) = \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial \mathbb{D}^n} \frac{\zeta_k}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] \frac{d\zeta}{\zeta - sz} \, ds - \int_0^1 \frac{s^2 z_k}{n} \, d\phi_0(sz).
\]

Hence representation (4) gets the form

\[
u(z) = \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial \mathbb{D}^n} \frac{\langle \zeta, z \rangle}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] \frac{d\zeta}{\zeta - sz} \, ds
\]

\[+ u_0(z) + \phi_0(z) - \int_0^1 \frac{\langle sz, sz \rangle}{n} \, d\phi_0(sz).
\]
If we take \( \phi_0(z) = \sum_{|\kappa| \geq 0} a_\kappa z^\kappa \) \((z \in \mathbb{D}^n)\), then

\[
\phi_0(z) - \int_0^1 \frac{\langle s, z\rangle^2}{n} \frac{d\phi_0(sz)}{n} = \sum_{|\kappa| \geq 0} a_\kappa z^\kappa - \int_0^1 \frac{s|z|^2}{n} \sum_{|\kappa| \geq 0} a_\kappa |\kappa| (sz)^\kappa ds
\]

\[
= a_0 + \sum_{|\kappa| \geq 0} a_\kappa \left[ 1 - \frac{|\kappa| |z|^2}{n(|\kappa| + 2)} \right] z^\kappa.
\]

Thus

\[
u(z) = \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \zeta, z\rangle}{n} \left[ \gamma(\zeta) - \sum_{j=1}^{n} \zeta_j \partial u_0/\partial \zeta_j \right] \frac{d\zeta}{\zeta - sz} ds + u_0(z)
\]

\[
+ \sum_{|\kappa| \geq 0} a_\kappa \left[ 1 - \frac{|\kappa| |z|^2}{n(|\kappa| + 2)} \right] z^\kappa \quad (z \in \mathbb{D}^n).
\]

**Theorem 1.** Problem \((N_2)\) is solvable if and only if its right-hand sides satisfy condition \((7)\) on \(\partial_0 \mathbb{D}^n\). The general solution can be given by \((9)\). The corresponding homogeneous problem has infinitely many linearly independent non-trivial solutions

\[
\left[ 1 - \frac{|\kappa| |z|^2}{n(|\kappa| + 2)} \right] z^\kappa \quad (|\kappa| > 0, z \in \mathbb{D}^n).
\]

Problem \((N_2)\) is not well-posed.

**3. The modified problem**

Since solution \((9)\) includes a free analytic function, clearly to get a fixed solution only a Schwarz problem is needed to be solved. So we introduce an additional boundary condition.

**Problem \((N_2^*)\)** Find a \(C^1(\overline{\mathbb{D}^n})\) solution to system \((1)\) satisfying the Neumann condition \((2)\) and

\[
\text{Re} \, u(\zeta) = \gamma^*(\zeta) \quad (\zeta \in \partial_0 \mathbb{D}^n).
\]

We call this problem the **modified Neumann problem** for system \((1)\).

Let \(f_{k\ell} = 0\) in \((1)\). Then the solvability condition \((7)\) takes the form

\[
\sum_{\nu=1}^{n} \sum_{\lambda=0}^{\nu-1} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \zeta, z\rangle}{n} \gamma(\zeta) \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} d\zeta = 0 \quad (\zeta \in \partial_0 \mathbb{D}^n; z \in \mathbb{D}^n \cup \partial_0 \mathbb{D}^n)
\]
and it means that every $\zeta_k \gamma(\zeta)$ on $\partial \Omega^n$ $(1 \leq k \leq n)$ belongs to $\partial H_D^n$. Actually, it is evident that $\gamma \in \partial H_D^n$. Note if $\zeta_1 \gamma(\zeta) = \varphi_1(\zeta)$ with $\varphi_1 \in \partial H_D^n$, then $\gamma(\zeta) = \overline{\zeta}_1 \varphi_1(\zeta)$. If $\gamma \notin \partial H_D^n$, then $\zeta_2 \zeta_1 \varphi_1(\zeta) \notin \partial H_D^n$. But by the condition above $\zeta_2 \gamma(\zeta) \in \partial H_D^n$. This is a contradiction. Hence condition (11) becomes

$$\sum_{\nu=1}^{n} \sum_{\lambda=0}^{\nu-1} \sum_{1 \leq k_1 < \ldots < k_\lambda \leq n} \frac{1}{(2\pi i)^{\nu}} \int_{\partial \Omega^n} \gamma(\zeta) \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{\overline{z}_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \frac{d\zeta}{\zeta} = 0. \quad (12)$$

Substituting (10) into (9) shows

$$\sum_{|\kappa| \geq 0} \frac{a_\kappa \overline{\zeta}^\kappa + a_\kappa \zeta^\kappa}{2 + |\kappa|} = \gamma^*(\zeta) - \text{Re} \int_{0}^{1} \frac{1}{(2\pi i)^{n}} \int_{\partial \Omega^n} \frac{\langle \eta, \zeta \rangle}{n} \eta - s\zeta \frac{d\eta}{\eta - s\zeta} ds =: 2\Gamma(\zeta),$$

i.e.

$$\text{Re} \sum_{|\kappa| \geq 0} \frac{a_\kappa \zeta^\kappa}{2 + |\kappa|} = \Gamma(\zeta) \quad (\zeta \in \partial \Omega^n). \quad (13)$$

Due to the character of the left-hand side of (13), the right-hand side $\Gamma$ on $\partial \Omega^n$ is also the boundary value of a function, pluriharmonic in $\mathbb{D}^n$. This means the given function $\Gamma$ on $\partial \Omega^n$ must satisfy the condition

$$\sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{1 \leq k_1 < \ldots < k_\lambda \leq n} \frac{1}{(2\pi i)^{n}} \int_{\partial \Omega^n} \Gamma(\zeta) \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{\overline{z}_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \frac{d\zeta}{\zeta} = 0. \quad (14)$$

In fact, due to $\gamma \in \partial H_D^n$, it follows that

$$2\Gamma(\zeta) = \gamma^*(\zeta) - \text{Re} \int_{0}^{1} \frac{s\zeta, \zeta}{n} \gamma(s\zeta) ds = \gamma^*(\zeta) - \text{Re} \int_{0}^{1} s\gamma(s\zeta) ds.$$

Hence $\text{Re} \int_{0}^{1} s\gamma(s\zeta) ds \in \partial \Phi \mathbb{D}^n$ and condition (14) implies that $\gamma^* \in \partial \Phi \mathbb{D}^n$, i.e.

$$\sum_{\nu=2}^{n} \sum_{\lambda=1}^{\nu-1} \sum_{1 \leq k_1 < \ldots < k_\lambda \leq n} \frac{1}{(2\pi i)^{n}} \int_{\partial \Omega^n} \gamma^*(\zeta) \prod_{\tau=1}^{\lambda} \frac{z_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \prod_{\tau=\lambda+1}^{\nu} \frac{\overline{z}_{k_\tau}}{\zeta_{k_\tau} - z_{k_\tau}} \frac{d\zeta}{\zeta} = 0. \quad (15)$$

So if this condition is satisfied, then the Schwarz problem (13) is solvable and the solution is given by

$$\sum_{|\kappa| \geq 0} \frac{a_\kappa z^\kappa}{|\kappa| + 2} = \frac{1}{(2\pi i)^{n}} \int_{\partial \Omega^n} \Gamma(\zeta) \left[ \frac{2}{\zeta - z} - 1 \right] \frac{d\zeta}{\zeta} + iC^0 \quad = \sum_{|\kappa| > 0} \frac{2}{(2\pi i)^{n}} \int_{\partial \Omega^n} \Gamma(\zeta)(z\overline{\zeta})^\kappa \frac{d\zeta}{\zeta} + \frac{1}{(2\pi i)^{n}} \int_{\partial \Omega^n} \Gamma(\zeta) \frac{d\zeta}{\zeta} + iC^0.$$
with an arbitrary real constant $C^0$, it is analytic in $\mathbb{D}^n$ and satisfies equation (13) (see [1]). One can see that
\[
a_0 = \frac{2}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Gamma(\zeta) \frac{d\zeta}{\zeta} + i2C^0
\]
\[
a_\kappa = \frac{2(2 + |\kappa|)}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \Gamma(\zeta) \zeta^\kappa \frac{d\zeta}{\zeta} \quad (|\kappa| > 0).
\]
Hence if conditions (12) and (15) are satisfied, i.e. if $\gamma \in \partial H_{\mathbb{D}^n}$ and $\gamma^* \in \partial P_h_{\mathbb{D}^n}$, then problem $(N_2^*)$ with $f_{k\ell} = 0$ is solvable and the solution is given by
\[
u(z) = \sum_{|\kappa| \geq 0} a_\kappa \left[1 - \frac{|\kappa| |z|^2}{n(|\kappa| + 2)}\right] z^\kappa + \int_0^1 \left(\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{\langle \zeta, z \rangle}{\zeta - sz} d\zeta \right) ds
\]
for $z \in \mathbb{D}^n$. But from
\[
\sum_{|\kappa| \geq 0} a_\kappa \left[1 - \frac{|\kappa| |z|^2}{n(|\kappa| + 2)}\right] z^\kappa
\]
\[
= a_0 + \sum_{|\kappa| > 0} \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[2 + \frac{n - |\zeta|^2}{n}\right] (z\zeta)^\kappa \frac{d\zeta}{\zeta}
\]
\[
= a_0 + \frac{2}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[1 - \frac{1}{1 - z\zeta} - 1\right] \frac{d\zeta}{\zeta}
\]
\[
+ \sum_{|\kappa| > 0} \frac{n - |\zeta|^2}{n(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) |\zeta|^\kappa (z\zeta)^\kappa \frac{d\zeta}{\zeta}
\]
\[
= a_0 + \frac{2}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{1}{1 - z\zeta} - 1\right] \frac{d\zeta}{\zeta}
\]
\[
+ \left.\frac{n - |\zeta|^2}{n(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \frac{\partial}{\partial t} (tz\zeta)^\kappa \right|_{t=1} \frac{d\zeta}{\zeta}
\]
\[
= a_0 + \frac{n - |\zeta|^2}{n} \frac{\partial}{\partial t} \left(\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{1}{1 - t\zeta} - 1\right] \frac{d\zeta}{\zeta}\right|_{t=1}
\]
\[
+ \frac{2}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{1}{1 - z\zeta} - 1\right] \frac{d\zeta}{\zeta}
\]
we get
\[
u(z) = iC_0 + \frac{n - |\zeta|^2}{n} \frac{\partial}{\partial t} \left(\frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{1}{1 - t\zeta} - 1\right] \frac{d\zeta}{\zeta}\right|_{t=1}
\]
\[
+ \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{2}{1 - z\zeta} - 1\right] \frac{d\zeta}{\zeta}
\]
where $C_0$ is an arbitrary real constant.

Next we make some simplifications. Let
\[
I_1 = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} 2\Gamma(\zeta) \left[\frac{2}{1 - z\zeta} - 1\right] \frac{d\zeta}{\zeta} \quad (z \in \mathbb{D}^n).
\]
Then

\[
I_1 = \frac{-1}{(2\pi i)^n} \int_{\partial D^n} \left\{ \text{Re} \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial D^n} \frac{\langle \eta, \zeta \rangle}{n} \gamma(\eta) \frac{d\eta}{\eta - s\zeta} \right\} \left[ \frac{2}{1 - z\zeta} - 1 \right] d\zeta \\
+ \frac{1}{(2\pi i)^n} \int_{\partial D^n} \gamma^*(\zeta) \left[ \frac{2}{1 - z\zeta} - 1 \right] d\zeta \\
= -I_{1a} - I_{1b} + I_{1c}
\]

where

\[
2I_{1a} = \frac{1}{(2\pi i)^n} \int_{\partial D^n} \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial D^n} \frac{\langle \eta, \zeta \rangle}{n} \gamma(\eta) \frac{d\eta}{\eta - s\zeta} \left[ \frac{2}{1 - z\zeta} - 1 \right] d\zeta \\
\]

By changing the order of integration, we get

\[
2I_{1a} = \frac{1}{(2\pi i)^n} \int_{\partial D^n} \int_0^1 \gamma(\eta) \left\{ \frac{1}{(2\pi i)^n} \int_{\partial D^n} \frac{\langle \eta, \zeta \rangle}{n} \frac{1}{1 - s\zeta\eta} \left[ \frac{2}{1 - z\zeta} - 1 \right] d\zeta \right\} ds \frac{d\eta}{\eta},
\]

but

\[
\frac{1}{(2\pi i)^n} \int_{\partial D^n} \frac{\langle \eta, \zeta \rangle}{n} \frac{1}{1 - s\zeta\eta} \left[ \frac{2}{1 - z\zeta} - 1 \right] d\zeta \\
= \sum_{k=1}^n \frac{1}{n(2\pi i)^n} \int_{\partial D^n} \frac{\eta_k \bar{\zeta}_k}{1 - s\zeta_k\eta_k} \\
\times \prod_{\tau=1, \tau \neq k}^n \frac{1}{1 - s\zeta_\tau\eta_\tau} \frac{2d\zeta}{\zeta - z} - \frac{1}{(2\pi i)^n} \int_{\partial D^n} \frac{\langle \zeta, \eta \rangle}{n} \frac{d\zeta}{\zeta - s\eta}
\]

\[
= \sum_{k=1}^n \frac{2}{n(2\pi i)^n} \int_{\partial D^n} \eta_k \left[ \frac{\bar{\zeta}_k}{1 - s\zeta_k\eta_k} + \frac{s\eta_k}{1 - s\zeta_k\eta_k} \right] \\
\times \prod_{\tau=1, \tau \neq k}^n \frac{1}{1 - s\zeta_\tau\eta_\tau} \frac{1}{1 - z\zeta} \frac{d\zeta}{\zeta} - \frac{\langle s\eta, \eta \rangle}{n}
\]

\[
= \sum_{k=1}^n \frac{2}{n} \frac{\eta_k}{1 - sz_k\eta_k} \prod_{\tau=1, \tau \neq k}^n \frac{1}{1 - sz_\tau\eta_\tau} - s
\]

\[
= s \left[ \frac{2}{1 - sz\eta} - 1 \right]
\]

leads to

\[
2I_{1a} = \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial D^n} \gamma(\eta) \left[ \frac{2}{\eta - sz} - 1 \right] \frac{d\eta}{\eta} ds ds.
\]

The second part of \( I_1 \) which has to be simplified is

\[
2I_{1b} = \frac{1}{(2\pi i)^n} \int_{\partial D^n} \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial D^n} \frac{\langle \eta, \zeta \rangle}{n} \gamma(\eta) \frac{d\eta}{\eta - s\zeta} ds \left[ \frac{2}{1 - z\zeta} - 1 \right] d\zeta.
\]
By changing the order of the integrals

$$2I_{1b} = \frac{1}{(2\pi i)^n} \int_{\partial_D^n} \int_0^1 \frac{1}{\gamma(n)} \left\{ \frac{1}{(2\pi i)^n} \int_{\partial_D^n} \langle \zeta, \eta \rangle \frac{1}{n} \frac{1}{1-s\zeta} \left[ \frac{2}{1-z\zeta} - 1 \right] d\zeta \right\} ds \frac{d\eta}{\eta}$$

and from

$$\frac{1}{(2\pi i)^n} \int_{\partial_D^n} \langle \zeta, \eta \rangle \frac{1}{n} \frac{1}{1-s\zeta} \left[ \frac{2}{1-z\zeta} - 1 \right] d\zeta$$

$$= \sum_{k=1}^n \frac{1}{(2\pi i)^n} \int_{\partial_D^n} \frac{\zeta_k \eta_k}{n} \left[ 1 + s\eta_k \zeta_k + \left( \frac{s\eta_k \zeta_k}{1-s\zeta_k} \right)^2 \right]$$

$$\times \prod_{\tau \neq k} \frac{1}{1-s\eta_k \zeta_k} \left\{ 2 \left[ 1 + \frac{z_k \zeta_k}{1-z_k \zeta_k} \right] \prod_{\tau \neq k} \frac{1}{1-sz_k \zeta_k} - 1 \right\} d\zeta_k$$

$$= \sum_{k=1}^n \frac{1}{(2\pi i)^n} \int_{\partial_D^n} \frac{1}{n} \left[ \zeta_k \eta_k + s + s \frac{\zeta_k}{1-s\zeta_k} \right] \left\{ 2 \left[ 1 + \frac{z_k \zeta_k}{1-z_k \zeta_k} \right] - 1 \right\} d\zeta_k$$

$$= \sum_{k=1}^n \frac{1}{2\pi i} \int_{\partial_D^n} \frac{1}{n} \left[ \zeta_k \eta_k + s + s \frac{\zeta_k}{1-s\zeta_k} \right] \left[ 1 + \frac{2z_k \zeta_k}{1-z_k \zeta_k} \right] d\zeta_k$$

$$= \sum_{k=1}^n \frac{1}{n} \left[ s + 2z_k \zeta_k \right]$$

we have

$$2I_{1b} = \frac{1}{(2\pi i)^n} \int_{\partial_D^n} \int_0^1 \frac{1}{\gamma(n)} \left[ 2 \frac{\langle z, \eta \rangle}{n} + s \right] ds \frac{d\eta}{\eta}$$

$$= \frac{2}{(2\pi i)^n} \int_{\partial_D^n} \langle z, \zeta \rangle \gamma(n) \frac{d\eta}{\eta} + \frac{1}{2(2\pi i)^n} \int_{\partial_D^n} \gamma(n) \frac{d\zeta}{\zeta}.$$

Thus we have got $I_1$ calculated as

$$I_1 = \frac{1}{(2\pi i)^n} \int_{\partial_D^n} \gamma^*(\zeta) \left[ 2 \frac{\zeta}{\zeta - z} - 1 \right] \frac{d\zeta}{\zeta}$$

$$- \frac{1}{2} \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_D^n} \gamma(\zeta) \left[ 2 \frac{\zeta}{\zeta - sz} - 1 \right] d\zeta \frac{d\zeta}{\zeta}.$$

Now let

$$I_2 := \frac{1}{(2\pi i)^n} \int_{\partial_D^n} 2\Gamma(\zeta) \left[ \frac{1}{1-tz\zeta} - 1 \right] \frac{d\zeta}{\zeta}.$$

Similar to $I_1$ it is easy to get

$$I_2 = \frac{1}{(2\pi i)^n} \int_{\partial_D^n} \gamma^*(\zeta) \left[ \frac{\zeta}{\zeta - tz} - 1 \right] \frac{d\zeta}{\zeta}$$

$$- \frac{1}{2} \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_D^n} \gamma(\zeta) \left[ \frac{\zeta}{\zeta - sz} - 1 \right] d\zeta \frac{d\zeta}{\zeta}.$$

$$- \frac{1}{2(2\pi i)^n} \int_{\partial_D^n} \frac{\langle tz, \zeta \rangle}{\gamma(\zeta)} d\zeta.$$
So we have
\[
 u(z) = iC_0 + \frac{n - |z|^2}{n} \frac{\partial}{\partial t} \left\{ \frac{1}{(2\pi i)^n} \int_{\partial_{0} \mathbb{D}^n} \gamma^*(\zeta) \left[ \frac{\zeta}{\zeta - tz} - 1 \right] \frac{d\zeta}{\zeta} \right\} \\
- \frac{1}{2} \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_{0} \mathbb{D}^n} \gamma(t\zeta) \left[ \frac{\zeta}{\zeta - stz} - 1 \right] \frac{d\zeta}{\zeta} s \, ds \\
- \frac{1}{2(2\pi i)^n} \int_{\partial_{0} \mathbb{D}^n} \overline{\gamma}(\zeta) \frac{d\zeta}{\zeta} t + 1 \\
+ \frac{1}{(2\pi i)^n} \int_{\partial_{0} \mathbb{D}^n} \gamma^*(\zeta) \left[ \frac{2}{\zeta} - 1 \right] \frac{d\zeta}{\zeta} \\
- \frac{1}{2} \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_{0} \mathbb{D}^n} \overline{\gamma}(\zeta) \frac{d\zeta}{\zeta} - \frac{1}{4(2\pi i)^n} \int_{\partial_{0} \mathbb{D}^n} \gamma(\zeta) \frac{d\zeta}{\zeta}
\]  
(16)
where \( C_0 \) is an arbitrary real constant.

**Lemma 2.** The modified Neumann problem \((N_{2}^* \mathbb{H})\) for pluriholomorphic functions in \(\mathbb{D}^n\) is uniquely solvable if and only if conditions (12) and (15) are satisfied, i.e. \(\gamma \in \partial H_{\mathbb{D}^n}\) and \(\gamma^* \in \partial Ph_{\mathbb{D}^n}\). The solution unique up to an arbitrary real constant is given by (16). The problem is well-posed.

Next we clarify the solution and the solvability conditions of the modified problem \((N_{2}^* \mathbb{H})\) for the inhomogeneous system (1). By substituting condition (10) into representation (9) we have

\[
\sum_{|\kappa| \geq 0} \left( a_{\kappa} \overline{\zeta}^\kappa + a_{\kappa} \zeta^\kappa \right) \frac{1}{|\kappa| + 2} \sum_{\nu = 2}^{n} \sum_{\lambda = 1}^{\nu - 1} \sum_{1 \leq k_1 < \cdots < k_\lambda \leq \nu \atop 1 \leq \lambda + 1 < \cdots < \nu \leq n} \frac{1}{(2\pi i)^n} \int_{\partial_{0} \mathbb{D}^n} F(\zeta) \prod_{\tau = 1}^{\lambda} \frac{z_{k_{\tau}}}{\zeta_{k_{\tau}} - z_{k_{\tau}}} \prod_{\tau = \lambda + 1}^{\nu} \frac{\overline{z}_{k_{\tau}}}{\zeta_{k_{\tau}} - \overline{z}_{k_{\tau}}} \frac{d\zeta}{\zeta} = 0
\]  
(18)
for \(z \in \mathbb{D}^n\). Then the Schwarz problem (17) is solvable and the solution can be given by

\[
\sum_{|\kappa| \geq 0} \frac{a_{\kappa} \zeta^\kappa}{|\kappa| + 2} = \frac{1}{(2\pi i)^n} \int_{\partial_{0} \mathbb{D}^n} F(\zeta) \left[ 2 \frac{\zeta}{\zeta - z} - 1 \right] \frac{d\zeta}{\zeta} + iC^1
\]
and from it one can derive that
\[ a_0 = \frac{2}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} F(\zeta) \frac{d\zeta}{\zeta} + i2C^1 \]
\[ a_\kappa = \frac{2(2 + |\kappa|)}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} F(\zeta) \zeta^{\kappa} \frac{d\zeta}{\zeta} \quad (|\kappa| > 0) \]
where \( C^1 \) is an arbitrary real constant. Substituting them into (9) we get
\[ u(z) = \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \langle \zeta, z \rangle \left[ \gamma(\zeta) - \sum_{j=1}^n \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] \frac{d\zeta}{\zeta - sz} ds + u_0(z) \]
\[ + \sum_{|\kappa| \geq 0} \frac{2 + |\kappa|}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \left[ 1 - \frac{|\kappa| |z^2|}{n(|\kappa| + 2)} \right] F(\zeta) (z\zeta)^n \frac{d\zeta}{\zeta} + i2C^1 \]
for all \( z \in \mathbb{D}^n \). Similarly to the case of the pluriholomorphic system we obtain
\[ u(z) = \int_0^1 \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \langle \zeta, z \rangle \left[ \gamma(\zeta) - \sum_{j=1}^n \zeta_j \frac{\partial u_0}{\partial \zeta_j} \right] \frac{d\zeta}{\zeta - sz} ds + u_0(z) + iC^* \]
\[ + \frac{\partial}{\partial t} \frac{n - |z|^2}{n(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} F(\zeta) \left[ \frac{1}{1 - tz\zeta} - 1 \right] \frac{d\zeta}{\zeta} \bigg|_{t=1} \]
\[ + \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} F(\zeta) \left[ \frac{2}{1 - z\zeta} - 1 \right] \frac{d\zeta}{\zeta} \]
where \( C^* \) is an arbitrary real constant.

**Theorem 2.** The modified Neumann problem \( (N^*_2) \) for the inhomogeneous pluriholomorphic system (1) in \( \mathbb{D}^n \) is solvable if and only if conditions (7) and (18) are satisfied. The solution which is unique up to an arbitrary real constant, is given by (19). The problem is well-posed.

**A simple application.** Find the sums
\[ \sum_{|k| > 0} \frac{x^k}{|k|} \quad \text{and} \quad \sum_{|k| > 0} |k| x^k \quad (|x_1| < 1, \ldots, |x_n| < 1). \]
By the above method we get
\[ \sum_{|k| > 0} x_1^{k_1} \cdots x_n^{k_n} = \int_0^1 \left( \frac{1}{1 - sx_1} \cdots \frac{1}{1 - sx_n} - 1 \right) ds \]
and
\[ \sum_{|k| > 0} (k_1 + \ldots + k_n) (x_1^{k_1} \cdots x_n^{k_n}) = \frac{\partial}{\partial s} \left( \frac{1}{1 - sx_1} \cdots \frac{1}{1 - sx_n} - 1 \right) \bigg|_{s=1}. \]
References


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