On a rigidity result for the first conformal eigenvalue of the Laplacian

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Abstract. Given \((M, g)\) a smooth compact Riemannian manifold without boundary of dimension \(n \geq 3\), we consider the first conformal eigenvalue which is by definition the supremum of the first eigenvalue of the Laplacian among all metrics conformal to \(g\) of volume 1. We prove that it is always greater than \(n \omega_n^2\), the value it takes in the conformal class of the round sphere, except if \((M, g)\) is conformally diffeomorphic to the standard sphere.

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Let \((M, g)\) be a smooth compact Riemannian manifold without boundary of dimension \(n \geq 3\) and let us define the first conformal eigenvalue of \((M, g)\) by

\[
\Lambda_1(M, [g]) = \sup_{\tilde{g} \in [g]} \lambda_1(M, \tilde{g}) \text{Vol}_{\tilde{g}}(M)^{\frac{2}{n}}
\]

where \(\lambda_1(M, g)\) is the first nonzero eigenvalue of the Laplacian \(\Delta_g = -\text{div}_g(\nabla)\) and \([g]\) is the conformal class of \(g\). In this paper, we aim at proving a rigidity result concerning this first conformal eigenvalue.

The maximisation on conformal classes is natural because the scale invariant quantity supremum is infinite among all metrics [3] (except in dimension 2, [16]), while El Soufi and Ilias [7] proved that it is always bounded among conformal metrics. Generalizing a result by Li and Yau [13] in dimension 2, they gave an explicit upper bound thanks to the \(m\)-conformal volume \(V_c(m, M, [g])\) of \((M, [g])\)

\[
\Lambda_1(M, [g]) \leq n V_c(m, M, [g])^{\frac{2}{n}}
\]  

(1)
These conformal invariants on the standard sphere \((S^n, [\text{can}])\) satisfy (cf. [7])
\[
\Lambda_1(S^n, [\text{can}]) = n\omega_n^{\frac{2}{n}} = nV_c(S^n, [\text{can}])^{\frac{2}{n}}
\]
and this value is achieved if and only if the metric is round. Here, \(\omega_n\) denotes the volume of the standard \(n\)-sphere. Colbois and El Soufi [4] also proved that, for any compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\),
\[
\Lambda_1(M, [g]) \geq \Lambda_1(S^n, [\text{can}]).
\]
We prove here that the case of equality characterizes the standard sphere:

**Theorem 1.** Let \((M, g)\) be a compact Riemannian manifold without boundary of dimension \(n \geq 3\). Then
\[
\Lambda_1(M, [g]) > \Lambda_1(S^n, [\text{can}])
\]
if \((M, [g])\) is not conformally diffeomorphic to \((S^n, [\text{can}])\).

This theorem answers the question raised in [2] and [11]. Note that a similar result was proved by the author in dimension 2 (see [14]). Note also that thanks to (1) and (2), the theorem implies
\[
V_c(m, M, [g]) > \omega_n = V_c(S^n, [\text{can}])
\]
if \((M, [g])\) is not conformally diffeomorphic to \((S^n, [\text{can}])\). This gives a positive answer to Question 2 in [13].

In the rest of this paper, we prove the theorem. Based on the idea of Ledoux [12] and Druet [5], we start from a sharp Sobolev inequality in dimensions \(n \geq 3\) (see [9, 5, 6]) which possesses extremal functions. These extremal functions give natural metrics \(\tilde{g} \in [g]\) with
\[
\text{Vol}_{\tilde{g}}(M) = 1 \quad \text{and} \quad \lambda_1(\tilde{g}) \geq n\omega_n^{\frac{2}{n}}.
\]
As in dimension 2, see [14], we deal with the degeneracy consequences of the hypothesis \(\lambda_1(\tilde{g}) = n\omega_n^{\frac{2}{n}}\).

Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 3\) with \(\text{Vol}_{g}(M) = 1\), which is not conformally diffeomorphic to the standard sphere. For an integer \(m \geq 1\), let \(h \in \mathcal{C}^m(M)\). We let \(J_{g,h}\) be the functional defined for \(u \in W^{1,2}(M) \setminus \{0\}\) by
\[
J_{g,h}(u) = \frac{\int_M |\nabla u|^2_g \, dv_g + \int_M hu^2 \, dv_g - K_n^{-2} \left(\int_M |u|^{2^*}_g \, dv_g\right)^{\frac{2}{2^*}}}{\int_M u^2 \, dv_g}
\]
(3)
where
\[ K_n = \frac{2}{\sqrt{n(n-2)}} \omega_n^{-\frac{1}{n}} \] (4)
is the sharp constant for the Sobolev inequality induced by the critical Sobolev embedding \( W_0^{1,2} \subset L^{2^*} \) for bounded domains of \( \mathbb{R}^n \), with \( 2^* = \frac{2n}{n-2} \). Hebey and Vaugon proved in [9] that
\[ -\alpha(g, h) = \inf_{u \in W^{1,2}(M) \setminus \{0\}} J_{g,h}(u) \] (5)
is finite. Note that \( J_{g,h} \) is scale invariant.

We will assume in the following that up to a conformal change, \( g \) is a metric in \([g]\) with volume 1 which has a constant scalar curvature \( S_g \). Since \( M \) is not conformally diffeomorphic to the standard sphere, by the resolution of the Yamabe problem by Aubin [1] and Schoen [15], it satisfies
\[ \mu(M, g) < K_n^{-2} \] (6)where \( \mu(M, g) \) is the Yamabe invariant of \((M, [g])\). Let \( V \) be an open neighbourhood of \( \frac{n-2}{4(n-1)} S_g \) in \( C^m(M) \) such that
\[ \left\| h - \frac{n-2}{4(n-1)} S_g \right\|_\infty \leq \frac{1}{2} (K_n^{-2} - \mu(M, g)), \quad \text{for all } h \in V. \] (7)
Let \( s \geq 0 \) be such that \( s + 2 > \frac{n}{2} \) and \( m \geq s + 2 \). By the Sobolev embedding
\[ W^{s+2,2} \hookrightarrow C^0, \]
the subset \( W_+^{s+2,2} \) of positive functions of \( W^{s+2,2} \) is open. We define
\[ F : W_+^{s+2,2} \times \mathbb{R} \times V \longrightarrow W^{s,2}, \]
\[ (u, \beta, h) \longmapsto \Delta_g u + (h + \beta)u - K_n^{-2} u^{2^*-1}, \]
which is well defined because of the Sobolev algebra property of \( W^{s+2,2} \) and \( F \) is a \( C^\infty \) map. By a result of Druet [5], thanks to (6) and (7), for any \( h \in V \), the functional \( J_{g,h} \) attains its infimum. Let \( u \in W^{1,2}(M) \) be such that
\[ J_{g,h}(u) = -\alpha(g, h). \]
Up to replace \( u \) by \( |u| \) and up to normalize, we can take
\[ u \geq 0 \quad \text{and} \quad \int_M u^{2^*} dv_g = 1. \]
Then, \( u \) satisfies the Euler–Lagrange equation
\[
F(u, \alpha(g, h), h) = \Delta_g u + (h + \alpha(g, h))u - K_n^{-2} u^{2^* - 1} = 0
\]
(8)
where, by elliptic regularity theory, \( u \in C^{m+2} \) and, by the maximum principle, \( u > 0 \).

Let \( v \in C^\infty(M) \) and \( t \in \mathbb{R} \) such that \( |t| < \|v\|^{-1}_\infty \). Since \( u \) is a minimum for (5),
\[
\int_M |\nabla (u + tv)|^2 d\nu_g + \int_M (h + \alpha(g, h))(u + tv)^2 d\nu_g
- K_n^{-2} \left( \int_M (u + tv)^{2^*} d\nu_g \right)^{\frac{2}{2^*}} \geq 0.
\]
(9)
Since \( u \) satisfies (8), the left term in (9) vanishes until the order 2 in the Taylor development as \( t \to 0 \). Computing the second-order coefficient as \( t \to 0 \), one gets
\[
\int_M |\nabla (uv)|^2 d\nu_g + \int_M (h + \alpha(g, h))(uv)^2 d\nu_g
- K_n^{-2}(2^* - 1) \int_M v^2 u^{2^*} d\nu_g + K_n^{-2}(2^* - 2) \left( \int_M vu^{2^*} d\nu_g \right)^2 \geq 0.
\]
(10)
We now use the conformal transformation of the conformal Laplacian
\[
u^{2^*-1} \Delta_{\bar{g}} v = \Delta_g (uv) - v \Delta_g u, \quad \text{for all } v \in C^\infty(M),
\]
(11)
where
\[
\bar{g} = u^{\frac{4}{n-2}} g.
\]
We integrate (11) against \( uv \) and with (8),
\[
\int_M |\nabla (uv)|^2 d\nu_g
= \int_M |\nabla v|^2 d\nu_{\bar{g}} + \int_M v^2 u \Delta_g u d\nu_g
= \int_M |\nabla v|^2 d\nu_{\bar{g}} - \int_M (h + \alpha(g, h))v^2 u^2 d\nu_g + K_n^{-2} \int_M v^2 u^{2^*} d\nu_g
\]
and with (4), (10) becomes
\[
\int_M |\nabla v|^2 d\nu_{\bar{g}} - n \omega_\frac{2}{n} \int_M \left( v - \int_M v d\nu_{\bar{g}} \right)^2 d\nu_{\bar{g}} \geq 0.
\]
(12)
This gives that \( \lambda_1(\tilde{g}) \geq n\omega_n^2 \). Note that if the inequality is strict for one solution \((h, u)\) of \( F(u, \alpha(g, h), h) = 0 \), the theorem is proved.

We now assume that for any solution \((h, u)\) of \( F(u, \alpha(g, h), h) = 0 \), we have \( \lambda_1(u^{\frac{4}{n-2}} g) = n\omega_n^2 \). We will apply the following theorem ([10], Theorem 5.4, p. 63) of Fredholm theory to \( F \), with \( U = W^{s+2,2}_+(M) \times \mathbb{R} \).

**Theorem 2.** Let \( X, Y \) be two separable Banach spaces, \( U \) an open set of \( X \), \( V \) a separable \( C^\infty \) Banach manifold and \( F \in C^\infty(U \times V, Y) \) which satisfy:

- for all \((u, v)\) in \( F^{-1}(0) \), \( DF(u) \) is surjective;
- for all \((u, v)\) in \( F^{-1}(0) \), \( D_u F(u, v) \) is a Fredholm operator.

Then there exists a countable intersection of open dense sets (a residual set) \( \Sigma \subset V \) such that for all \( v \in \Sigma \), and for all \( u \in F(., v)^{-1}(0) \), \( D_u F(u, v) \) is surjective.

Using (11) and (4), one gets for \((u, \beta, h)\) in \( F^{-1}(0) \),

\[
D_{(u, \beta)} F(u, \beta, h)(\theta, \mu) = u^{2^n-1} \left( \Delta_{\tilde{g}} \left( \frac{\theta}{u} \right) - n\omega_n^2 \frac{\theta}{u} \right) + \mu u, \tag{13}
\]

where \( \tilde{g} = u^{\frac{4}{n-2}} g \). Then, \( D_{(u, \beta)} F(u, \beta, h) \) is a Fredholm operator. It remains to prove that if \((u, \beta, h)\) in \( F^{-1}(0) \), \( DF(u, \beta, h) \) is surjective. We have

\[
DF(u, \beta, h)(\theta, \mu, \tau) = u^{2^n-1} \left( \Delta_{\tilde{g}} \left( \frac{\theta}{u} \right) - n\omega_n^2 \frac{\theta}{u} \right) + \mu u + \tau u. \tag{14}
\]

\( \text{Im}(D_{(u, \beta)} F(u, \beta, h)) \) is a closed space in \( W^{s,2} \) of finite codimension. Thus, since \( \text{Im}(DF(u, \beta, h)) \) contains \( \text{Im}(D_{(u, \beta)} F(u, \beta, h)) \), it is a closed space in \( W^{s,2} \) by the following lemma.

**Lemma.** Let \( X \) a Banach space, and \( E \subset F \subset X \) some subspaces. If \( E \) is a closed finite codimensional subspace of \( X \), then \( F \) is a closed subspace of \( X \).

**Proof.** Let \( G \) a finite dimensional subspace of \( X \) such that \( X = E \oplus G \). We set \( H = G \cap F \). Then, \( F = E \oplus H \). Let \( x_k \in F \) such that \( x_k \to x \) as \( k \to +\infty \). We denote \( x_k = y_k + z_k \) with \( y_k \in E \) and \( z_k \in H \).

We suppose that \((z_k)_{k \geq 0}\) is not bounded. Then, up to the extraction of a subsequence, \( |z_k| \to +\infty \) as \( k \to +\infty \). By Bolzano’s theorem, up to the extraction of a subsequence, there exists \( z \in H \) such that

\[
\frac{z_k}{|z_k|} \to z, \quad \text{as} \quad k \to +\infty.
\]
Since \((x_k)\) converges as \(k \to +\infty\),

\[
\frac{y_k}{|z_k|} = \frac{x_k}{|z_k|} - \frac{z_k}{|z_k|} \longrightarrow -z, \quad \text{as } k \to +\infty.
\]

Since \(E\) is closed, we get \(z \in E \cap H = 0\), which contradicts \(|z| = 1\).

Then \((z_k)_{k \geq 0}\) is bounded and by Bolzano’s theorem, up to the extraction of a subsequence, we can suppose that \(z_k \to z \in H\) as \(k \to +\infty\). Then,

\[
y_k = x_k - z_k \longrightarrow x - z, \quad \text{as } k \to +\infty.
\]

and \(y = x - z \in E\) since \(E\) is closed. Therefore \(x = y + z \in E + H = F\) and the proof of the lemma is complete.

Now, it suffices to prove that \(\text{Im}(DF(u, \beta, h))^\perp = 0\), where \(\perp\) refers to the orthogonal in \(W^{s,2}\). Let \(\phi \in \text{Im}(DF(u, \beta, h))^\perp\). Then, with (14),

\[
\langle \phi, u\tau \rangle_{W^{s,2}} = 0, \quad \text{for all } \tau \in C^m.
\]

Since \(u \in C^m\) is positive and \(C^m\) is dense in \(W^{s,2}\), we get \(\phi = 0\).

By Theorem 2, there exists \(h \in V\) such that for all couple \((u, \beta)\) satisfying \(F(u, \beta, h) = 0\), \(DF(u, \beta)(u, \beta, h)\) is surjective. We take in particular \(\beta = \alpha(g, h)\) and we will deduce that for a minimal function \(u\), \(\lambda_1(\tilde{g}) = n\omega_n^{\frac{2}{n}}\) is simple with \(\tilde{g} = u^{\frac{4}{n-2}}g\). We claim that

\[
\int_M u^2\phi dv_g \neq 0, \quad \text{for all } \phi \in E_1(\tilde{g}) \setminus \{0\}. \tag{15}
\]

Indeed, if \(\phi\) is an eigenfunction for \(\lambda_1(\tilde{g})\) such that this integral vanishes, one easily checks with (13) that \(u\phi\) is orthogonal to the image of \(D_{(u, \beta)}F(u, \alpha(h, g), h)\) in \(L^2(g)\). It implies \(\phi = 0\) and we obtain (15). Since a bounded linear form vanishes on a one-codimensional space, we get that \(\lambda_1(\tilde{g})\) is simple. Thus, \(\lambda_1(\tilde{g})\) cannot be an extremal eigenvalue in the sense of [8] and as a result, \(\lambda_1(\tilde{g}) = n\omega_n^{\frac{2}{n}}\) is not locally maximal. The proof of Theorem 1 for \(n \geq 3\) is complete.

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References


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