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Invariant densities for random $\beta$-expansions

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Abstract. Let $\beta > 1$ be a non-integer. We consider expansions of the form $\sum_{i=1}^{\infty} d_i / \beta^i$, where the digits $(d_i)_{i>1}$ are generated by means of a Borel map $K_\beta$ defined on $[0, 1]^{\mathbb{N}} \times [0, \lfloor \beta \rfloor / (\beta - 1)]$. We show existence and uniqueness of a $K_\beta$-invariant probability measure, absolutely continuous with respect to $m_p \otimes \lambda$, where $m_p$ is the Bernoulli measure on $\{0, 1\}^\mathbb{N}$ with parameter $p (0 < p < 1)$ and $\lambda$ is the normalized Lebesgue measure on $[0, \lfloor \beta \rfloor / (\beta - 1)]$. Furthermore, this measure is of the form $m_p \otimes \mu_{\beta,p}$, where $\mu_{\beta,p}$ is equivalent to $\lambda$. We prove that the measure of maximal entropy and $m_p \otimes \lambda$ are mutually singular. In case the number 1 has a finite greedy expansion with positive coefficients, the measure $m_p \otimes \mu_{\beta,p}$ is Markov. In the last section we answer a question concerning the number of universal expansions, a notion introduced in [EK].

Keywords. Greedy expansions, lazy expansions, absolutely continuous invariant measures, measures of maximal entropy, Markov chains, universal expansions

1. Introduction

Let $\beta > 1$ be a non-integer, and denote by $\lfloor \beta \rfloor$ the integer part of $\beta$. In this paper we consider expansions of numbers $x$ in $J_\beta := [0, \lfloor \beta \rfloor / (\beta - 1)]$ of the form

$$x = \sum_{i=1}^{\infty} a_i \beta^{-i}$$

with $a_i \in \{0, 1, \ldots, \lfloor \beta \rfloor\}$, $i \in \mathbb{N}$. We shall refer to expansions of this form as $(\beta)$-expansions or expansions in base $\beta$. The largest expansion in lexicographical order of a number $x \in J_\beta$ is the greedy expansion of $x$ ([P], [R1], [R2]), and the smallest is the lazy expansion of $x$ ([JS], [EJK], [DK1]). The greedy expansion is obtained by iterating the greedy
transformation $T_\beta : J_\beta \to J_\beta$, defined by

$$T_\beta(x) = \beta x - d \quad \text{for } x \in C(d),$$

where

$$C(j) = \left[ \frac{j}{\beta}, \frac{j + 1}{\beta} \right), \quad j \in \{0, \ldots, \lfloor \beta \rfloor - 1\},$$

and

$$C(\lfloor \beta \rfloor) = \left[ \frac{[\beta]}{\beta}, \frac{[\beta]}{\beta} - 1 \right].$$

The greedy expansion of $x \in J_\beta$ is given by $x = \sum_{i=1}^{\infty} d_i(x) / \beta^i$, where $d_i(x) = d$ if and only if $T_{\beta - 1}^i(x) \in C(d)$. Let $\ell : J_\beta \to J_\beta$ be given by

$$\ell(x) = \frac{[\beta]}{\beta - 1} - x.$$

Then the lazy transformation $L_\beta : J_\beta \to J_\beta$ is defined by

$$L_\beta(x) = \beta x - d \quad \text{for } x \in \Delta(d) = \ell(C([\beta] - d)), \ d \in \{0, \ldots, \lfloor \beta \rfloor \}.$$

The lazy expansion of $x \in J_\beta$ is given by $x = \sum_{i=1}^{\infty} \tilde{d}_i(x) / \beta^i$, where $\tilde{d}_i(x) = d$ if and only if $L_{\beta - 1}^i(x) \in \Delta(d)$.

We denote by $\mu_\beta$ the extended $T_\beta$-invariant Parry measure (see [P], [G]) on $J_\beta$ which is absolutely continuous with respect to Lebesgue measure, and with density

$$h_\beta(x) = \begin{cases} \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{1}{\beta^n} 1_{[0, T_\beta^n(1))}(x), & 0 \leq x < 1, \\ 0, & 1 \leq x \leq [\beta]/(\beta - 1), \end{cases}$$

where $F(\beta)$ is the normalizing constant. Define the lazy measure $\rho_\beta$ on $J_\beta$ by setting $\rho_\beta = \mu_\beta \circ \ell^{-1}$. It is easy to see ([DK1]) that $\ell$ is a continuous isomorphism between $(J_\beta, \mu_\beta, T_\beta)$ and $(J_\beta, \rho_\beta, L_\beta)$.

In order to produce other expansions in a dynamical way, a new transformation $K_\beta$ was introduced in [DK2]. The expansions generated by iterating this map are random mixtures of greedy and lazy expansions. This is done by superimposing the greedy map and the corresponding lazy map on $J_\beta$. In this way one obtains $[\beta]$ intervals on which the greedy map and the lazy map differ. These intervals are given by

$$S_k = \left[ \frac{k}{\beta}, \frac{k}{\beta(\beta - 1)} + \frac{k - 1}{\beta} \right], \quad k = 1, \ldots, \lfloor \beta \rfloor,$$

which one refers to as switch regions. On $S_k$, the greedy map assigns the digit $k$, while the lazy map assigns the digit $k - 1$. Outside these switch regions both maps are identical, and hence they assign the same digits. Now define other expansions in base $\beta$ by randomizing the choice of the map used in the switch regions. So, whenever $x$ belongs to a switch
region, flip a coin to decide which map will be applied to \( x \), and hence which digit will be assigned. To be more precise, partition the interval \( J_\beta \) into switch regions \( S_k \) and equality regions \( E_k \), where

\[
E_k = \left( \frac{\lfloor \beta \rfloor}{\beta(\beta - 1)} + \frac{k - 1}{\beta}, \frac{k + 1}{\beta} \right), \quad k = 1, \ldots, \lfloor \beta \rfloor - 1,
E_0 = \left[ 0, \frac{1}{\beta} \right) \quad \text{and} \quad E_{\lfloor \beta \rfloor} = \left( \frac{\lfloor \beta \rfloor}{\beta(\beta - 1)} + \frac{\lfloor \beta \rfloor - 1}{\beta}, \frac{\lfloor \beta \rfloor}{\beta} \right).
\]

Let

\[
S = \bigcup_{k=1}^{\lfloor \beta \rfloor} S_k \quad \text{and} \quad E = \bigcup_{k=0}^{\lfloor \beta \rfloor} E_k,
\]

and consider \( \Omega = \{0, 1\}^\mathbb{N} \) with product \( \sigma \)-algebra \( \mathcal{A} \). Let \( \sigma : \Omega \to \Omega \) be the left shift, and define \( K_\beta : \Omega \times J_\beta \to \Omega \times J_\beta \) by

\[
K_\beta(\omega, x) = \begin{cases} 
(\omega, \beta x - k), & x \in E_k, k = 0, 1, \ldots, \lfloor \beta \rfloor, \\
(\sigma(\omega), \beta x - k), & x \in S_k \text{ and } \omega_1 = 1, k = 1, \ldots, \lfloor \beta \rfloor, \\
(\sigma(\omega), \beta x - k + 1), & x \in S_k \text{ and } \omega_1 = 0, k = 1, \ldots, \lfloor \beta \rfloor.
\end{cases}
\]

The elements of \( \Omega \) represent the coin tosses (‘heads’ = 1 and ‘tails’ = 0) used every time the orbit \( \{K_\beta^n(\omega, x) : n \geq 0\} \) hits \( \Omega \times S \). Let

\[
d_1 = d_1(\omega, x) = \begin{cases} 
k & \text{if } x \in E_k, k = 0, 1, \ldots, \lfloor \beta \rfloor, \\
1 & \text{or } (\omega, x) \in \{\omega_1 = 1\} \times S_k, k = 1, \ldots, \lfloor \beta \rfloor, \\
0 & \text{if } (\omega, x) \in \{\omega_1 = 0\} \times S_k, k = 1, \ldots, \lfloor \beta \rfloor.
\end{cases}
\]

Then

\[
K_\beta(\omega, x) = \begin{cases} 
(\omega, \beta x - d_1) & \text{if } x \in E, \\
(\sigma(\omega), \beta x - d_1) & \text{if } x \in S.
\end{cases}
\]

Set \( d_n = d_n(\omega, x) = d_1(K_\beta^{n-1}(\omega, x)) \), and let \( \pi_2 : \Omega \times J_\beta \to J_\beta \) be the canonical projection onto the second coordinate. Then

\[
\pi_2(K_\beta^n(\omega, x)) = \beta^n x - \beta^{n-1} d_1 - \cdots - \beta d_{n-1} - d_n.
\]

and rewriting yields

\[
x = \frac{d_1}{\beta} + \cdots + \frac{d_n}{\beta^n} + \frac{\pi_2(K_\beta^n(\omega, x))}{\beta^n}.
\]

This shows that for all \( \omega \in \Omega \) and for all \( x \in J_\beta \) one has

\[
x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = \sum_{i=1}^{\infty} \frac{d_i(\omega, x)}{\beta^i}.
\]

The random procedure just described shows that to each \( \omega \in \Omega \) corresponds an algorithm that produces an expansion in base \( \beta \). Furthermore, if we identify the point \( (\omega, x) \) with
(ω, (d₁(ω, x), d₂(ω, x), . . .)), then the action of $K_β$ on the second coordinate corresponds to the left shift.

Let $<_{\text{lex}}$ and $\leq_{\text{lex}}$ denote the lexicographical ordering on both $\Omega$ and $\{0, \ldots, \lfloor β \rfloor\}^\mathbb{N}$. We recall from [DdV] the following basic properties of random $β$-expansions.

**Theorem 1.** Suppose $ω, ω' ∈ Ω$ are such that $ω <_{\text{lex}} ω'$. Then

$$(d₁(ω, x), d₂(ω, x), \ldots) ≤_{\text{lex}} (d₁(ω', x), d₂(ω', x), \ldots).$$

**Theorem 2.** Let $x ∈ J_β$ and let $x = ∑_{i=1}^{∞} a_i/β^i$ with $a_i ∈ \{0, 1, \ldots, \lfloor β \rfloor\}$ be an expansion of $x$ in base $β$. Then there exists an $ω ∈ Ω$ such that $a_i = d_i(ω, x)$ for all $i ≥ 1$.

In [DdV] it is shown that there exists a unique measure of maximal entropy $ν_β$ for the map $K_β$. It is the main goal of this paper to investigate the relationship between this measure and the measure $m_p ⊗ \lambda$, where $λ$ is the normalized Lebesgue measure on $J_β$ and $m_p$ is the Bernoulli measure on $Ω$ with parameter $p$ ($0 < p < 1$):

$$m_p((ω_1 = i_1, \ldots, ω_n = i_n)) = p^{∑_{i=1}^{n} i_j} (1 - p)^{n - ∑_{i=1}^{n} i_j}, \quad i_1, \ldots, i_n ∈ \{0, 1\}.$$

In this paper, the parameter $p ∈ (0, 1)$ is fixed but arbitrary, unless stated otherwise. In order to prove that the measures $ν_β$ and $m_p ⊗ \lambda$ are mutually singular, we introduce in the next section another $K_β$-invariant probability measure. It is a product measure $m_p ⊗ μ_β$, and we show in Section 3 that $K_β$ is ergodic with respect to it. Furthermore, the measures $m_p ⊗ λ$ and $m_p ⊗ μ_β$ are shown to be equivalent. These facts enable us to conclude that the measures $ν_β$ and $m_p ⊗ λ$ are mutually singular. Moreover, it follows that $m_p ⊗ μ_β$ is the unique absolutely continuous $K_β$-invariant probability measure with respect to $m_p ⊗ λ$. The measure $μ_β$ satisfies the important relationship

$$μ_β = p · μ_β T^{-1}_β + (1 - p) · μ_β L^{-1}_β.$$

In Section 4 we show that if 1 has a finite greedy expansion with positive coefficients, then the measure $m_p ⊗ μ_β$ is Markov, and we determine the measure $μ_β$ explicitly. In Section 5 we discuss some open problems. As an application of some of the results in this paper, we also show that for $λ$-a.e. $x ∈ J_β$, there exist $2^{ℵ_0}$ so-called universal expansions of $x$ in base $β$.

2. The skew product transformation $R_β$

Define the skew product transformation $R_β$ on $Ω × J_β$ as follows:

$$R_β(ω, x) = \begin{cases} (σ(ω), T_β x) & \text{if } ω_1 = 1, \\ (σ(ω), L_β x) & \text{if } ω_1 = 0. \end{cases}$$

On the set $Ω × J_β$, we consider the $σ$-algebra $A ⊗ B$, where $A$ is the product $σ$-algebra on $Ω$ and $B$ is the Borel $σ$-algebra on $J_β$. Let $μ$ be an arbitrary probability measure on $J_β$. The following result shows that a product measure of the form $m_p ⊗ μ$ is $K_β$-invariant if and only if it is $R_β$-invariant.
Lemma 1. \( m_p \otimes \mu \circ K_\beta^{-1} = m_p \otimes \mu \circ R_\beta^{-1} = m_p \otimes v \), where
\[
v = p \cdot \mu \circ T_\beta^{-1} + (1 - p) \cdot \mu \circ L_\beta^{-1}.
\]

Proof. Denote by \( C \) an arbitrary cylinder in \( \Omega \) and let \([a, b]\) be an interval in \( J_\beta \). It suffices to verify that the measures coincide on sets of the form \( C \times [a, b] \), because the collection of these sets forms a generating \( \pi \)-system. Furthermore, let \([i, C] = \{\omega_1 = i\} \cap \sigma^{-1}(C)\) for \( i = 0, 1 \). Note that \( E \cap T_\beta^{-1}[a, b] = E \cap L_\beta^{-1}[a, b] \), and that
\[
K_\beta^{-1}(C \times [a, b]) = C \times (E \cap T_\beta^{-1}[a, b]) \cup [0, C] \times (S \cap L_\beta^{-1}[a, b])
\]
\[
\cup [1, C] \times (S \cap \beta^{-1}[a, b]).
\]
Hence,
\[
m_p \otimes \mu \circ K_\beta^{-1}(C \times [a, b]) = p \cdot m_p(C) \cdot \mu(T_\beta^{-1}[a, b])
\]
\[
+ (1 - p) \cdot m_p(C) \cdot \mu(L_\beta^{-1}[a, b])
\]
\[
= m_p \otimes v(C \times [a, b]).
\]

On the other hand,
\[
R_\beta^{-1}(C \times [a, b]) = [0, C] \times L_\beta^{-1}[a, b] \cup [1, C] \times T_\beta^{-1}[a, b],
\]
and the result follows. \( \square \)

Let \( \mathcal{D} = \mathcal{D}(J_\beta, B, \lambda) \) denote the space of probability density functions on \( J_\beta \) with respect to \( \lambda \). A measurable transformation \( T : J_\beta \to J_\beta \) is called nonsingular if \( \lambda(T^{-1}B) = 0 \) whenever \( \lambda(B) = 0 \).

If \( \mu \) is absolutely continuous with respect to \( \lambda \) with probability density \( f = d\mu/d\lambda \), and if \( T \) is a nonsingular transformation, then \( \mu \circ T^{-1} \) is absolutely continuous with respect to \( \lambda \) with probability density \( P_T f \) (say). Equivalently, the Frobenius–Perron operator \( P_T : \mathcal{D} \to \mathcal{D} \) is defined as a linear operator such that for \( f \in \mathcal{D} \), \( P_T f \) is the function for which
\[
\int_B P_T f \, d\lambda = \int_{T^{-1}B} f \, d\lambda \quad \text{for all } B \in \mathcal{B}.
\]

Existence and uniqueness (\( \lambda \)-a.e.) follow from the Radon–Nikodým theorem. A nonsingular transformation \( T : J_\beta \to J_\beta \) is said to be a Lasota–Yorke type map (L-Y map) if \( T \) is piecewise monotone and \( C^2 \). Piecewise monotone and \( C^2 \) means that there exists a partition \( \mathcal{P} = \{[a_i-1, a_i] : i = 1, \ldots, k \} \) such that for each \( i = 1, \ldots, k \), the restriction of \( T \) to \([a_i-1, a_i]\) is monotone and extends to a \( C^2 \) map on \([a_i-1, a_i]\). For such a transformation the Frobenius–Perron operator can be computed explicitly (see [BG, p. 86]) by the formula
\[
P_T f(x) = \sum_{T(y) = x} \frac{f(y)}{|T'(y)|}.
\]
If, in addition, \( |T'(x)| \geq \alpha > 1 \) for each \( x \in (a_{i-1}, a_i), i = 1, \ldots, k \), then we say that \( T \) is a piecewise expanding L-Y map. Let \( T_1, \ldots, T_n \) be L-Y maps on \( J_\beta \) with common partition of joint monotonicity \( \mathcal{P} = \{[a_{i-1}, a_i] : i = 1, \ldots, k \} \). For \( f \in \mathcal{D} \), define \( Pf = \sum_{i=1}^n p_i \cdot P_{T_i} f \), where \((p_1, \ldots, p_n)\) is a probability vector. We recall the following important theorem, due to Pelikan [Pel]. For more results concerning invariant densities of L-Y maps see [LY], [LY], [Pel].

**Theorem 3.** Suppose that for all \( x \in J_\beta \setminus \{a_0, \ldots, a_k\} \), \( \sum_{i=1}^n p_i / |T'_i(x)| \leq \gamma < 1 \). Then for all \( f \in \mathcal{D} \), the limit

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j f = f^*
\]

exists in \( L_1(J_\beta, \lambda) \). Furthermore, \( Pf^* = f^* \) and one can choose \( f^* \) to be of bounded variation.

Since \( T_\beta \) and \( L_\beta \) are both piecewise expanding L-Y maps, it follows at once from Theorem 3 that for all \( f \in \mathcal{D} \), the limit

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j f = f^*
\]

exists in \( L_1(J_\beta, \lambda) \), where

\[
P f = p \cdot P_{T_\beta} f + (1 - p) \cdot P_{L_\beta} f.
\]

Define for \( f \in \mathcal{D} \) the probability measure \( \mu_f \) by

\[
\mu_f(B) = \int_B f \, d\lambda \quad [B \in \mathcal{B}].
\]

Observe that \( Pf = f \) if and only if

\[
\mu_f = p \cdot \mu_f \circ T_\beta^{-1} + (1 - p) \cdot \mu_f \circ L_\beta^{-1},
\]

i.e., if and only if \( m_p \otimes \mu_f \) is \( R_\beta \)-invariant (cf. Lemma 1).

Let \( 1 \) denote the constant function equal to 1 on \( J_\beta \) and consider the function \( 1^* \), given by

\[
1^* = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} P^j 1 \quad \text{in } L_1(J_\beta, \lambda).
\]

We shall assume that the function \( 1^* \) is of bounded variation. Note that this is possible by Theorem 3. It follows easily from the definition of bounded variation that the left- and right-hand limits of \( 1^* \) at every point \( x \in J_\beta \) exist and that the function \( 1^* \) is continuous except maybe at countably many points. Now we modify the function \( 1^* \) in such a way that it becomes lower semicontinuous. Replace \( 1^*(x) \) at every discontinuity point \( x \) in the interior of \( J_\beta \) by setting

\[
1^*(x) = \min\{1^*(x^-), 1^*(x^+)\}
\]
and replace $1^*(x)$ by its left- or right-hand limit if $x$ is an endpoint of $J_\beta$. In the remainder of this section we work with this modified version of $1^*$ which we denote again by $1^*$.

In the next theorem, we show that this function is bounded below by a positive constant $d > 0$, everywhere on $J_\beta$.

**Theorem 4.** The skew product transformation $R_\beta$ is ergodic with respect to the measure $m_p \otimes \mu_1$. Furthermore, the measures $m_p \otimes \mu_1$ and $m_p \otimes \lambda$ are equivalent and the density $1^*$ is bounded below by a positive constant $d$, everywhere on $J_\beta$.

**Proof.** Since $P1^* = 1^*$, it follows from Lemma 1 that the measure $m_p \otimes \mu_1$ is $R_\beta$-invariant. It is well known that the greedy transformation $T_\beta$ is ergodic with respect to its unique absolutely continuous invariant measure, which is the Parry measure $\mu_\beta$ (see Section 1). Similarly, the lazy transformation is ergodic with respect to its unique absolutely continuous invariant measure. This implies [Pel, Corollary 7] that the skew product transformation $R_\beta$ is ergodic with respect to $m_p \otimes \mu_1$. Since the random Frobenius–Perron operator $P$ is integral preserving with respect to $\lambda$, we have

$$\int_{J_\beta} 1^* d\lambda = 1.$$  

In particular, there exists a point $x_0$ in the interior of $J_\beta$ for which $1^*(x_0) > 0$. By lower semicontinuity of $1^*$, there exist an open interval $(a, b) \subset J_\beta$ and a constant $c > 0$ such that $1^*(x) > c$ for each $x \in (a, b)$. Rewriting (1) one gets, for $\lambda$-a.e. $x$,

$$P_{T_\beta} f(x) = \frac{1}{\beta} \sum_{T_\beta y = x} f(y), \quad P_{L_\beta} f(x) = \frac{1}{\beta} \sum_{L_\beta y = x} f(y)$$

(see also [P, Theorem 1]), and thus

$$1^*(x) = \frac{p}{\beta} \sum_{T_\beta y = x} 1^*(y) + \frac{1-p}{\beta} \sum_{L_\beta y = x} 1^*(y).$$

Hence, for $\lambda$-a.e. $x \in T_\beta(a, b)$, we have

$$1^*(x) > \frac{pc}{\beta}.$$  

By induction, for each $n$ and for $\lambda$-a.e. $x \in T_\beta^n(a, b)$, we have

$$1^*(x) > \frac{p^n c}{\beta^n}.$$  

It is easy to verify that there exist a number $\delta > 0$ and a positive integer $n$ such that

$$T_\beta^n(a, b) \supset [z, z + \delta),$$

where $z$ is a discontinuity point of $T_\beta$. Hence,

$$T_\beta^{n+1}(a, b) \supset [0, \beta \delta).$$
Moreover, there exists a positive integer \( m \) such that
\[
L_\beta^m (\{0, \beta \delta\}) = J_\beta.
\]
Using the same argument as before, we conclude that for \( \lambda \)-a.e. \( x \in J_\beta \),
\[
\mathbf{1}^\ast(x) > d := \frac{p^{n+1} (1 - p)^m c}{\beta^{n+m+1}}.
\]
Hence, the function \( \mathbf{1}^\ast \) is larger than or equal to \( d \) at every continuity point of \( \mathbf{1}^\ast \). Due to our modification of \( \mathbf{1}^\ast \) at discontinuity points, the function \( \mathbf{1}^\ast \) is everywhere larger than or equal to \( d \). The equivalence of \( m_p \otimes \mu_\ast \) and \( m_p \otimes \lambda \) is an immediate consequence. \( \square \)

Since any invariant probability measure absolutely continuous with respect to an ergodic invariant probability measure coincides with this measure, we deduce from Theorems 3 and 4 that for all \( f \in D \),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} P_j f = \mathbf{1}^\ast \text{ in } L_1 (J_\beta, \lambda).
\]

**Remarks 1.**

(i) From now on we write \( \mu_{\beta, p} \) instead of \( \mu_{\mathbf{1}^\ast} \), since the measure depends on both \( \beta \) and \( p \). It is the unique probability measure, absolutely continuous with respect to \( \lambda \), satisfying the relationship
\[
\mu_{\beta, p} = p \cdot \mu_{\beta, p} \circ T_\beta^{-1} + (1 - p) \cdot \mu_{\beta, p} \circ L_\beta^{-1}.
\]

(ii) Recall that \( \ell : J_\beta \to J_\beta \) given by \( \ell(x) = \lfloor \beta \rfloor / (\beta - 1) \) satisfies \( T_\beta \circ \ell = \ell \circ L_\beta \). It follows from the previous remark that \( \mu_{\beta, p} \circ \ell^{-1} = \mu_{\beta, 1-p} \). In particular, we see that the invariant density \( \mathbf{1}^\ast \) is symmetric on \( J_\beta \) if \( p = 1/2 \).

(iii) Let \( T_1, \ldots, T_n \) be piecewise expanding L-Y maps on \( J_\beta \) and let \( (p_1, \ldots, p_n) \) be a probability vector. Recently it has been shown by Boyarsky, Góra and Islam (see [BGI]) that functions \( f \in D \) satisfying \( f = P f = \sum_{i=1}^n p_i \cdot P_i f \) are bounded below by a positive constant on their support (\( \lambda \)-a.e.). Hence, the fact that \( \mathbf{1}^\ast \) is bounded below by a positive constant on \( J_\beta \) can also be deduced from their result combined with the equivalence of \( m_p \otimes \lambda \) and \( m_p \otimes \mu_{\beta, p} \).

(iv) It is well known that the Parry measure \( \mu_\beta \) is the unique probability measure, absolutely continuous with respect to \( \lambda \) and satisfying equation (3) with \( p = 1 \). Note however that \( \mu_\beta \) and \( \lambda \) are not equivalent on \( J_\beta \). Similarly, the lazy measure \( \rho_\beta \) and \( \lambda \) are not equivalent. For this reason, we restrict ourselves to values of the parameter \( p \) in the open interval \((0, 1)\).

3. Main Theorem

It is the object of this section to show that the measure \( \nu_\beta \) of maximal entropy for the map \( K_\beta \) and the measure \( m_p \otimes \lambda \) are mutually singular.
Let $D = [0, 1, \ldots, [\beta]]^\mathbb{N}$ be equipped with the product \(\sigma\)-algebra \(D\) and let \(\sigma'\) be the left shift on \(D\). Define the function \(\varphi : \Omega \times J_\beta \to D\) by

\[
\varphi(\omega, x) = (d_1(\omega, x), d_2(\omega, x), \ldots).
\]

Clearly, \(\varphi\) is measurable and \(\varphi \circ K_\beta = \sigma' \circ \varphi\). Furthermore, Theorem 2 implies that \(\varphi\) is surjective. Let

\[
Z = \{(\omega, x) \in \Omega \times J_\beta : K_n(\omega, x) \in \Omega \times S \text{ for infinitely many } n \geq 0\},
\]

\[
D' = \{(a_1, a_2, \ldots) \in D : \sum_{i=1}^\infty a_i = 0, 1, [\beta] \text{ for infinitely many } j \geq 1\}.
\]

Observe that \(K_\beta^{-1}(Z) = Z\), \((\sigma')^{-1}(D') = D'\) and that the restriction \(\varphi' : Z \to D'\) of \(\varphi\) to \(Z\) is a bimeasurable bijection. Let \(P\) denote the uniform product measure on \(D\). We recall from [DdV] that the measure \(\nu_\beta\) defined on \(A \otimes B\) by

\[
\nu_\beta(A) = P(\varphi(Z \cap A))
\]

is the unique \(K_\beta\)-invariant measure of maximal entropy \(\log(1 + [\beta])\). It was also shown that the projection of \(\nu_\beta\) on the second coordinate is an infinite convolution of Bernoulli measures (see [E1], [E2]). More precisely, consider the purely discrete probability measures \(\delta_i\) defined on \(J_\beta\) and determined by

\[
\delta_i(\{k\beta - i\}) = \frac{1}{[\beta] + 1} \text{ for } k = 0, 1, \ldots, [\beta].
\]

Let \(\delta_\beta\) be the corresponding infinite Bernoulli convolution, i.e.,

\[
\delta_\beta = \lim_{n \to \infty} \delta_1 * \cdots * \delta_n.
\]

Then \(\nu_\beta \circ \pi_1^{-1} = \delta_\beta\).

For \(\omega \in \Omega\), let \(\overline{\omega}\) be given by

\[
\overline{\omega} = (\overline{\omega_1}, \overline{\omega_2}, \ldots) = (1 - \omega_1, 1 - \omega_2, \ldots).
\]

Concerning the projection \(\pi_1 : \Omega \times J_\beta \to \Omega\) of the measure \(\nu_\beta\) on the first coordinate, we have the following lemma.

**Lemma 2.** For \(n \geq 1\) and \(i_1, \ldots, i_n \in \{0, 1\}\), we have

\[
\nu_\beta(\pi_1^{-1}([\omega_1 = i_1, \ldots, \omega_n = i_n])) = \nu_\beta(\pi_1^{-1}([\overline{\omega}_1 = i_1, \ldots, \overline{\omega}_n = i_n])).
\]

**Proof.** Define the map \(r : D \to D\) by

\[
r(a_1, a_2, \ldots) = ([\beta] - a_1, [\beta] - a_2, \ldots).
\]

It follows easily by induction that for \(i \geq 1\) and \((\omega, x) \in \Omega \times J_\beta\),

\[
d_i(\omega, x) = [\beta] - d_i(\overline{\omega}, \ell(x)).
\]

Hence,

\[
\varphi(\omega, x) = r \circ \varphi(\overline{\omega}, \ell(x)).
\]

Since the map \(r\) is clearly invariant with respect to \(P\), the assertion follows. \(\square\)
In particular, it follows from Lemma 2 that $v_\beta \circ \pi_1^{-1}([\omega_i = 1]) = 1/2$ for all $i \geq 1$. However, in general, the measure $v_\beta \circ \pi_1^{-1}$ is not the uniform Bernoulli measure on $[0, 1]^\mathbb{N}$. For instance, using the techniques in [DdV, Section 4], one easily shows that if the greedy expansion of $1$ in base $\beta$ satisfies $1 = 1/\beta + 1/\beta^2$, then $v_\beta \circ \pi_1^{-1}$ provides a counterexample. In the case that $1$ has a finite greedy expansion with positive coefficients, it has been shown in [DdV, Theorem 8] that $v_\beta \circ \pi_1^{-1}$ is the uniform Bernoulli measure.

The next lemma shows that the $K_\beta$-invariant measures $v_\beta$ and $m_\rho \otimes \mu_{\beta,p}$ are different.

**Lemma 3.** $v_\beta \neq m_\rho \otimes \mu_{\beta,p}$.

**Proof.** According to Theorem 4 there exists a constant $c > 0$ such that $I^*(x) \geq c$ for all $x \in J_\beta$. Choose $n \in \mathbb{N}$ such that $1/\beta + 1/\beta^n \in S_1$. Now, suppose the converse is true, i.e., that the measures $v_\beta$ and $m_\rho \otimes \mu_{\beta,p}$ coincide. In particular, $v_\beta$ is a product measure and $\delta_\beta = \mu_{\beta,p}$.

On the one hand, we infer from Lemma 2 that

$$v_\beta \left( \{\omega_1 = 1\} \times J_\beta \right) = \frac{1}{2}.$$

On the other hand, since the digits $(d_i)_{i \geq 1}$ form a uniform Bernoulli process under $v_\beta$,

$$v_\beta \left( \{\omega_1 = 1\} \times J_\beta \right) = \frac{1}{2} \leq \frac{1}{c} \left( \frac{\beta}{1+\beta} \right)^n \delta_\beta([0,1]).$$

Passing to the limit, we get a contradiction. \hfill \Box

Define the map $F : \Omega \times J_\beta \to D$ by

$$F(\omega, x) = (d_1(\omega, x), d_1(R_\beta(\omega, x)), d_1(R_\beta^2(\omega, x)), \ldots).$$

We have $\sum_{i=1}^{\infty} d_1(R_\beta^{i-1}(\omega, x))/\beta^i = x$ for all $(\omega, x) \in \Omega \times J_\beta$. Moreover, the map $F$ is surjective and $\sigma' \circ F = F \circ R_\beta$. Hence $F$ is a factor map and $\sigma'$ is ergodic with respect to the measure $\rho = m_\rho \otimes \mu_{\beta,p} \circ F^{-1}$. Note, however, that $F$ is not injective, even if we restrict it to the set for which $R_\beta$ hits $\Omega \times S$ infinitely many times; this is due to the fact that in equality regions only one digit can be assigned. It follows from Theorem 4 and Birkhoff’s ergodic theorem that $\rho$ is concentrated on $D'$. Therefore, the measure $\rho'$ defined on $A \otimes B$ by $\rho'(A) = \rho(\varphi(A \cap Z))$ is a $K_\beta$-invariant probability measure and $K_\beta$ is ergodic with respect to $\rho'$.

**Lemma 4.** $\rho' = m_\rho \otimes \mu_{\beta,p}$. 
Proof. Let

\[ A_{00} = \{ \omega_1 = 0 \} \times S_1, \quad A_{1\beta j} = \{ \omega_1 = 1 \} \times S_{\beta j}, \]
\[ A_{02} = \Omega \times E_0, \quad A_{1\beta 2} = \Omega \times E_{1\beta}, \]

and

\[ A_{i0} = \{ \omega_1 = 0 \} \times S_{i+1}, \]
\[ A_{i1} = \{ \omega_1 = 1 \} \times S_i, \]
\[ A_{i2} = \Omega \times E_i, \]

for \( 1 \leq i \leq \lfloor \beta \rfloor - 1 \). Note that for all \( 0 \leq i \leq \lfloor \beta \rfloor \), \( \varphi^{-1}(\{d_1 = i\}) \) is the union of the sets \( A_{ij} \). It is enough to show that \( \rho' = m_p \otimes \mu_{\beta, p} \) on sets of the form

\[ \varphi^{-1}(\{d_1 = i_1, \ldots, d_n = i_n\}), \quad i_1, \ldots, i_n \in \{0, \ldots, \lfloor \beta \rfloor \}. \]

Now,

\[ \varphi^{-1}(\{d_1 = i_1, \ldots, d_n = i_n\}) = \bigcup_{j_1, \ldots, j_n} A_{i_1 j_1} \cap \cdots \cap K^{-n+1}_\beta A_{i_n j_n}, \]

where the union is taken over all \( j_1, \ldots, j_n \) for which the sets \( A_{i_1 j_1}, \ldots, A_{i_n j_n} \) are defined. Hence, it is enough to show that

\[ \rho'(A_{i_1 j_1} \cap \cdots \cap K^{-n+1}_\beta A_{i_n j_n}) = m_p \otimes \mu_{\beta, p}(A_{i_1 j_1} \cap \cdots \cap K^{-n+1}_\beta A_{i_n j_n}). \]

It is easy to see that \( A_{i_1 j_1} \cap \cdots \cap K^{-n+1}_\beta A_{i_n j_n} \) is a product set. Denote its projection on the second coordinate by \( V_{i_1 j_1 \cdots i_n j_n} \). Define

\[ U = \{ (0, 0), ([\beta], 1) \} \cup \{(i, j) : 1 \leq i \leq \lfloor \beta \rfloor - 1, j \in \{0, 1\} \} \]

and

\[ \{\ell_1, \ldots, \ell_L\} = \{\ell : (i_\ell, j_\ell) \in U \} \subset \{1, \ldots, n\}, \quad \ell_1 < \cdots < \ell_L. \]

Then

\[ A_{i_1 j_1} \cap \cdots \cap K^{-n+1}_\beta A_{i_n j_n} = \{\omega_1 = j_{\ell_1}, \ldots, \omega_L = j_{\ell_L}\} \times V_{i_1 j_1 \cdots i_n j_n}. \quad (4) \]

Note that for all \( x \in V_{i_1 j_1 \cdots i_n j_n} \),

\[ F^{-1} \circ \varphi(\{\omega_1 = j_{\ell_1}, \ldots, \omega_L = j_{\ell_L}\} \times \{x\}) = \{\omega_{\ell_1} = j_{\ell_1}, \ldots, \omega_{\ell_L} = j_{\ell_L}\} \times \{x\}. \]

Therefore,

\[ F^{-1} \circ \varphi(A_{i_1 j_1} \cap \cdots \cap K^{-n+1}_\beta A_{i_n j_n}) = \{\omega_1 = j_{\ell_1}, \ldots, \omega_L = j_{\ell_L}\} \times V_{i_1 j_1 \cdots i_n j_n}. \quad (5) \]

The assertion follows immediately from (4) and (5). \( \Box \)

From Theorem 4, Lemmas 3 and 4, and the ergodicity of \( K_\beta \) with respect to \( \rho' \) and \( \nu_\beta \), we arrive at the following theorem.

**Theorem 5.** The measures \( \nu_\beta \) and \( m_p \otimes \lambda \) are mutually singular.

**Remark 2.** If \( \beta \in (1, 2) \) is a Pisot number, the mutual singularity of \( \nu_\beta \) and \( m_p \otimes \lambda \) is a simple consequence of the fact that in this case \( \delta_\beta \) and \( \lambda \) are mutually singular (see [E1], [E2]).
4. Finite greedy expansion of \(1\) with positive coefficients, and the Markov property of the random \(\beta\)-expansion

In this section we assume that the greedy expansion of \(1\) in base \(\beta\) satisfies \(1 = b_1/\beta + b_2/\beta^2 + \cdots + b_n/\beta^n\) with \(b_i \geq 1\) for \(i = 1, \ldots, n\) and \(n \geq 2\) (note that \(\lfloor \beta \rfloor = b_1\)). It has been shown in [DdV] that in this case the dynamics of \(K_\beta\) can be identified with a subshift of finite type with an irreducible adjacency matrix.

We exhibit the measure \(\nu_\beta \otimes \mu_{\beta, p}\) obtained in the previous section explicitly. Moreover, it turns out that \(K_\beta\) is exact with respect to \(m_\beta \otimes \mu_{\beta, p}\). The mutual singularity of \(v_\beta\) and \(m_\beta \otimes \lambda\), i.e., Theorem 5, will be derived by elementary means, independent of the results established in the previous sections.

The analysis of the case \(\beta^2 = b_1\beta + 1\) needs some adjustments. For this reason, we assume here that \(\beta^2 \neq b_1\beta + 1\), and refer the reader to [DdV, Remarks 6(2)] for the appropriate modifications needed for the case \(\beta^2 = b_1\beta + 1\). We first briefly recall some results obtained in [DdV].

We begin with a proposition which plays a crucial role in finding the Markov partition describing the dynamics of \(K_\beta\).

Proposition 1. Suppose \(1\) has a finite greedy expansion of the form \(1 = b_1/\beta + b_2/\beta^2 + \cdots + b_n/\beta^n\). If \(b_j \geq 1\) for \(1 \leq j \leq n\), then

(i) \(T^i_\beta 1 = L^i_\beta 1 \in E_{b_{i+1}},\quad 0 \leq i \leq n - 2\).

(ii) \(T^{n-1}_\beta 1 = L^{n-1}_\beta b_n \in S_{b_n},\quad T^n_\beta 1 = 0\), and \(L^n_\beta 1 = 1\).

(iii) \(T^i_\beta \left( b_1 / (\beta - 1) - 1 \right) = L^i_\beta \left( b_1 / (\beta - 1) - 1 \right) \in E_{b_i - b_{i+1}},\quad 0 \leq i \leq n - 2\).

(iv) \(T^{n-1}_\beta \left( b_1 / (\beta - 1) - 1 \right) = L^{n-1}_\beta \left( b_1 / (\beta - 1) - 1 \right) = \frac{b_1}{\beta(\beta - 1)} + \frac{b_1 - b_n}{\beta} \in S_{b_i - b_{i+1}}\),
\(T^n_\beta \left( b_1 / (\beta - 1) - 1 \right) = b_1 / (\beta - 1),\quad \text{and} \quad L^n_\beta \left( b_1 / (\beta - 1) - 1 \right) = b_1 / (\beta - 1)\).

To find the Markov chain behind the map \(K_\beta\), one starts by refining the partition
\[E = \{E_0, S_1, E_1, \ldots, S_{b_n}, E_{b_n}\}\]
of \([0, b_1/(\beta - 1)]\), using the orbits of \(1\) and \(b_1/(\beta - 1) - 1\) under the transformation \(T_\beta\).

We place the endpoints of \(E\) together with \(T^n_\beta 1, T^n_\beta (b_1/(\beta - 1) - 1), \ldots, n - 2\), in increasing order. We use these points to form a new partition \(C\) which is a refinement of \(E\), consisting of intervals. We write \(C\) as
\[C = \{C_0, C_1, \ldots, C_k\}\]
We choose \(C\) to satisfy the following. For \(0 \leq i \leq n - 2\),

- \(T^i_\beta 1 \in C_j\) if and only if \(T^i_\beta 1\) is a left endpoint of \(C_j\),
- \(T^n_\beta (b_1/(\beta - 1) - 1) \in C_j\) if and only if \(T^n_\beta (b_1/(\beta - 1) - 1)\) is a right endpoint of \(C_j\).
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Note that this choice is possible, because the points $T^i_{\psi}$, $T^i_{\psi}(b_1/(\beta - 1) - 1)$ for $0 \leq i \leq n - 2$ are all different. From the dynamics of $K_{\beta}$ on this refinement, one reads the following properties of $C$.

**p1.** $C_0 = [0, b_1/(\beta - 1) - 1]$ and $C_L = [1, b_1/(\beta - 1)]$.

**p2.** For $i = 0, 1, \ldots, b_1$, $E_i$ can be written as a finite disjoint union of the form $E_i = \bigcup_{j \in M} C_j$ with $M_0, M_1, \ldots, M_{b_1}$ disjoint subsets of $[0, 1, \ldots, L]$. Further, the number of elements in $M_i$ equals the number of elements in $M_{b_1-i}$.

**p3.** For each $S_i$ there is exactly one $j \in [0, 1, \ldots, L] \setminus \bigcup_{k=0}^{b_1} M_k$ such that $S_i = C_j$.

**p4.** If $C_j \subset E_i$, then $T_\beta(C_j) = L_\beta(C_j)$ is a finite disjoint union of elements of $C$, say $T_\beta(C_j) = C_i \cup \cdots \cup C_{j_i}$. Since $\ell(C_j) = C_{L-j} \subset E_{b_1-i}$, it follows that $T_\beta(C_{L-j}) = C_{L-i} \cup \cdots \cup C_{L-j}$.

**p5.** If $C_j = S_i$, then $T_\beta(C_j) = C_0$ and $L_\beta(C_j) = C_L$.

To define the underlying subshift of finite type associated with the map $K_{\beta}$, we consider the $(L+1) \times (L+1)$ matrix $A = (a_{i,j})$ with entries in $[0, 1]$ defined by

$$a_{i,j} = \begin{cases} 1 & \text{if } i \in \bigcup_{k=0}^{b_1} M_k \text{ and } \lambda(C_j \cap T_\beta(C_i)) = \lambda(C_j), \\ 0 & \text{if } i \in \bigcup_{k=0}^{b_1} M_k \text{ and } C_i \cap T_\beta^{-1}C_j = \emptyset, \\ 1 & \text{if } i \in [0, \ldots, L] \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j = 0, L, \\ 0 & \text{if } i \in [0, \ldots, L] \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j \neq 0, L. \end{cases}$$

Let $Y$ denote the topological Markov chain (or the subshift of finite type) determined by the matrix $A$. That is, $Y = \{y = (y_1) \in [0, 1, \ldots, L]^\mathbb{N} : a_{y_i, y_{i+1}} = 1\}$. We let $\sigma_Y$ be the left shift on $Y$. For ease of notation, we denote by $s_1, \ldots, s_{b_1}$ the states $j \in [0, \ldots, L] \setminus \bigcup_{k=0}^{b_1} M_k$ corresponding to the switch regions $S_1, \ldots, S_{b_1}$ respectively.

To each $y \in Y$, we associate a sequence $(e_i) \in [0, 1, \ldots, b_1]^\mathbb{N}$ and a point $x \in [0, b_1/(\beta - 1)]$ as follows. Let

$$e_j = \begin{cases} i & \text{if } y_j \in M_i, \\ i & \text{if } y_j = s_i \text{ and } y_{j+1} = 0, \\ i-1 & \text{if } y_j = s_i \text{ and } y_{j+1} = L. \end{cases} \quad (6)$$

Now set

$$x = \sum_{j=1}^{\infty} \frac{e_j}{\beta^j}. \quad (7)$$

Our aim is to define a map $\psi : Y \to \Omega \times [0, b_1/(\beta - 1)]$ that intertwines the actions of $K_\beta$ and $\sigma_Y$. Given $y \in Y$, equations (6) and (7) describe what the second coordinate of $\psi$ should be. In order to be able to associate an $\omega \in \Omega$, one needs that $y_1 \in [s_1, \ldots, s_{b_1}]$ infinitely often. For this reason it is not possible to define $\psi$ on all of $Y$, but only on an invariant subset. To be more precise, let

$$Y' = \{y = (y_1, y_2, \ldots) \in Y : y_1 \in [s_1, \ldots, s_{b_1}] \text{ for infinitely many } i \text{'s}\}.$$
Define $\psi : Y' \to \Omega \times [0, b_1/(\beta - 1)]$ as follows. Let $y = (y_1, y_2, \ldots) \in Y'$, and define $x$ as in (ii). To define a point $\omega \in \Omega$ corresponding to $y$, we first locate the indices $n_1 = n_1(y)$ where the realization $y$ of the Markov chain is in state $s_r$ for some $r \in \{1, \ldots, b_1\}$. Then, let $n_1 < n_2 < \cdots$ be the indices such that $y_{n_i} = s_r$ for some $r = 1, \ldots, b_1$. Define $\omega_j = \begin{cases} 1 & \text{if } y_{n_j + 1} = 0, \\ 0 & \text{if } y_{n_j + 1} = L. \end{cases}$

Now set $\psi(y) = (\omega, x)$.

The following two lemmas reflect the fact that the dynamics of $K_\beta$ is essentially the same as that of the Markov chain $Y$.

**Lemma 5.** Let $y \in Y'$ be such that $\psi(y) = (\omega, x)$. Then:

(i) $y_1 = k$ for some $k \in \bigcup_{i=0}^{b_1} M_i \Rightarrow x \in C_k$.

(ii) $y_1 = s_i, y_2 = 0 \Rightarrow x \in S_i$ and $\omega_1 = 1$ for $i = 1, \ldots, b_1$.

(iii) $y_1 = s_i, y_2 = L \Rightarrow x \in S_i$ and $\omega_1 = 0$ for $i = 1, \ldots, b_1$.

**Lemma 6.** For $y \in Y'$, we have $\psi \circ \sigma_Y(y) = K_\beta \circ \psi(y)$.

We now consider on $Y$ the Markov measure $Q_{\beta, P}$ with transition matrix $P = (p_{i,j})$, given by

$$p_{i,j} = \begin{cases} \lambda(C_i \cap T^{-1}_\beta C_j) / \lambda(C_i) & \text{if } i \in \bigcup_{k=0}^{b_1} M_k, \\ p & \text{if } i \in [0, \ldots, L] \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j = 0, \\ 1 - p & \text{if } i \in [0, \ldots, L] \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j = L, \\ 0 & \text{if } i \in [0, \ldots, L] \setminus \bigcup_{k=0}^{b_1} M_k \text{ and } j \neq 0, L, \end{cases}$$

and initial distribution the corresponding stationary distribution $\pi$.

**Theorem 6.** $Q_{\beta, P} \circ \psi^{-1}$ is a product measure of the form $m_P \otimes \mu$.

**Proof.** Define the measure $\mu$ on $[0, b_1/(\beta - 1)]$ by

$$\mu(B) = \sum_{j=0}^{L} \frac{\lambda(B \cap C_j)}{\lambda(C_j)} \cdot \pi(j) \quad [B \in \mathcal{B}].$$

Define the Markov partition $\mathcal{P}_0$ of $\Omega \times [0, b_1/(\beta - 1)]$ by

$$\mathcal{P}_0 = \left\{ \Omega \times C_j : j \in \bigcup_{k=0}^{b_1} M_k \right\} \cup \{[\omega_1 = i] \times S_j : i = 0, 1, j = 1, \ldots, b_1\},$$

and let $\mathcal{P}_n = \mathcal{P}_0 \vee K_{\beta}^{-1} \mathcal{P}_0 \vee \cdots \vee K_{\beta}^{-n} \mathcal{P}_0$. It is straightforward to see that the inverse images of elements in $\mathcal{P}_n$ under $\psi$ are cylinders in $Y$ and that for each element $P \in \mathcal{P}_n$, $m_P \otimes \mu(P) = Q_{\beta, P} \circ \psi^{-1}(P)$. It follows that $Q_{\beta, P} \circ \psi^{-1} = m_P \otimes \mu$. □
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Since $P$ is an irreducible transition matrix, $σ_Y$ is ergodic with respect to $Q_{β,p}$ and $π(i) > 0$ for all $i \in \{0, \ldots, L\}$. It follows from Lemma 6 that $K_β$ is ergodic with respect to $m_p \otimes µ$. Furthermore, it is immediately seen from the definition that $µ$ is equivalent to $λ$. Hence, the measure $Q_{β,p} \circ ψ^{-1}$ is equivalent to $m_p \otimes λ$.

**Proposition 2.** The map $K_β$ is exact with respect to $m_p \otimes µ_{β,p}$. Moreover, $µ = µ_{β,p}$.

**Proof.** It follows from Lemma 1 and Remarks 1(i) that $µ = µ_{β,p}$. Since the transition matrix $P$ is also aperiodic, $σ_Y$ is exact with respect to $Q_{β,p}$. It follows from Lemma 6 that $K_β$ is exact with respect to $m_p \otimes µ_{β,p}$. $\Box$

It also follows from the above proposition that the density $1^*$ assumes the constant value $π(j)/λ(C_j)$ on the interval $C_j$, $j \in \{0, \ldots, L\}$.

**Example 1.** Let $β = G = \frac{1}{2}(1 + \sqrt{5})$ and let $g = G - 1 = \frac{1}{2}(\sqrt{5} - 1)$. Note that $1 = 1/β + 1/β^2$. In this case, we let $C = E$, since $1$ and $1/(β - 1) - 1$ are already endpoints of intervals in $E$. Using the techniques in this section it is easily verified that the dynamical system $(Ω \times J_β, A \otimes B, m_p \otimes µ_{β,p}, K_β)$ is measurably isomorphic to the Markov chain with transition matrix $P$, given by

$$P = \begin{pmatrix} g & g^2 & 0 & 1 - p \\ p & 0 & 1 \end{pmatrix},$$

and stationary distribution $π$ determined by $πP = π$.

It remains to prove that $Q_{β,p} \circ ψ^{-1}$ and $ν_β$ are mutually singular. Since $K_β$ is ergodic with respect to both measures, it suffices to show that the measures do not coincide.

**Lemma 7.** $ν_β \neq Q_{β,p} \circ ψ^{-1}$.

**Proof.** We distinguish between the cases $p = 1/2$ and $p \neq 1/2$.

Suppose $p = 1/2$. On the one hand, for all $i \in \{1, \ldots, [β]\}$ we have

$$\frac{i}{β} + \sum_{i=2}^{∞} \frac{d_i}{β^i} \in S_i \Leftrightarrow \sum_{i=1}^{∞} \frac{d_{i+1}}{β^i} \in C_0,$$

$$\frac{i - 1}{β} + \sum_{i=2}^{∞} \frac{d_i}{β^i} \in S_i \Leftrightarrow \sum_{i=1}^{∞} \frac{d_{i+1}}{β^i} \in C_L.$$

Using the fact that the digits $(d_i)_{i=1}^{∞}$ form a uniform Bernoulli process under $ν_β$, a simple calculation yields

$$ν_β(Ω \times S) = \frac{[β]}{[β] + 1} : ν_β(Ω \times C_0) + \frac{[β]}{[β] + 1} : ν_β(Ω \times C_L).$$

Since $ν_β(Ω \times C_0) = ν_β(Ω \times C_L)$, it follows that

$$\frac{ν_β(Ω \times S)}{ν_β(Ω \times C_0)} = \frac{2[β]}{[β] + 1}.$$
On the other hand, it follows from $\pi P = \pi$ that

$$\pi(0) = \frac{1}{\beta} \pi(0) + \frac{1}{2} (\pi(s_1) + \cdots + \pi(s_{b_1})).$$

Rewriting one gets

$$\frac{\pi(s_1) + \cdots + \pi(s_{b_1})}{\pi(0)} = \frac{Q_{\beta,p} \circ \psi^{-1}(\Omega \times S)}{Q_{\beta,p} \circ \psi^{-1}(\Omega \times C_0)} = \frac{2(\beta - 1)}{\beta}.$$  

However,

$$\frac{2(\beta - 1)}{\beta} \neq \frac{2[\beta]}{[\beta] + 1}$$

for all non-integer $\beta$, in particular for the $\beta$'s under consideration.

Suppose $p \neq 1/2$. In this case, the assertion follows from the fact that the projection of $\nu_{\beta}$ on the first coordinate is the uniform Bernoulli measure on $\{0, 1\}^\mathbb{N}$ [DdV Theorem 8]. Note that this result is applicable because 1 has a finite greedy expansion with positive coefficients.

The mutual singularity of $\nu_{\beta}$ and $m_{p} \otimes \lambda$ follows as before.

5. Open problems and final remarks

1. We have not been able to find an explicit formula for $1^*$. Recall that the Parry density $h_{\beta} = P_{T_{\beta}} h_{\beta}$ is given by

$$h_{\beta}(x) = \frac{1}{F(\beta)} \sum_{x \in T_{\beta}(1)} \frac{1}{\beta^x}$$

(see Section 1). We expect that the density $1^*$ can be expressed in a similar way, but now the random orbits of 1 as well as the random orbits of the complementary point $[\beta]/(\beta - 1) - 1$ are involved. Let us consider an example.

**Example 2.** Let $p = 1/2$ and $\beta = 3/2$. Note that in this case $[\beta]/(\beta - 1) - 1 = 1$.

Rewriting (2) one gets

$$P_{T_{\beta}} f(x) = \frac{1}{\beta} \sum_{i=0}^{1} f\left(\frac{x + i}{\beta}\right) \cdot 1_{[0,1]}(x) + \frac{1}{\beta} f\left(\frac{x + 1}{\beta}\right) \cdot 1_{[1,2]}(x),$$

$$P_{L_{\beta}} f(x) = \frac{1}{\beta} f\left(\frac{x}{\beta}\right) \cdot 1_{[0,1]}(x) + \frac{1}{\beta} \sum_{i=0}^{1} f\left(\frac{x + i}{\beta}\right) \cdot 1_{[1,2]}(x).$$

It is easy to verify that $1 \in \mathcal{D}$ satisfies $P1 = 1$, hence $1^* = 1$. It follows that $m_{1/2} \otimes \lambda$ is $K_{3/2}$-invariant.
2. We have not been able to give an explicit formula for \( h_{mp \otimes \mu_{\beta}, p}(K_\beta) \). However, in the special case that \( \beta^2 = b_1 \beta + 1 \), the entropy is already calculated in [DK2]:

\[
h_{mp \otimes \mu_{\beta}, p}(K_\beta) = \log \beta - \frac{b_1}{1 + \beta^2}(p \log p + (1 - p) \log(1 - p)).
\]

Since in this case \( \pi(s_i) = \frac{1}{1 + \beta^2}, i = 1, \ldots, b_1 \), it follows that

\[
h_{mp \otimes \mu_{\beta}, p}(K_\beta) = \log \beta - \mu_{\beta, p}(S)(p \log p + (1 - p) \log(1 - p)).
\]

One might conjecture that this formula holds in general.

3. Fix \( p \in (0, 1) \). It is a direct consequence of Birkhoff’s ergodic theorem, Theorem 4 and the ergodicity of \( K_\beta \) with respect to \( m_p \otimes \mu_{\beta, p} \) that for \( m_p \otimes \lambda \)-a.e. \((\omega, x) \in \Omega \times J_\beta\),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(K_\beta^i(\omega, x)) = \mu_{\beta, p}(S) > 0.
\]  

(8)

In particular, we infer from (8) that the set

\[ G = \{ x \in J_\beta : x \text{ has a unique expansion in base } \beta \} \]

has Lebesgue measure zero, since \( K_\beta^n(\omega, x) \in \Omega \times E \) for all \((\omega, x) \in \Omega \times G\) and all \( n \geq 0 \). Let \( T_0 = L_\beta, T_1 = T_\beta\), and let

\[ N = \bigcup_{n=1}^{\infty} \{ x \in J_\beta : T_{u_1} \circ \cdots \circ T_{u_n} x \in G \text{ for some } u_1, \ldots, u_n \in \{0, 1\} \}. \]

Since the greedy map and the lazy map are nonsingular, \( \lambda(N) = 0 \). Note that \( \Omega \times J_\beta \setminus N \subset Z \) and that for \( x \in J_\beta \setminus N \), different elements of \( \Omega \) give rise to different expansions of \( x \) in base \( \beta \). We conclude that for \( \lambda \)-a.e. \( x \in J_\beta \), there exist \( 2^{b_0} \) expansions of \( x \) in base \( \beta \).

For a more elementary proof of this fact in case \( \beta \in (1, 2) \), we refer to [S1].

4. Erdős and Komornik introduced in [EK] the notion of universal expansions. They called an expansion \((d_1, d_2, \ldots)\) in base \( \beta \) of some \( x \in J_\beta \) **universal** if for each (finite) block \( b_1 \ldots b_n \), consisting of digits in the set \([0, \ldots, [\beta]]\), there exists an index \( k \geq 1 \) such that \( d_k \ldots d_{k+n-1} = b_1 \ldots b_n \). They proved that there exists a number \( \beta_0 \in (1, 2) \) such that for each \( \beta \in (1, \beta_0) \), every \( x \in (0, 1/(\beta - 1)) \) has a universal expansion in base \( \beta \).

Subsequently, Sidorov proved in [S2] that for a given \( \beta \in (1, 2) \) and for \( \lambda \)-a.e. \( x \in J_\beta \), there exists a universal expansion of \( x \) in base \( \beta \). We now strengthen his result and the conclusion of the preceding remark by the following theorem.

**Theorem 7.** For any non-integer \( \beta > 1 \), and for \( \lambda \)-a.e. \( x \in J_\beta \), there exist \( 2^{b_0} \) universal expansions of \( x \) in base \( \beta \).

In order to prove Theorem 7 we need the following lemma.
Lemma 8. Let $\beta > 1$ be a non-integer and let $p \in (0, 1)$. Then, for $n \geq 1$ and $i_1, \ldots, i_n \in \{0, \ldots, [\beta]\}$, we have

$$m_p \otimes \mu_{\beta,p}(\{d_1 = i_1, \ldots, d_n = i_n\}) > 0.$$  

Proof. By Theorem [4], it suffices to show that

$$m_p \otimes \lambda(\{d_1 = i_1, \ldots, d_n = i_n\}) > 0.$$  

It is easy to verify that there exists a sequence $(j_1, j_2, \ldots) \in D$, starting with $i_1 \ldots i_n$, such that the numbers $x_1, \ldots, x_n$, given by

$$x_r = \sum_{i=1}^{\infty} \frac{j_{i+r-1}}{\beta^i}, \quad r = 1, \ldots, n,$$

are elements of $J_\beta \setminus \partial(S)$, where $\partial(S)$ denotes the boundary of $S$. For $m \geq 1$, consider the set

$$I_m = \left[ \sum_{i=1}^{n+m} \frac{i_1}{\beta^i}, \sum_{i=1}^{n+m} \frac{i_2}{\beta^i} + \sum_{i=n+m+1}^{\infty} \frac{\lfloor \beta \rfloor}{\beta^i} \right].$$

Let $y \in I_m$ and let $(a_1, a_2, \ldots)$ be an expansion of $y$, starting with $j_1 \ldots j_{n+m}$. Define

$$y_r = \sum_{i=1}^{\infty} \frac{d_{i+r-1}}{\beta^i}, \quad r = 1, \ldots, n.$$

Choose $m$ large enough, so that for each $r = 1, \ldots, n$, $x_r$ and $y_r$ are elements of the same equal or switch region, regardless of the values of the digits $a_\ell, \ell > n + m$, and hence regardless of the chosen element $y \in I_m$. Note that this is possible because $x_r \notin \partial(S)$ for $r = 1, \ldots, n$. Denote the set of indices $r \in \{1, \ldots, n\}$ for which $x_r \in S$ by $\{r_1, \ldots, r_\ell\}$. Then, for suitably chosen $u_1, \ldots, u_\ell \in [0, 1]$, we have

$$[\omega_1 = u_1, \ldots, \omega_\ell = u_\ell] \times I_m \subset \{d_1 = i_1, \ldots, d_n = i_n\}$$

and the conclusion follows. \qed

Proof of Theorem [7]. Fix $p \in (0, 1)$ and let $b_1 \ldots b_n$ be an arbitrary block. Using Birkhoff’s ergodic theorem, Theorem [4], Lemma [8] and the ergodicity of $K_\beta$ with respect to $m_p \otimes \mu_{\beta,p}$, we may conclude that for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$, the block $b_1 \ldots b_n$ occurs in

$$(d_1(\omega, x), d_2(\omega, x), \ldots)$$

with positive limiting frequency $m_p \otimes \mu_{\beta,p}(\{d_1 = b_1, \ldots, d_n = b_n\})$. In particular, for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$, the block $b_1 \ldots b_n$ occurs in [9]. Since there are only countably many blocks, we deduce that for $m_p \otimes \lambda$-a.e. $(\omega, x) \in \Omega \times J_\beta$, the expansion [9] is universal in base $\beta$. An application of Fubini’s theorem shows that there exists a Borel set $B \subset J_\beta \setminus N$ of full Lebesgue measure and there exist sets $A_x \in A$ with $m_p(A_x) = 1 (x \in B)$ such that for all $x \in B$ and $(\omega, x) \in A_x \times \{x\}$, the expansion [9] is universal in base $\beta$. Since the sets $A_x$ have necessarily the cardinality of the continuum and since different elements of $\Omega$ give rise to different expansions of $x$ in base $\beta$ for any $x \in J_\beta \setminus N$, the assertion follows. \qed
5. An expansion \((a_1, a_2, \ldots)\) in base \(\beta\) of some number \(x \in \mathbb{J}_\beta\) is called normal if each block \(i_1 \ldots i_n\) with digits in \([0, \ldots, \lfloor\beta\rfloor]\) occurs in \((a_1, a_2, \ldots)\) with limiting frequency \((\lfloor\beta\rfloor + 1)^{-n}\). Note that a normal expansion is in particular universal.

Fix \(p \in (0, 1)\). Since \(\nu_\beta \neq m_p \otimes \mu_{\beta,p}\) and since both measures \(\nu_\beta\) and \(m_p \otimes \mu_{\beta,p}\) are concentrated on \(\mathbb{Z}\), there exists a block \(i_1 \ldots i_n\) such that

\[
m_p \otimes \mu_{\beta,p}(\{d_1 = i_1, \ldots, d_n = i_n\}) \neq (\lfloor\beta\rfloor + 1)^{-n}.
\]

Hence, for \(m_p \otimes \lambda\text{-a.e. } (\omega, x) \in \Omega \times \mathbb{J}_\beta\), the expansion (9) is universal but not normal.

On the other hand, Sidorov proved in [S2] that there exists a Borel set \(V \subset (1, 2)\) of full Lebesgue measure such that for each \(\beta \in V\) and for \(\lambda\text{-a.e. } x \in \mathbb{J}_\beta\), there exists a normal expansion of \(x\) in base \(\beta\).

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References


