
Abstract. — The existence of at least two nonnegative smooth solutions to a homogeneous Dirichlet problem with \(p\)-Laplacian and reaction \((p / 2 \cdot 1)\)-linear, but asymmetric, at \(\pm \infty\) is investigated through variational and truncation techniques. The case \(p = 2\) is separately examined, obtaining a third nontrivial smooth solution via Morse’s theory.

Key words: \(p\)-Laplacian, asymmetric nonlinearity, critical groups, Morse identity.

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1. Introduction

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^N\), \(N \geq 3\), with a smooth boundary \(\partial \Omega\), let \(1 < p < +\infty\), and let \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) be a Carathéodory function such that \(f(x, 0) = 0\). Consider the homogeneous Dirichlet problem

\[
\begin{cases}
-\Delta_p u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(\Delta_p\) denotes the \(p\)-Laplace differential operator, namely \(\Delta_p u := \text{div}([|\nabla u|^{p-2}\nabla u])\). As usual, a function \(u \in W^{1,p}_0(\Omega)\) is called a (weak) solution to (1.1) provided

\[
\int_{\Omega} |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x, u(x))v(x) \, dx \quad \forall v \in W^{1,p}_0(\Omega).
\]

The literature concerning (1.1) is by now very wide and many existence or multiplicity results are already available. Quite popular hypotheses are the following:

\[
\lim_{|t| \to +\infty} \frac{f(x, t)}{|t|^{p-2}t} = \alpha \quad \text{uniformly in } x \in \Omega,
\]

\[
\lim_{t \to 0} \frac{f(x, t)}{|t|^{p-2}t} = \beta \neq 0 \quad \text{uniformly in } x \in \Omega.
\]

If \(\alpha \in \mathbb{R}\setminus\{0\}\) then one usually says that \(t \mapsto f(x, t)\) exhibits a symmetric \((p - 1)\)-linear growth at infinity; see [3, 17] and the references therein. The recent paper
[8] treats the case \( \alpha \leq 0 \) and \( f(x, t) := \lambda g(x, t) \) with \( \lambda > 0 \) large enough, while \( \alpha = +\infty \) in [9, 10, 11].

Let \( \lambda_1 \) (respectively, \( \lambda_2 \)) be the first (respectively, second) eigenvalue of the operator \(-\Delta_p\) in \( W_0^{1,p}(\Omega) \). Roughly speaking, in this paper, we shall consider a reaction term \( f \) whose behavior is \((p-1)\)-linear, but asymmetric, near \(-\infty\) and \(+\infty\), in the sense that the graph of the function \( t \mapsto \frac{f(x, t)}{|t|^{p-2}t} \) crosses \( \lambda_1 \) as \( t \) moves from \(-\infty\) to \(+\infty\). Such an \( f \) is usually called crossing or jumping nonlinearity.

The existence of two solutions to (1.1) lying in \( C_0^1(\overline{\Omega}) \setminus \{0\} \) is established via variational and truncation methods; see Theorem 3.3. Section 4 investigates the case \( p = 2 \). A third nontrivial \( C_0^1(\overline{\Omega}) \)-solution is obtained through Morse’s theory.

Equations with \( p \)-Laplacian and \((p-1)\)-linear asymmetric reactions have previously been studied by mainly using the so-called Fučík spectrum of \(-\Delta_p\) in \( W_0^{1,p}(\Omega) \); see [16], besides the seminal work [1]. This approach depends on the knowledge of the Fučík spectrum and requires that the limit (1.2) exists.

Our arguments are patterned after those of [6] (cf. also [15]) where, however, a further sign condition on \( f \) is taken on and the semi-linear case is not separately treated. Accordingly, (1.2)–(1.3) become here

\[
\limsup_{t \to +\infty} \frac{f(x, t)}{|t|^{p-2}t} \leq a_1 < \lambda_1 < a_2 \leq \liminf_{t \to -\infty} \frac{f(x, t)}{|t|^{p-2}t} \leq \limsup_{t \to -\infty} \frac{f(x, t)}{|t|^{p-2}t} \leq b_2
\]

and

\[
\lambda_2 < a_3 \leq \liminf_{t \to 0} \frac{f(x, t)}{|t|^{p-2}t} \leq \limsup_{t \to 0} \frac{f(x, t)}{|t|^{p-2}t} \leq b_3
\]

uniformly in \( x \in \Omega \), with \( a_i, b_j \) being nonnegative constants. It should be noted that none of limits (1.2)–(1.3) needs to exist. Moreover, (1.2) and (1.4) are mutually independent, whereas (1.3) forces (1.5) as soon as \( \lambda_2 < \beta < +\infty \).

2. Preliminaries

Let \( (X, \| \cdot \|) \) be a real Banach space. Given a set \( V \subseteq X \), write \( \overline{V} \) for the closure of \( V \), \( \partial V \) for the boundary of \( V \), and \( \text{int}(V) \) for the interior of \( V \). If \( x \in X \) and \( \delta > 0 \) then

\[
B_\delta(x) := \{ z \in X : \| z - x \| < \delta \}.
\]

The symbol \( (X^*, \| \cdot \|_{X^*}) \) denotes the dual space of \( X \), \( \langle \cdot, \cdot \rangle \) indicates the duality pairing between \( X \) and \( X^* \), while \( x_n \to x \) (respectively, \( x_n \rightharpoonup x \)) in \( X \) means “the sequence \( \{x_n\} \) converges strongly (respectively, weakly) in \( X^* \).

Let \( T \) be a topological space and let \( L \) be a multifunction from \( T \) into \( X \) (briefly, \( L : T \to 2^X \)), namely a function which assigns to each \( t \in T \) a nonempty subset \( L(t) \) of \( X \). We say that \( L \) is lower semi-continuous when \( \{ t \in T : L(t) \cap V = \emptyset \} \) turns out to be open in \( T \) for every open set \( V \subseteq X \). A function \( l : T \to X \) is called a selection of \( L \) provided \( l(t) \in L(t) \) for all \( t \in T \).
We say that $\Phi : X \to \mathbb{R}$ is coercive when
\[
\lim_{\|x\| \to +\infty} \Phi(x) = +\infty.
\]
The function $\Phi$ is called weakly sequentially lower semi-continuous if $x_n \to x$ in $X$ implies $\Phi(x) \leq \liminf_{n \to \infty} \Phi(x_n)$. Let $\Phi \in C^1(X)$. The classical Cerami compactness condition for $\Phi$ reads as follows.

(C) Every sequence $\{x_n\} \subseteq X$ such that $\{\Phi(x_n)\}$ is bounded and
\[
\lim_{n \to +\infty} (1 + \|x_n\|)\|\Phi'(x_n)\|_{X^*} = 0
\]
possesses a convergent subsequence.

Define, provided $c \in \mathbb{R}$,
\[
\Phi^c := \{x \in X : \Phi(x) \leq c\}, \quad K_c(\Phi) := K(\Phi) \cap \Phi^{-1}(c),
\]
where, as usual, $K(\Phi)$ denotes the critical set of $\Phi$, i.e., $K(\Phi) := \{x \in X : \Phi'(x) = 0\}$. Given a topological pair $(A, B)$ fulfilling $B \subset A \subseteq X$, the symbol $H_q(A, B)$, $q \in \mathbb{N}_0$, indicates the $q$th-relative singular homology group of $(A, B)$ with integer coefficients. Let $x_0 \in K_c(\Phi)$ be an isolated point of $K(\Phi)$. Then
\[
C_q(\Phi, x_0) := H_q(\Phi^c \cap V, \Phi^c \cap V \setminus \{x_0\}), \quad q \in \mathbb{N}_0,
\]
are the critical groups of $\Phi$ at $x_0$. Here, $V$ stands for any neighborhood of $x_0$ such that $K(\Phi) \cap \Phi^c \cap V = \{x_0\}$. By excision, this definition does not depend on the choice of $V$. Suppose $\Phi$ satisfies condition (C). When $\Phi|_{K(\Phi)}$ is bounded below and $c < \inf_{x \in K(\Phi)} \Phi(x)$, we define
\[
C_q(\Phi, \infty) := H_q(X, \Phi^c), \quad q \in \mathbb{N}_0.
\]
The second deformation lemma [4, Theorem 5.1.33] implies that this definition does not depend on the choice of $c$. If $K(\Phi)$ is finite then, setting
\[
M(t, x) := \sum_{q=0}^{+\infty} \text{rank} \ C_q(\Phi, x)t^q, \quad \forall (t, x) \in \mathbb{R} \times K(\Phi),
\]
\[
P(t, \infty) := \sum_{q=0}^{+\infty} \text{rank} \ C_q(\Phi, \infty)t^q
\]
the following Morse relation holds:
\[
(2.1) \quad \sum_{x \in K(\Phi)} M(t, x) = P(t, \infty) + (1 + t)Q(t),
\]
where $Q(t)$ denotes a formal series with nonnegative integer coefficients; see for instance [14, Theorem 6.62].

Now, let $X$ be a Hilbert space, let $x \in K(\Phi)$, and let $\Phi$ be $C^2$ in a neighborhood of $x$. If $\Phi''(x)$ turns out to be invertible then $x$ is called non-degenerate. The Morse index $d$ of $x$ is the supremum of the dimensions of the vector subspaces of $X$ on which $\Phi''(x)$ turns out to be negative definite. When $x$ is non-degenerate and with Morse index $d$ one has

$$C_q(\Phi, x) = \delta_{q,d} \mathbb{Z}, \quad q \in \mathbb{N}_0.$$  

The monographs [12, 14] represent general references on the subject.

Throughout the paper, $\Omega$ is a bounded domain of the real euclidean $N$-space $(\mathbb{R}^N, | \cdot |)$ with a smooth boundary $\partial \Omega$, $m$ stands for the Lebesgue measure, $p \in (1, +\infty)$, $p' := p/(p-1)$, $\| \cdot \|_{L^p(\Omega)}$ with $q \geq 1$ indicates the usual norm of $L^q(\Omega)$, and $W^{1,p}_0(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. On $W^{1,p}_0(\Omega)$ we introduce the norm

$$\|u\|_{1,p} := \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}, \quad u \in W^{1,p}_0(\Omega).$$

Write $p^*$ for the critical exponent of the Sobolev embedding $W^{1,p}_0(\Omega) \subset L^q(\Omega)$. Recall that $p^* = Np/(N-p)$ if $p < N$, $p^* = +\infty$ otherwise, and the embedding is compact whenever $1 \leq q < p^*$.

Define $C^1_0(\overline{\Omega}) := \{ u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial \Omega \}$. Obviously, $C^1_0(\overline{\Omega})$ turns out to be an ordered Banach space with positive cone

$$C^1_0(\overline{\Omega})_+ := \{ u \in C^1_0(\overline{\Omega}) : u(x) \geq 0 \ \forall x \in \overline{\Omega} \}.$$  

Moreover, one has

$$\text{int}(C^1_0(\overline{\Omega})_+) = \left\{ u \in C^1_0(\overline{\Omega}) : u > 0 \text{ in } \Omega, \frac{\partial u}{\partial n} < 0 \text{ on } \partial \Omega \right\},$$

where $n(x)$ is the outward unit normal vector to $\partial \Omega$ at the point $x \in \partial \Omega$; see, for example, [4, Remark 6.2.10].

Let $W^{-1,p}_0(\Omega)$ be the dual space of $W^{1,p}_0(\Omega)$ and let $A_p : W^{1,p}_0(\Omega) \to W^{-1,p}_0(\Omega)$ be the nonlinear operator stemming from the negative $p$-Laplacian, i.e.,

$$\langle A_p(u), v \rangle := \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) \, dx \quad \forall u, v \in W^{1,p}_0(\Omega).$$

The Liusternik-Schnirelman theory gives a strictly increasing sequence $\{ \lambda_n \}$ of eigenvalues for the operator $-\Delta_p$ in $W^{1,p}_0(\Omega)$. The following assertions involving $\lambda_1$, $\lambda_2$, and $A_p$ can be found in [4, Section 6.2]; see also [14, Sections 9.1–9.2].
(p₁) $0 < \lambda_1 < \lambda_2$.

(p₂) $\|u\|_{L^p(\Omega)} \leq \frac{1}{\lambda_1} \|u\|_{L^p(\Omega)}$ for all $u \in W_0^{1,p}(\Omega)$.

(p₃) There exists an eigenfunction $\phi_1$ corresponding to $\lambda_1$ such that $\phi_1 \in \text{int}(C_0^1(\Omega))$ as well as $\|\phi_1\|_{L^p(\Omega)} = 1$.

(p₄) If $U := \{u \in W_0^{1,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1\}$ and

$$\Gamma_0 := \{\gamma \in C^0([-1, 1], U) : \gamma(-1) = -\phi_1, \gamma(1) = \phi_1\},$$

then

$$\lambda_2 = \inf_{\gamma \in \Gamma_0} \max_{u \in U([-1, 1])} \|u\|_{L^p}^p.$$

(p₅) $u_n \to u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n \to +\infty} \langle A_p(u_n), u_n - u \rangle \leq 0$ imply $u_n \to u$ in $W_0^{1,p}(\Omega)$.

Let $\alpha \in L^\infty(\Omega) \setminus \{0\}$ satisfy $\alpha \geq 0$. Consider the weighted eigenvalue problem

$$-\Delta_p u = \lambda \alpha(x) |u|^{p-2} u \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega.$$  

As before, there exists a strictly increasing sequence $\{\lambda_n(x)\}$ of eigenvalues for (2.4) enjoying the properties [4, Section 6.2]:

(p₆) $0 < \lambda_1(x) < \lambda_2(x)$.

(p₇) If $\alpha, \beta \in L^\infty(\Omega) \setminus \{0\},$ $0 \leq \alpha \leq \beta,$ and $\alpha \neq \beta$ then $\lambda_1(\beta) < \lambda_1(\alpha)$. If $0 \leq \alpha < \beta$ then $\lambda_2(\beta) < \lambda_2(\alpha)$.

Obviously, $\lambda_n = \lambda_n(1)$, $n \in \mathbb{N}$. Now, suppose $p = 2$ and denote by $E(\lambda_n)$ the eigenspace associated with $\lambda_n$. It is known (see e.g. [4, Section 6.2]) that:

(p₈) $E(\lambda_n) \subseteq C_0^1(\overline{\Omega})$ for all $n \in \mathbb{N}$.

(p₉) If $u$ lies in $E(\lambda_n)$ and vanishes on a set of positive Lebesgue measure then $u = 0$.

Setting, for every integer $m \geq 1$, $H_m := \bigoplus_{n=1}^{m} E(\lambda_n)$ and $\hat{H}_m := \overline{H}_m^\perp$, we get

$$H_0^1(\Omega) = H_m \oplus \hat{H}_m.$$ 

Consequently, each $u \in H_0^1(\Omega)$ can uniquely be written as $u = \tilde{u} + \hat{u}$, where $\tilde{u} \in H_m$, $\hat{u} \in \hat{H}_m$. A simple argument, based on orthogonality and (p₉), yields the next result.

**Lemma 2.1.** Let $m \in \mathbb{N}$ and let $\theta \in L^\infty(\Omega) \setminus \{\lambda_m\}$ satisfy $\theta \geq \lambda_m$. Then there exists a constant $\tilde{c} > 0$ such that

$$\|\tilde{u}\|_{1,2}^2 - \int_{\Omega} \theta(x)\tilde{u}(x)^2 \, dx \leq -\tilde{c}\|\tilde{u}\|_{1,2}^2 \quad \forall \tilde{u} \in H_m.$$
Let \( m \in \mathbb{N}_0 \) and let \( \theta \in L^\infty(\Omega)\setminus\{\lambda_{m+1}\} \) satisfy \( \theta \leq \lambda_{m+1} \). Then there exists a constant \( \hat{c} > 0 \) such that
\[
\|\hat{u}\|^2_{1,2} - \int_\Omega \theta(x)\hat{u}(x)^2 \, dx \geq \hat{c}\|\hat{u}\|^2_{1,2} \quad \forall \hat{u} \in H_m.
\]

Define \( U_C := \{ u \in C^1_0(\overline{\Omega}) : \|u\|_{L^p(\Omega)} = 1 \} \). Evidently, \( U_C \) turns out to be dense in the set \( U \) given by (p4). Moreover, if \( \Gamma_C := \{ \gamma \in C^0([-1, 1], U_C) : \gamma(-1) = -\phi_1, \gamma(1) = \phi_1 \} \) then the following result holds.

**Lemma 2.2.** The set \( \Gamma_C \) is dense in \( \Gamma_0 \).

**Proof.** Pick any \( \gamma_0 \in \Gamma_0 \). We shall prove that there exists a sequence \( \{\gamma_n\} \subseteq \Gamma_C \) fulfilling
\[
\lim_{n \to +\infty} \max_{t \in [-1, 1]} \|\gamma_n(t) - \gamma_0(t)\| = 0.
\]

The multifunction \( L_n : [-1, 1] \to 2C^1_0(\overline{\Omega}) \) defined by
\[
L_n(t) := \begin{cases} 
\{-\phi_1\} & \text{if } t = -1, \\
\{u \in C^1_0(\overline{\Omega}) : \|u - \gamma_0(t)\| < 1/n\} & \text{if } t \in (-1, 1), \\
\{\phi_1\} & \text{if } t = 1
\end{cases}
\]
takes nonempty convex values and is lower semi-continuous. So, Theorem 3.1 in [13] provides a continuous selection \( l_n : [-1, 1] \to C^1_0(\overline{\Omega}) \) of \( L_n \). This entails
\[
\|l_n(t) - \gamma_0(t)\| < \frac{1}{n} \quad \forall t \in (-1, 1), \quad l_n(-1) = -\phi_1, \quad l_n(1) = \phi_1.
\]

Consequently,
\[
\lim_{n \to +\infty} \|l_n(t)\|_{L^p(\Omega)} = \|\gamma_0(t)\|_{L^p(\Omega)} = 1
\]
uniformly with respect to \( t \in [-1, 1] \). For any \( n \) large enough we can thus set
\[
\gamma_n(t) := \frac{l_n(t)}{\|l_n(t)\|_{L^p(\Omega)}}, \quad t \in [-1, 1].
\]

On account of (2.6) and (p3) one has \( \gamma_n \in \Gamma_C \). Moreover, thanks to (2.6),
\[
\|\gamma_n(t) - \gamma_0(t)\| \leq \|\gamma_n(t) - l_n(t)\| + \|l_n(t) - \gamma_0(t)\| < |1 - \|l_n(t)\|_{L^p(\Omega)}| \frac{\|l_n(t)\|}{\|l_n(t)\|_{L^p(\Omega)}} + \frac{1}{n} \quad \forall t \in [-1, 1].
\]
Recall that $\gamma_0 \in G_0$. Since, by (2.6) again,
\[
\max_{t \in [-1, 1]} |1 - \|l_n(t)\|_{L^p(\Omega)}| = \max_{t \in [-1, 1]} \|\gamma_0(t)\|_{L^p(\Omega)} - \|l_n(t)\|_{L^p(\Omega)}
\leq \max_{t \in [-1, 1]} \|\gamma_0(t) - l_n(t)\|_{L^p(\Omega)}
\leq c \max_{t \in [-1, 1]} \|\gamma_0(t) - l_n(t)\| \leq \frac{c}{n}
\]
for some $c > 0$, (2.5) immediately follows from (2.6)–(2.8). \hfill \Box

Finally, put, provided $t \in \mathbb{R}$, $u : \Omega \to \mathbb{R}$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$,
\[
t^- := \max\{-t, 0\}, \quad t^+ := \max\{t, 0\},
\]
\[
u^-(x) := u(x)^-, \quad u^+(x) := u(x)^+, \quad N_g(u)(x) := g(x, u(x)).
\]

### 3. Existence results

To avoid unnecessary technicalities, ‘for every $x \in \Omega$’ will take the place of ‘for almost every $x \in \Omega$’ and the variable $x$ will be omitted when no confusion can arise.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(x, 0) \equiv 0$ and let
\begin{equation}
F(x, z) := \int_0^z f(x, t) \, dt, \quad (x, z) \in \Omega \times \mathbb{R}.
\end{equation}
We will posit the following assumptions, where $a_i$ and $b_i$ denote appropriate nonnegative constants.

\begin{enumerate}
\item[(f_0)] $|f(x, t)| \leq a_0(1 + |t|^{p-1})$ for every $(x, t) \in \Omega \times \mathbb{R}$.
\item[(f_1)] $\limsup_{t \to +\infty} \frac{f(x, t)}{t^{p-1}} \leq a_1 < \lambda_1$ uniformly with respect to $x \in \Omega$.
\item[(f_2)] $\lambda_1 < a_2 \leq \limsup_{t \to -\infty} \frac{f(x, t)}{|t|^{p-2} t} \leq \limsup_{t \to -\infty} \frac{f(x, t)}{|t|^{p-2} t} \leq b_2$ uniformly in $x \in \Omega$.
\item[(f_3)] $\lambda_2 < a_3 \leq \liminf_{t \to 0} \frac{f(x, t)}{|t|^{p-2} t} \leq \liminf_{t \to 0} \frac{f(x, t)}{|t|^{p-2} t} \leq b_3$ uniformly with respect to $x \in \Omega$.
\item[(f_4)] There exists $a_4 > \lambda_1$ such that $\frac{a_4}{p} |z|^p \leq F(x, z)$ for all $(x, z) \in \Omega \times \mathbb{R}^-$.
\end{enumerate}

On account of (f_0) and (f_3), to every $\rho > 0$ there corresponds $\mu_\rho > 0$ satisfying
\begin{equation}
f(x, t) + \mu_\rho t^{p-1} \geq 0, \quad (x, t) \in \Omega \times [0, \rho].
\end{equation}

**Remark 3.1.** The constants that appear in (f_0)–(f_4) can evidently be replaced by suitable functions belonging to $L^\infty(\Omega)$. In particular, we might have $a_1, a_2, a_4 \in L^\infty(\Omega) \setminus \{\lambda_1\}$ with $0 \leq a_1 \leq \lambda_1 \leq \min\{a_2, a_4\}$. 

Write $X := W_{0}^{1,p}(\Omega)$ and $C_{+} := C_{0}^{1}(\Omega)_{+}$. The energy functional $\varphi : X \to \mathbb{R}$ stemming from Problem (1.1) is

\begin{equation}
\varphi(u) := \frac{1}{p} \| u \|_{1,p}^{p} - \int_{\Omega} F(x, u(x)) \, dx \quad \forall u \in X,
\end{equation}

with $F$ as in (3.1). Obviously, $\varphi \in C^{1}(X)$. Moreover, if

$$f_{+}(x, t) := f(x, t^{+}), \quad F_{+}(x, z) := \int_{0}^{z} f_{+}(x, t) \, dt$$

then $F_{+}(x, z) = F(x, z^{+})$ and the corresponding truncated function

$$\varphi_{+}(u) := \frac{1}{p} \| u \|_{1,p}^{p} - \int_{\Omega} F_{+}(x, u(x)) \, dx, \quad u \in X,$$

turns out to be $C^{1}$ as well.

**Lemma 3.1.** Under hypotheses $(f_{0})-(f_{1})$, the functional $\varphi_{+}$ is weakly sequentially lower semi-continuous and coercive.

**Proof.** The space $X$ compactly embeds in $L^{p}(\Omega)$ while the Nemitskii operator $N_{f_{+}}$ turns out to be continuous on $L^{p}(\Omega)$. Thus, a standard argument ensures that $\varphi_{+}$ is weakly sequentially lower semi-continuous.

Pick $\varepsilon \in (0, \lambda_{1} - a_{1})$. By $(f_{0})-(f_{1})$ there exists $c_{0} > 0$ fulfilling

$$F(x, z) < \frac{a_{1} + \varepsilon}{p} z^{p} + c_{0} \quad \forall (x, z) \in \Omega \times \mathbb{R}_{0}^{+}.$$ 

Consequently, on account of (p$_{2}$),

$$\varphi_{+}(u) \geq \frac{1}{p} \| u \|_{1,p}^{p} - (a_{1} + \varepsilon) \| u^{+} \|_{L^{p}(\Omega)}^{p} - c_{0} m(\Omega)$$

$$\geq \frac{1}{p} \| u \|_{1,p}^{p} - (a_{1} + \varepsilon) \| u \|_{L^{p}(\Omega)}^{p} - c_{0} m(\Omega)$$

$$\geq \frac{1}{p} \left( 1 - \frac{a_{1} + \varepsilon}{\lambda_{1}} \right) \| u \|_{1,p}^{p} - c_{0} m(\Omega)$$

for any $u \in X$. Since $a_{1} + \varepsilon < \lambda_{1}$, the conclusion follows. \hfill \Box

**Theorem 3.1.** Let $(f_{0})$, $(f_{1})$, and $(f_{3})$ be satisfied. Then Problem (1.1) admits a solution $u_{0} \in \text{int}(C_{+})$, which is a local minimizer of $\varphi$.

**Proof.** Thanks to Lemma 3.1 we can find $u_{0} \in X$ such that

\begin{equation}
\varphi_{+}(u_{0}) = \inf_{u \in X} \varphi_{+}(u).
\end{equation}

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Bearing in mind \((p_1)\), fix \(\varepsilon \in (0, a_3 - \hat{\lambda}_1)\). By \((f_3)\) one has

\[
(3.5) \quad F(x, z) \geq \frac{a_3 - \varepsilon}{p} |z|^p \quad \text{in } \Omega \times [-\delta, \delta]
\]

for appropriate \(\delta > 0\). If \(t > 0\) is so small that

\[
0 \leq t\phi_1(x) \leq \delta \quad \forall x \in \overline{\Omega},
\]

where \(\phi_1\) comes from \((p_3)\), then \((3.5)\) yields

\[
(3.6) \quad \varphi_+(t\phi_1) \leq \frac{t^p}{p} [\varepsilon - (a_3 - \hat{\lambda}_1)] \|\phi_1\|_{L^p(\Omega)}^p < 0.
\]

Hence,

\[
(3.7) \quad \varphi_+(u_0) < 0 = \varphi_+(0),
\]

which clearly means \(u_0 \neq 0\). Now, through \((3.4)\) we get \(\varphi'_+(u_0) = 0\), namely

\[
\langle A_p(u_0), v \rangle = \int_\Omega f_+(x, u_0(x))v(x) \, dx, \quad v \in X.
\]

Choosing \(v := -u_0^-\) leads to \(\|u_0^-\|_{L^p(\Omega)}^p = 0\). Thus, \(u_0 \geq 0\) and, a fortiori, the function \(u_0\) solves \((1.1)\). Standard regularity results [5, Theorems 1.5.5–1.5.6] ensure that \(u_0 \in C_+ \setminus \{0\}\). Let \(\rho := \|u_0\|_{L^\infty(\Omega)}\). Due to \((3.2)\) one has

\[
-\Delta_p u_0(x) + \mu_p u_0(x)^{p-1} = f(x, u_0(x)) + \mu_p u_0(x)^{p-1} \geq 0 \quad \text{a.e. in } \Omega.
\]

Therefore, by Theorem 5 in [18], \(u_0 \in \text{int}(C_+)\). This also implies that \(u_0\) is a local \(C_0^1(\overline{\Omega})\)-minimizer of \(\varphi\), because \(\varphi|_{C_+} = \varphi_+|_{C_+}\). Finally, owing to [2, Theorem 1.1], the same holds true with \(C_0^1(\Omega)\) replaced by \(X\).

**Lemma 3.2.** Under hypotheses \((f_0)-(f_2)\), the functional \(\varphi\) fulfills condition \((C)\).

**Proof.** Since \(X\) compactly embeds in \(L^p(\Omega)\), the Nemitskii operator \(N_f\) is continuous on \(L^p(\Omega)\), and \(A_p\) enjoys property \((p_3)\), it suffices to show that every sequence \(\{u_n\} \subseteq X\) satisfying

\[
(3.8) \quad |\varphi(u_n)| \leq c_1 \quad \forall n \in \mathbb{N},
\]

\[
(3.9) \quad \lim_{n \to +\infty} (1 + \|u_n\|_{L^p(\Omega)}) \varphi'(u_n) = 0
\]

turns out to be bounded. Obviously, this happens once the same holds for both \(\{u_n^+\}\) and \(\{u_n^-\}\). We are thus reduced to verifying two claims.

**Claim 1.** The sequence \(\{u_n^+\}\) is bounded in \(X\).
If the assertion were false then, up to subsequences, \( \|u_n^+\|_{1,p} \to +\infty \). Write \( v_n := u_n^+ / \|u_n^+\|_{1,p} \). From \( \|v_n\|_{1,p} \equiv 1 \) it follows, along a subsequence when necessary,

\[
(3.10) \quad v_n \to v \quad \text{in } X, \quad v_n \to v \quad \text{in } L^p(\Omega), \quad v_n \to v \geq 0 \quad \text{a.e. in } \Omega.
\]

Through (3.9) one has \( \langle \varphi'(u_n^+), u_n^+ \rangle \to 0 \), which, dividing by \( \|u_n^+\|_{1,p}^p \), easily entails

\[
(3.11) \quad \|v_n\|_{1,p}^p \leq \varepsilon_n + \int_{\Omega} \frac{f(x, u_n^+(x))}{\|u_n^+\|_{1,p}^{p-1}} v_n(x) \, dx \quad \forall n \in \mathbb{N},
\]

where \( \varepsilon_n \to 0^+ \). Because of \( (f_0) \) the sequence \( \{\|u_n^+\|_{1,p}^{p+1} N_f(u_n^+)\} \subseteq L^{p'}(\Omega) \) is bounded. Via the same reasoning made in [14, pp. 302–303] we thus get a function \( \alpha \in L^\infty(\Omega) \) such that \( 0 \leq \alpha \leq \alpha_1 \) and

\[
\frac{1}{\|u_n^+\|_{1,p}^{p-1}} N_f(u_n^+) \to \alpha v^{p-1} \quad \text{in } L^{p'}(\Omega).
\]

Thanks to (3.10)–(3.11) this produces, as \( n \to +\infty \),

\[
(3.12) \quad \|v\|_{1,p}^p \leq \int_{\Omega} \alpha(x) v(x)^p \, dx \leq \lambda_1 \|v\|_{L^{p}(\Omega)}^p.
\]

Consequently, \( v = t \phi_1 \) for some \( t \geq 0 \). If \( t = 0 \) then, by (3.10)–(3.11) again, \( v_n \to 0 \) in \( X \), which contradicts \( \|v_n\|_{1,p} = 1 \) for all \( n \in \mathbb{N} \). Otherwise, on account of (3.12) and \( (f_1) \),

\[
\|\phi_1\|_{1,p}^p = \frac{1}{t^p} \|v\|_{1,p}^p \leq \frac{1}{t^p} \int_{\Omega} \alpha(x) v(x)^p \, dx < \int_{\Omega} \lambda_1 \phi_1(x)^p \, dx = \lambda_1 \|\phi_1\|_{L^{p}(\Omega)}^p,
\]

but this is impossible; cf. \( (p_3) \).

**Claim 2.** The sequence \( \{u_n^-\} \) is bounded in \( X \).

If the assertion were false then, up to subsequences, \( \|u_n^-\|_{1,p} \to +\infty \). Write, like before, \( w_n := u_n^- / \|u_n^-\|_{1,p} \). From \( \|w_n\|_{1,p} \equiv 1 \) it follows, along a subsequence when necessary,

\[
(3.13) \quad w_n \to w \quad \text{in } X, \quad w_n \to w \quad \text{in } L^p(\Omega), \quad w_n \to w \geq 0 \quad \text{a.e. in } \Omega.
\]

Through (3.9) one has

\[
(3.14) \quad \left| \langle A_p(u_n), v \rangle - \int_{\Omega} f(x, u_n(x)) v(x) \, dx \right| \leq \varepsilon_n \|v\|_{1,p} \quad \forall v \in X,
\]

where \( \varepsilon_n \to 0^+ \). Assumption \( (f_0) \) and the boundedness of \( \{u_n^+\} \) readily lead to

\[
(3.15) \quad \left| \langle A_p(u_n^+), v \rangle - \int_{\Omega} f(x, u_n^+(x)) v(x) \, dx \right| \leq c_2 \|v\|_{1,p}
\]
for appropriate $c_2 > 0$. Since $u_n = u_n^+ - u_n^-$, inequalities (3.14)–(3.15) produce, after dividing by $\|u_n\|_{1,p}^{p-1}$,

\begin{equation}
(3.16) \quad \langle A_p(-w_n), v \rangle - \frac{1}{\|u_n\|_{1,p}^{p-1}} \int_{\Omega} f(x, -u_n^- (x)) v(x) \, dx \leq e_n' v \|v\|_{1,p}, \quad v \in X,
\end{equation}

with $e_n' \to 0^+$. Observe next, by $(f_0)$ besides (3.13),

\[
\lim_{n \to +\infty} \frac{1}{\|u_n\|_{1,p}^{p-1}} \int_{\Omega} f(x, -u_n^- (x)) (w_n(x) - w(x)) \, dx = 0.
\]

So, (3.16) written for $v := w_n - w$ and (3.13) again provide

\[
\lim_{n \to +\infty} \langle A_p(w_n), w_n - w \rangle = 0,
\]

namely, because of (p5),

\begin{equation}
(3.17) \quad \lim_{n \to +\infty} w_n = w \quad \text{in } X,
\end{equation}

whence $\|w\|_{1,p} = 1$. Thanks to $(f_0)$ the sequence $\{\|u_n\|_{1,p}^{p+1} N_f(-u_n^-)\}$ $\subseteq L^{p'}(\Omega)$ is bounded. Using the arguments made in [14, pp. 302–303] we thus obtain a function $x \in L^\infty(\Omega)$ such that $a_2 \leq x \leq b_2$ and

\[
\frac{1}{\|u_n\|_{1,p}^{p-1}} N_f(-u_n^-) \rightharpoonup -x w^{p-1} \quad \text{in } L^{p'}(\Omega).
\]

On account of (3.16)–(3.17) this implies, as $n \to +\infty$,

\[
\langle A_p(w), v \rangle = \int_{\Omega} x(x) w(x)^{p-1} \, dx \quad \forall v \in X,
\]

i.e., $w$ turns out to be a weak positive solution of the problem

\[-\Delta_p u = x(x) |u|^{p-2} u \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial \Omega.
\]

Now, recalling $(f_1)$, from (p7) it follows

\[\lambda_1(x) < \lambda_1(\lambda_1) = 1 = \lambda_2(\lambda_2) < \lambda_2(x).\]

Therefore $w = 0$, which contradicts $\|w\|_{1,p} = 1$.

A further nontrivial smooth solution to (1.1) can now be found.

**Theorem 3.2.** Let $(f_0)$–$(f_4)$ be satisfied. Then Problem (1.1) possesses a nontrivial solution $u_1 \in C_0^1(\bar{\Omega}) \setminus \{u_0\}$.
Proof. We may evidently assume that the local minimizer $u_0$ of $\varphi$ given by Theorem 3.1 is proper. Thus, for sufficiently small $\rho > 0$ one has

$$\varphi(u_0) < c_\rho := \inf_{u \in \partial \mathcal{B}_\rho(u_0)} \varphi(u). \tag{3.18}$$

Since, due to (f$_2$),

$$\lim_{t \to -\infty} \varphi(t\phi_1) = -\infty,$$

there exists $t_1 > 0$ such that

$$\|t_1 \phi_1 + u_0\|_{1,p} > \rho, \quad \varphi(-t_1 \phi_1) < c_\rho.$$

On account of Lemma 3.2, the Mountain-Pass Theorem can be applied, which yields a point $u_1 \in X$ complying with $\varphi'(u_1) = 0$ and

$$c_\rho \leq \varphi(u_1) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)), \tag{3.19}$$

where

$$\Gamma := \{ \gamma \in C^0([0,1],X) : \gamma(0) = -t_1 \phi_1, \gamma(1) = u_0 \}.$$

Obviously, the function $u_1$ solves (1.1). Through (3.18)--(3.19) we get $u_1 \neq u_0$, while standard regularity arguments ensure that $u_1 \in C^1_0(\overline{\Omega})$. The proof is thus completed once one verifies that $u_1 \neq 0$. This will follow from the inequality

$$\varphi(u_1) < 0, \tag{3.20}$$

which, in view of (3.19), can be shown by constructing a path $\tilde{\gamma} \in \Gamma$ such that

$$\varphi(\tilde{\gamma}(t)) < 0 \quad \forall t \in [0,1]. \tag{3.21}$$

By (f$_3$) to every $\eta > 0$ small there corresponds $\delta > 0$ such that

$$\frac{\lambda_2 + \eta}{p} |z|^p \leq F(x,z), \quad (x,z) \in \Omega \times [-\delta,\delta]. \tag{3.22}$$

Combining (p$_4$) with Lemma 2.2 entails

$$\max_{t \in [-1,1]} \|\gamma_\eta(t)\|_{1,p}^p < \lambda_2 + \eta \tag{3.23}$$

for appropriate $\gamma_\eta \in \Gamma_c$. Since $\gamma_\eta([-1,1])$ is compact in $C^1_0(\overline{\Omega})$ and $t_1 \phi_1, u_0 \in \text{int}(C_+)$ we can find $\varepsilon > 0$ so small that

$$-t_1 \phi_1(x) \leq \varepsilon \gamma_\eta(t)(x) \leq u_0(x), \quad |\varepsilon \gamma_\eta(t)(x)| \leq \delta$$
whenever $x \in \Omega$, $t \in [-1, 1]$. Thanks to (3.22)–(3.23) one has

$$
\varphi(\varepsilon \gamma(t)) = \frac{e^p}{p} \|\gamma(t)\|_{1,p}^p - \int_{\Omega} F(x, e^p \gamma(t)(x)) \, dx
$$

$$
< \frac{e^p}{p} (\lambda_2 + \eta) - \frac{e^p}{p} (\lambda_2 + \eta) \int_{\Omega} |\gamma(t)(x)|^p \, dx = 0 \quad \forall t \in [-1, 1],
$$

because $\gamma(t) \in U_C$. Consequently,

(3.24) \quad $\varphi|_{\varepsilon \gamma([0,1])} < 0$.

Next, write $a := \varphi_+(u_0)$. From (3.7) it follows $a < 0$. We may suppose

$$
K(\varphi_+) = \{0, u_0\},
$$

otherwise the conclusion is straightforward. Hence, no critical value of $\varphi_+$ lies in $(a, 0)$ while

$$
K_a(\varphi_+) = \{u_0\}.
$$

Due to the second deformation lemma [4, Theorem 5.1.33], there exists a continuous function $h : [0, 1] \times (\varphi^0 \setminus \{0\}) \to \varphi^0$ satisfying

$$
h(0, u) = u, \quad h(1, u) = u_0, \quad \text{and} \quad \varphi_+(h(t, u)) \leq \varphi_+(u)
$$

for all $(t, u) \in [0, 1] \times (\varphi^0 \setminus \{0\})$. Let $\gamma_+(t) := h(t, e\phi_1^+)$, $t \in [0, 1]$. Then $\gamma_+(0) = e\phi_1$, $\gamma_+(1) = u_0$, as well as

(3.25) \quad $\varphi(\gamma_+(t)) = \varphi_+(\gamma(t)) \leq \varphi_+(h(t, e\phi_1)) \leq \varphi_+(e\phi_1) = \varphi_+(\varepsilon \gamma(t)) < 0$;

cf. (3.24). Finally, define

$$
\gamma_-(t) := -(t_1 t + \varepsilon(1 - t))\phi_1, \quad t \in [0, 1].
$$

By $(f_4)$ and $(p_2)$–$(p_3)$ we easily have

(3.26) \quad $\varphi(\gamma_-(t)) \leq \frac{\frac{p}{p} \left( t_1 t + \varepsilon(1 - t) \right)^p}{p} (\lambda_1 - a_4) \|\phi_1\|_{L^p(\Omega)}^p < 0$.

Concatenating $\gamma_-, e \gamma$, and $\gamma_+$ one obtains a path $\gamma \in \Gamma$ which, in view of (3.24)–(3.26), fulfills (3.21). This shows (3.20), whence $u_1 \neq 0$.

The next multiplicity result directly stems from Theorems 3.1–3.2.

**Theorem 3.3.** Let $(f_0)$–$(f_4)$ be satisfied. Then Problem (1.1) possesses at least two nontrivial solutions $u_0 \in \text{int}(C_+)$ and $u_1 \in C_0^1(\overline{\Omega})$. 
4. The case $p = 2$

Suppose $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(x, 0) \equiv 0$ and $f(x, \cdot)$ belongs to $C^1(\mathbb{R})$ for every $x \in \Omega$, while $f(\cdot, t)$ and $f_i(\cdot, t)$ are measurable for all $t \in \mathbb{R}$. The following assumptions will be made in the sequel, where $a_i$ and $b_j$ denote appropriate nonnegative constants.

$(f_5)$ $|f'_i(x, t)| \leq a_0(1 + |t|^{r-2})$ for every $(x, t) \in \Omega \times \mathbb{R}$, being $2 \leq r < 2^*$.

$(f_6)$ $\lim_{t \to +\infty} \frac{f(x, t)}{t} = a_1 < \lambda_1$ uniformly with respect to $x \in \Omega$.

$(f_7)$ $\lambda_1 < a_2 \leq \liminf_{t \to -\infty} \frac{f(x, t)}{t} \leq \limsup_{t \to -\infty} \frac{f(x, t)}{t} \leq b_2$ uniformly in $x \in \Omega$.

$(f_8)$ $f'_i(x, 0) = \lim_{t \to 0} \frac{f(x, t)}{t}$ uniformly with respect to $x \in \Omega$. Moreover, for some $m \geq 2$ one has $\lambda_m < a_3 \leq f'_i(x, 0) \leq b_3 < \lambda_{m+1}$ in $\Omega$.

$(f_9)$ There exists $a_4 > \lambda_1$ fulfilling $\frac{a_4}{2}z^2 \leq F(x, z)$ for all $(x, z) \in \Omega \times \mathbb{R}_0$.

A comment analogous to that made in Remark 3.1 is true here.

Consider the semi-linear problem

\[
\begin{cases}
-\Delta u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

If $X := H_0^1(\Omega)$ and, to simplify notation, $\| \cdot \| := \| \cdot \|_{1, 2}$ then the energy functional $\varphi : X \to \mathbb{R}$ stemming from (4.1) is

\[
\varphi(u) := \frac{1}{2}\|u\|^2 - \int_\Omega F(x, u(x)) \, dx \quad \forall u \in X,
\]

with $F$ as in (3.1). Obviously, $\varphi \in C^2(X)$.

Adapting the arguments of Section 3 we see that $\varphi$ satisfies condition (C) and the following result holds.

**Theorem 4.1.** Let $(f_5) - (f_9)$ be satisfied. Then (4.1) admits at least two nontrivial solutions $u_0 \in \text{int}(C_+)$ and $u_1 \in C_0^1(\Omega)$.

A further nontrivial smooth solution to (4.1) will be found via Morse’s theory.

**Lemma 4.1.** Under hypotheses $(f_5) - (f_7)$ one has $C_q(\varphi, \infty) = 0$ for all $q \in \mathbb{N}_0$.

**Proof.** Pick any $\beta \in L^\infty(\Omega) \setminus \{0\}$ such that $\beta \geq 0$. Define, provided $u \in X$, $t \in [0, 1]$,

\[
\psi(u) := \frac{1}{2}\|u\|^2 - \frac{a_2}{2}\|u^\perp\|^2_{L^2(\Omega)} + \int_\Omega \beta(x)u(x) \, dx,
\]

\[
h(t, u) := t\psi(u) + (1 - t)\psi(u).
\]
On account of (f₅) the function $h : [0, 1] \times X \to \mathbb{R}$ maps bounded sets into bounded sets, while $h(0, \cdot)$ and $h(1, \cdot)$ evidently comply with condition (C). Since $u \mapsto h'_u(t, u)$ and $u \mapsto h''_u(t, u)$ are locally Lipschitz continuous, as a simple computation shows, Proposition 3.2 in [7] can be applied once we prove that there exist $c \in \mathbb{R}$, $\delta > 0$ fulfilling

$$h(t, u) \leq c \Rightarrow (1 + \|u\|)\|h'_u(t, u)\|_{X'} \geq \delta \|u\|^2.$$ 

If the assertion were false then one might construct two sequences $\{t_n\} \subseteq [0, 1]$, $\{u_n\} \subseteq X$ such that $t_n \to t$, $h(t_n, u_n) \to -\infty$, and

$$h(t_n, u_n) - h(0, u_n) = (1 + \|u_n\|)\|h'_u(t_n, u_n)\|_{X'} \leq \frac{1}{n} \|u_n\|^2, \quad n \in \mathbb{N}. \tag{4.3}$$

By the properties of $h$, from $h(t_n, u_n) \to -\infty$ it follows

$$\lim_{n \to +\infty} \|u_n\| = +\infty. \tag{4.4}$$

Set $w_n := \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Passing to a subsequence when necessary, we may suppose

$$w_n \to w \quad \text{in} \ X, \quad w_n \to w \quad \text{in} \ L^2(\Omega), \quad w_n(x) \to w(x) \quad \text{a.e. in} \ \Omega,$$

because $\|w_n\| = 1$ for all $n \in \mathbb{N}$. Inequality (4.3) yields

$$\left| \langle A_2(w_n), v \rangle - t_n \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|} v \, dx + (1 - t_n) a_2 \int_{\Omega} \frac{u_n^-}{\|u_n\|} v \, dx \right| + (1 - t_n) \int_{\Omega} \frac{\beta}{\|u_n\|} v \, dx \leq \frac{1}{n} \|v\| \quad \forall v \in X. \tag{4.5}$$

Now observe that, on account of (f₅)–(f₇), the sequence $\{|\|u_n\|^{-1} N_f(u_n)\}$ is bounded in $L^2(\Omega)$. Choosing $v := w_n - w$ and letting $n \to +\infty$ in (4.5) easily leads to

$$\lim_{n \to +\infty} \langle A_2(w_n), w_n - w \rangle = 0,$$

whence $w_n \to w$ in $X$ by (p₅). Through (f₆)–(f₇) we get

$$\frac{N_f(u_n)}{\|u_n\|} \to a_1 w^+ - \alpha w^- \quad \text{in} \ L^2(\Omega)$$

for appropriate $\alpha \in L^2(\Omega)$ such that $a_2 \leq \alpha \leq b_2$; see [6, pp. 1377–1378] or [14, pp. 302–303]. By (4.5) this implies, as $n \to +\infty$,

$$\langle A_2(w), v \rangle = \int_{\Omega} \left\{ t a_1 w^+(x) - [ta(x) + (1 - t)a_2] w^-(x) \right\} v(x) \, dx, \quad v \in X,$$
namely \( w \) turns out to be a weak solution of the problem

\[
-\Delta u = ta_1 u^+ - \alpha_t(x) u^- \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\]

where \( \alpha_t(x) := t\alpha(x) + (1 - t)a_2 \). Since \( ta_1 < \lambda_1 \) while

\[
\lambda_m < a_2 \leq \alpha_t(x) \leq b_2 < \lambda_{m+1},
\]

one has \( w = 0 \), which however contradicts \( \|w\| = 1 \). Hence, Proposition 3.2 in [7] provides

\[
C_q(\varphi, \infty) = C_q(\psi, \infty) \quad \forall q \in \mathbb{N}_0.
\]

(4.6) The conclusion is achieved once we show that \( C_q(\psi, \infty) = 0 \). If \( u \in K(\psi) \) then

\[
\langle A_2(u), v \rangle = -\int_\Omega [a_2 u^-(x) + \beta(x)] v(x) \, dx, \quad v \in X.
\]

Letting \( v := u^+ \) immediately leads to \( u \leq 0 \). So, \( u \) solves the problem

(4.7) \[-\Delta u = a_2 u - \beta(x) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial \Omega.
\]

Since \( \beta \in L^\infty(\Omega) \setminus \{0\} \) and \( \beta \geq 0 \), standard regularity results [5, Theorems 1.5.5–1.5.6], besides [18, Theorem 5], yield \( -u \in \text{int}(C_+) \). Define, for every \( v \in \text{int}(C_+) \),

\[
R(v, -u) := |\nabla v|^2 - \nabla(-u) \cdot \nabla \left( \frac{v^2}{-u} \right).
\]

From the classical Picone identity (see, e.g., [14, Proposition 9.60]), (4.7), the sign properties of \( u \) and \( \beta \), as well as \( (f_7) \) it follows

\[
0 \leq \int_\Omega R(v, -u)(x) \, dx = \|v\|^2 - \int_\Omega (-\Delta u) \frac{v^2}{u} \, dx
\]

\[
= \|v\|^2 - a_2 \|v\|^2_{L^2(\Omega)} + \int_\Omega \frac{v^2}{u} \beta \, dx
\]

\[
\leq \|v\|^2 - a_2 \|v\|^2_{L^2(\Omega)} < \|v\|^2 - \lambda_m \|v\|^2_{L^2(\Omega)}.
\]

Bearing in mind \( (p_3) \) this entails, for \( v := \phi_1 \),

\[
0 < \lambda_1 - \lambda_m \leq 0,
\]

which is clearly impossible. So, \( K(\psi) = \emptyset \) and, a fortiori, \( C_q(\psi, \infty) = 0 \).

\[\square\]

**Lemma 4.2.** Suppose \( (f_5) \) and \( (f_8) \) hold true. Then \( C_q(\varphi, 0) = \delta_{q,d_m} \mathbb{Z} \) for all \( q \in \mathbb{N}_0 \), where \( d_m := \dim \bigoplus_{i=1}^m E(\lambda_i) \).
Proof. Recall that \( j \in C^2(X) \) and one has

\[
\langle \phi''(u)(v), w \rangle = \int_{\Omega} \nabla v(x) \cdot \nabla w(x) \, dx
- \int_{\Omega} f_t'(x, u(x)) v(x) w(x) \, dx \quad \forall u, v, w \in X.
\]

Thanks to (f_8), Lemma 2.1 can be applied. Thus, \( u = 0 \) is a non-degenerate critical point of \( \phi \) with Morse index \( d_m \). Now, the conclusion follows from (2.2). \( \square \)

**Theorem 4.2.** Let \((f_5)-(f_9)\) be satisfied. Then Problem \((4.1)\) possesses at least three nontrivial solutions \( u_0 \in \text{int}(C_+) \) and \( u_1, u_2 \in C^1_0(\overline{\Omega}) \).

**Proof.** Theorem 4.1 directly gives the solutions \( u_0 \in \text{int}(C_+), u_1 \in C^1_0(\overline{\Omega}) \setminus \{0\} \). Through Theorem 3.1 we next infer

\[
(4.9) \quad C_q(\phi, u_0) = \delta_{q, 0} Z, \quad q \in \mathbb{N}_0;
\]

see [14, Example 6.45]. The proof of Theorem 3.2 ensures that \( u_1 \) is a Mountain-Pass type critical point for \( \phi \). Hence, taking into account (4.8), Corollary 6.102 in [14] yields

\[
(4.10) \quad C_q(\phi, u_1) = \delta_{q, 1} Z, \quad q \in \mathbb{N}_0.
\]

If the assertion were false then \( K(\phi) = \{0, u_0, u_1\} \). Lemmas 4.1–4.2, (4.9), (4.10), and Morse’s relation (2.1) written for \( t = -1 \) would imply

\[
(-1)^{d_m} + (-1)^0 + (-1)^1 = 0,
\]

which is absurd. Therefore, there exists a further point \( u_2 \in K(\phi) \setminus \{0, u_0, u_1\} \). Standard regularity arguments lead to the conclusion. \( \square \)

**References**


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