\section*{1. Introduction and main results}

Suppose that $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) is an aperture domain (see Figure 1) with smooth boundary, i.e.

$$\Omega \cup B_r(0) = \mathbb{R}_+^n \cup \mathbb{R}_-^n \cup B_r(0) \quad (r > 0)$$

with

$$\mathbb{R}_+^n = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0 \}$$

$$\mathbb{R}_-^n = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n < -d \} \quad (d > 0).$$

We consider the homogeneous non-stationary Stokes equations in $(0, \infty) \times \Omega$ concerning the velocity field $u(t, x)$ and the scalar pressure $p(t, x)$:

$$\partial_t u - \Delta u + \nabla p = f \quad \text{in } (0, \infty) \times \Omega \quad (1)$$

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\[ \text{div} u = 0 \quad \text{in} \ (0, \infty) \times \Omega \quad (2) \]

\[ u|_{\partial \Omega} = 0 \quad \text{on} \ (0, \infty) \times \partial \Omega \quad (3) \]

\[ \Phi(u) = \alpha \quad \text{in} \ (0, \infty) \quad (4) \]

\[ u|_{t=0} = u_0 \quad \text{in} \ \Omega \quad (5) \]

where \( \partial_t = \frac{\partial}{\partial t} \), \( \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \), \( \nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)^T \),

\[ \Phi(u(t)) = \int_{M} N \cdot u(t, x) \, d\sigma(x) = \alpha(t) \]

is the flux through a smooth, bounded \((n - 1)\)-dimensional manifold \(M\) with normal vector \(N\) directed downwards dividing \(\Omega\) into two unbounded connected components. This flux has to be prescribed in order to get a unique solution with \(u(t) \in L_q(\Omega)\) with \(\frac{n}{n-1} < q < \infty\). In the case \(1 < q \leq \frac{n}{n-1}\) the flux has to vanish, i.e. \(\Phi(u) = 0\) (see [4] for the corresponding resolvent problem).

Figure 1: An aperture domain

In this paper we only deal with the case \(f = 0\) and \(\Phi(u) = 0\). We consider the asymptotic behaviour of the solutions \(u(t)\). The general case can be derived from this case depending on the asymptotic behaviour of \(f(t)\) and \(\alpha(t)\). Since the Stokes operator \(A_q\) generates a bounded semigroup in \(J_q(\Omega) = \{u \in C_0^\infty(\Omega)^n : \text{div}u = 0\} \|\cdot\|_q\) the estimate \(\|u(t)\|_q \leq C\|u_0\|_q\) holds.

The goal of this paper is to prove the following decay rate measuring \(u(t)\) and \(u_0\) in the norm of \(L_q\) for different \(1 < q < \infty\).

**Theorem 1.1.** Let \(1 < q \leq r < \infty\). Then there is a constant \(C = C(\Omega, q, r)\) such that

\[ \|u(t)\|_{L_r(\Omega)} \leq Ct^{-\sigma} \|u_0\|_{L_q(\Omega)} \quad (6) \]

with \(\sigma = \frac{n}{2} \left( \frac{1}{q} - \frac{1}{r} \right)\) for all \(t > 0\) and \(u_0 \in J_q(\Omega)\).
Theorem 1.2. Let \( 1 < q \leq r < n \). Then there is a constant \( C = C(\Omega, q, r) \) such that

\[
\|\nabla u(t)\|_{L^r(\Omega)} \leq Ct^{-\sigma - \frac{1}{2}} \|u_0\|_{L^q(\Omega)}
\]

with \( \sigma = \frac{n}{2} \left( \frac{1}{q} - \frac{1}{r} \right) \) for all \( t > 0 \) and \( u_0 \in J_q(\Omega) \).

These inequalities are known for other unbounded domains. In [12] Ukai showed these estimates for \( 1 < q < \infty \) if the domain is the half-space \( \mathbb{R}^n_+ \). This is done by using an explicit solution formula in terms of Riesz operators and the heat kernel in \( \mathbb{R}^n_+ \). In the case of an exterior domain, Iwashita [8] showed the validity of (6) for \( 1 < q \leq r < \infty \) and that of (7) for \( 1 < q \leq r \leq n \).

The proof of Theorems 1.1 and 1.2 uses a similar technique as in [8]. It consists of first showing a local estimate of the \( L^q \)-norm of \( u(t) \) and then comparing the full \( L^q \)-norm with suitable solutions of the non-stationary Stokes equations in \( \mathbb{R}^n_+ \). The local estimate is derived from an asymptotic expansion of the resolvent of the Stokes operator in the aperture domain around 0 in special weighted \( L^q \)-spaces. The resolvent expansion is constructed by using a similar resolvent expansion of the Stokes operator in the half-space \( \mathbb{R}^n_+ \). For the latter expansion we combine Ukai’s solution formula [12] with an resolvent expansion of the Laplace operator \( \Delta \) in \( \mathbb{R}^n \), based on the results of Murata [9].

Remark 1.3. With the methods of this article we can not prove Theorem 1.2 for the case \( r = n \), which is done by Iwashita in the case of the exterior domain. This is due to a slightly weaker estimate of the local part of the \( L^q \)-norm (see Corollary 6.2 and [8: Theorem 1.2/(i)]). We get this condition because we have to deal with weighted \( L^q \)-spaces of the kind \( L^q(\Omega; \omega^{sq}) \) such that \( \omega^{sq} \) is a Muckenhoupt weight (see preliminaries); this condition on the weights is not needed in [8].

The \( L^q \- L^r \)-estimate can be used to construct solutions of the instationary Navier-Stokes equations with arbitrary flux \( \Phi(u) \) as perturbation of steady-state solution. For the case \( n = 2 \) this problem is still unsolved. Unfortunately, the used approach can not be applied to a two-dimensional aperture domain. The reason is that we can not prove Theorem 4.1 since there is no number \( \sigma \) with \( 1 < \sigma < \frac{n}{2} \), \( n = 2 \). The restriction \( \sigma < \frac{n}{2} \) is due to the restriction to Muckenhoupt weights. The condition \( \sigma > 1 \) is necessary for the perturbation argument used in the proof of Theorem 4.1. – We have to assure that the resolvent of the Stokes operator in \( \mathbb{R}^n_+ \) considered as map between different weighted \( L^q \)-spaces exists for \( z = 0 \).
2. Preliminaries and notation

We will consider the resolvent expansion in a scale of weighted \( L_q \)-spaces

\[
L_q(\Omega; \omega^{sq}) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable} \left| \| f \|_{L_q(\Omega; \omega^{sq})} < \infty \right. \right\} \quad (s \in \mathbb{R})
\]

where

\[
\| f \|_{L_q(\Omega; \omega^{sq})} = \left( \int_{\Omega} |f(x)|^q \omega^{sq}(x) \, dx \right)^{\frac{1}{q}}.
\]

Analogously we define the weighted Sobolev spaces as

\[
W^m_q(\Omega; \omega^{sq}) = \left\{ f \in L^1_{1, \text{loc}}(\Omega) \left| D^\alpha f \in L_q(\Omega; \omega^{sq}) \forall |\alpha| \leq m \right. \right\}
\]

and

\[
W^m_{0,q}(\Omega; \omega^{sq}) = C_0^\infty(\Omega) W^m_q(\Omega; \omega^{sq})
\]

Recall that \( f \in L^1_{1, \text{loc}}(\Omega) \) means that \( f \in L^1(\Omega \cap B) \) for all balls \( B \) with \( \Omega \cap B \neq \emptyset \). Moreover,

\[
D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f(x) \quad (\alpha \in \mathbb{N}_0^n).
\]

By \( \dot{W}_q^m(\Omega; \omega^{sq}) \) we denote the corresponding homogeneous Sobolev space of \( L^1_{1, \text{loc}} \)-functions \( f \) with \( D^\alpha f \in L_q(\Omega; \omega^{sq}) \) for all \( |\alpha| = m \). Finally,

\[
J_q(\Omega; \omega^{sq}_n) = \left\{ u \in C_0^\infty(\Omega)^n : \text{div} u = 0 \right\} L_q(\Omega; \omega^{sq}_n).
\]

For simplicity we often will skip the exponent \( n \) if we deal with spaces of vector fields, e.g. we write \( f \in L_q(\Omega) \) instead of \( f \in L_q(\Omega)^n \). If \( X \) and \( Y \) are two Banach spaces, we denote by \( \mathcal{L}(X, Y) \) the space of all bounded linear maps \( T : X \to Y \). Furthermore, \( \mathcal{L}(X) = \mathcal{L}(X, X) \).

In [8, 9] the simple weight \( \omega(x) = \langle x \rangle := (1 + |x|^2)^{\frac{1}{2}} \) is used. For \( -\frac{n}{q} < s < \frac{n}{q} \) the weight \( \langle x \rangle^q_s \) is an element of the Muckenhoupt class \( A_q \). This is the class of all measurable functions \( \omega : \mathbb{R}^n \to [0, \infty) \) with

\[
\frac{1}{|B|} \int_B \omega(x) \, dx \left( \frac{1}{|B|} \int_B \omega(x)^{-\frac{q}{q'}} \, dx \right)^{\frac{q'}{q}} \leq A < \infty
\]

where \( B \) is an arbitrary ball in \( \mathbb{R}^n \) and \( A \) is independent of \( B \). The weights \( \omega \in A_q \) have the important property that singular integral operators like the Riesz transforms

\[
R_j f(x) := \mathcal{F}^{-1} \left[ \frac{i \xi_j}{|\xi|^s} \hat{f}(\xi) \right] = c_n \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_{\varepsilon}(x)} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) \, dy
\]
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(j = 1, . . . , n) are continuous on $L_q(\mathbb{R}^n; \omega)$ into itself. Here $\mathcal{F}[u](\xi) = \hat{u}(\xi)$ denotes the Fourier transform with respect to $x$. See, for example, [11: Chapter V, §4.2/Theorem 2] for the continuity and [10: Chapter III, Section 1] for Riesz transforms.

We will also use the partial Riesz transforms

$$S_j f(x) = \mathcal{F}_{\xi' \to x'}^{-1} \left[ \frac{i\xi_j}{|\xi'|} \tilde{f}(\xi', x_n) \right] = c_{n-1} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n-1} \setminus B_\varepsilon(x')} \frac{x_j' - y_j'}{|x' - y'|^n} f(y', x_n) \, dy$$

(j = 1, . . . , n − 1; $x = (x', x_n), \xi = (\xi', \xi_n)$) for functions $f$ defined on $\mathbb{R}^n_+$ or $\mathbb{R}^n$. These partial Riesz transforms are used in Ukai’s solution formula.

Unfortunately, the weight $\langle x \rangle^{sq}$ considered for fixed $x_n$ as weight in $\mathbb{R}^{n-1}$ is in the class $A_q$ only if $-\frac{n-1}{q} < s < \frac{n-1}{q'}$. Therefore we will use the slightly weaker weight

$$\omega_n(x) = \prod_{i=1}^n \langle x_i \rangle^{\frac{1}{n}}.$$ 

For this weight $\omega_n(x)^{sq}$ considered for fixed $x_n$ is in $A_q$ on $\mathbb{R}^n$ for $-\frac{n}{q} < s < \frac{n}{q'}$. This is easily derived from the special product structure and the fact that $\langle x_i \rangle^\frac{n}{n}$ is a one-dimensional weight in $A_q$.

Therefore we get

**Lemma 2.1.** Let $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$, $1 < q < \infty$, $-\frac{n}{q} < s < \frac{n}{q'}$ and $\omega_n(x) = \prod_{i=1}^n \langle x_i \rangle^{\frac{1}{n}}$. Then the (partial) Riesz transforms are continuous from $L_q(\Omega; \omega_n^{sq})$ into itself.

Moreover, we introduce

$$\Sigma_\delta = \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| < \delta \}$$

$$\Sigma_{\delta, \varepsilon} = \Sigma_\delta \cap B_\varepsilon(0).$$

Recall the Helmholtz decomposition of a vector field $f \in L_q(\Omega; \omega_n^{sq})^n$, i.e. the unique decomposition $f = f_0 + \nabla p$ with $f_0 \in J_q(\Omega; \omega_n^{sq})$ and $p \in W^{1, q}_q(\Omega; \omega_n^{sq})$. The existence and continuity of the corresponding Helmholtz projection

$$P_q : L_q(\Omega; \omega_n^{sq})^n \to J_q(\Omega; \omega_n^{sq}), \quad f \mapsto P_q f = f_0$$

is proved in [3: Theorem 5] for the case that $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$, or that $\Omega$ is a bounded domain. For the case of an aperture domain and $s = 0$ the result is proved in [4: Theorem 2.6].

Furthermore, we define the Stokes operator

$$A_q = -P_q \Delta$$
in $J_q(\Omega)$ with $\mathcal{D}(A_q) = W^2_q(\Omega) \cap W^1_{0,q}(\Omega) \cap J_q(\Omega)$. Note that the resolvent of $A_q$ satisfies the estimate

$$
\|(z + A_q)^{-1} f\|_{L_q(\Omega)} \leq C_\delta |z|^{-1} \|f\|_{L_q(\Omega)}
$$

(8)

for $z \in \Sigma_\delta$ ($\delta \in (0, \pi)$) if $\Omega$ is an aperture domain (see [9: Theorem 2.5]). Therefore $-A_q$ generates an analytic semigroup.

### 3. The resolvent expansion in $\mathbb{R}^n_+$

We consider the resolvent equations system

$$
\begin{align*}
(z - \Delta)u + \nabla p &= f \quad \text{in } \mathbb{R}^n_+ \quad (9) \\
\text{div} u &= 0 \quad \text{in } \mathbb{R}^n_+ \quad (10) \\
u|_{\partial \mathbb{R}^n_+} &= 0 \quad \text{on } \partial \mathbb{R}^n_+ \quad (11)
\end{align*}
$$

Let $R_0(z) = (z - \Delta)^{-1}$ denote the resolvent of the Laplace operator in $\mathbb{R}^n$.

**Lemma 3.1.** Let $1 \leq p \leq \infty$, $0 < \delta < \pi$, $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq 2$, $\frac{|\alpha|}{2} < \sigma < \frac{n + |\alpha|}{2}$, $-\frac{n}{p} < s' < s < \frac{n}{p'}$ and $s' = s - 2\sigma + |\alpha|$. Then

$$
D^\alpha R_0(z) = \sum_{j=0}^{[\sigma]-1} z^j D^\alpha G_{0j} + G_{0r}(z)
$$

where

$$
G_{0r}(z) = O(z^{\sigma-1}) \quad \text{in } \mathcal{L}(W^m_p(\mathbb{R}^n; \omega^{s'p}_n), W^{m+2-|\alpha|}_p(\mathbb{R}^n; \omega^{s'p}_n))
$$

for $z \to 0$ with $z \in \Sigma_\delta$.

**Proof.** The proof is the same as [9: Lemma 2.3/(i)]. It is based on the estimate for the convolution operator with the heat kernel $E_0(t)$

$$
\|D^\alpha E_0(t)\|_{\mathcal{L}(L_p(\mathbb{R}^n; \omega^{s'p}_n), L_p(\mathbb{R}^n; \omega^{s'p}_n))} \leq |t|^{-\frac{|\alpha|}{2}} \langle t \rangle^{-\sigma}
$$

(12)

for $\omega(x) = \omega_n(x)$, $t \in \Sigma_{\delta_0}$, $0 < \delta_0 < \frac{\pi}{2}$, $\alpha \in \mathbb{N}^n_0$, $0 \leq \sigma < \frac{n}{2}$ and $-\frac{n}{p} < s' < s < \frac{n}{p'}$ with $s' = s - 2\sigma$. This estimate is proved in [9: Lemma 2.2] for the
case \( \omega(x) = \langle x \rangle \). But this case implies the estimate for \( \omega(x) = \omega_n(x) \) since
\[
\|D^\alpha E_0(t) f\|_{L_p(\mathbb{R}^n; \omega_n^{s'p})} \\
\leq \left\| \int_{\mathbb{R}^{n-1}} \left| D^\alpha e^{-|x'-y'|^2/(4\pi t)} \right| \right\|_{L_p(\mathbb{R}; |x_n|^{s'p})} \left\| f(y', \cdot) \right\|_{L_p(\mathbb{R}; |x_n|^{s'p})} \left( \int_{\mathbb{R}^{n-1}} \|f(y', \cdot)\|_{L_p(\mathbb{R}; |x_n|^{s'p})} \, dy' \right) \\
\leq C |t|^{-\alpha/2} \langle t \rangle^{-\sigma} \|f\|_{L_p(\mathbb{R}^n; \omega_n^{sp})}
\]
with \( \alpha = (\alpha', \alpha_n) \).

**Remark 3.2.** The operators \( G_{0j} \) and \( G_{0r}(z) \) are given by
\[
G_{0j} = \int_0^\infty E_0(t) \frac{(-t)^j}{j!} \, dt \\
G_{0r}(z) = \int_0^\infty E_0(t) f_{[\sigma]}(zt) \, dt \quad \text{with} \quad f_{[\sigma]}(zt) = e^{-zt} - \sum_{j=0}^{[\sigma]-1} \frac{(-zt)^j}{j!}.
\]

We recall Ukai's solution formula for the homogeneous non-stationary Stokes equations in \( \mathbb{R}_+^n \) (see [13]), i.e. (1) - (3) and (5) for \( \Omega = \mathbb{R}_+^n, f = 0 \) with compatibility condition \( \text{div} u_0 = 0 \) in \( \mathbb{R}_+^n \) and \( u_0^n = 0, u_0 = (u_0, u_0^0) \) on \( \partial \mathbb{R}_+^n \). Let \( R_j \) and \( S_j \) be as above. Moreover, let \( rf = f|_{\mathbb{R}_+^n}, \gamma f = f|_{\partial \mathbb{R}_+^n} \) and \( e \) be the extension operator from \( \mathbb{R}_+^n \) to \( \mathbb{R}^n \) with value 0. Finally, let \( E(t) \) be the solution operator for the heat equation in \( \mathbb{R}_+^n \), which is derived from \( E_0(t) \) by odd extension from \( \mathbb{R}_+^n \) to \( \mathbb{R}^n \). Then the solution \( (u(t), p(t)) \) of the non-stationary Stokes equations in \( \mathbb{R}_+^n \) is
\[
u(t) = W E(t) V u_0 \\
p(t) = -D\gamma \partial_n E(t)V_1 u_0
\]
where
\[
W = \begin{pmatrix} I & -SU \\ 0 & U \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} V_2 \\ V_1 \end{pmatrix}
\]
with 
\[ S = (S_1, \ldots, S_{n-1})^T \]
\[ U = R' \cdot S(R' \cdot S + R_n)e \]
\[ V_1 u_0 = -S \cdot u'_0 + u_n^0 \]
\[ V_2 u_0 = u'_0 + Su_0^n \]
\[ R' = (R_1, \ldots, R_{n-1})^T \]

and \( D \) is the Poisson operator for the Dirichlet problem of the Laplace equation in \( \mathbb{R}^n_+ \).

Using this result, we get:

**Theorem 3.3.** Let \( 1 < q < \infty, 0 < \delta < \pi, n \geq 3, \frac{1}{2} < \sigma < \frac{n+1}{2} \), \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq 2, \frac{n}{q} < s' \leq 0 \leq s < \frac{n}{q} \) and \( s' = s - 2\sigma + |\alpha| \). Then there exist operators \( R_+(z) \) and \( P_+(z) \) with

\[ D^\alpha R_+(z) \in \mathcal{L}(L_q(\mathbb{R}^n_+; \omega_n^{s,q}), W_q^{2-|\alpha|}(\mathbb{R}^n_+; \omega_n^{s'+q})) \]
\[ P_+(z) \in \mathcal{L}(L_q(\mathbb{R}^n_+; \omega_n^{s,q}), \dot{W}_q^1(\mathbb{R}^n_+; \omega_n^{s'+q})) \]

depending continuously on \( z \in \Sigma_\delta \cup \{0\} \) such that:

1. \( u = R_+(z)f \) and \( p = P_+(z)f \) with \( f \in L_q(\mathbb{R}^n_+; \omega_n^{s,q}) \) is a solution of problem (9) - (11) for \( z \in \Sigma_\delta \).
2. \( R_+(z) \in \mathcal{L}(L_q(\mathbb{R}^n_+; \omega_n^{s,q}), W_q^2(\mathbb{R}^n_+)) \) and \( P_+(z) \in \mathcal{L}(L_q(\mathbb{R}^n_+; \omega_n^{s,q}), \dot{W}_q^1(\mathbb{R}^n_+)) \)
   for every \( z \in \Sigma_\delta \).
3. The asymptotic expansions

\[ D^\alpha R_+(z) = \sum_{j=0}^{[\sigma]-1} z^j D^\alpha G_j + O(z^{\sigma-1}) \quad \text{in} \quad \mathcal{L}(L_q(\mathbb{R}^n_+; \omega_n^{s,q}), W_q^{2-|\alpha|}(\mathbb{R}^n_+; \omega_n^{s'+q})) \]
\[ P_+(z) = \sum_{j=0}^{[\sigma]-1} z^j P_{+,j} + O(z^{\sigma-1}) \quad \text{in} \quad \mathcal{L}(L_q(\mathbb{R}^n_+; \omega_n^{s,q}), \dot{W}_q^1(\mathbb{R}^n_+; \omega_n^{s'+q})) \] if \( |\alpha| = 2 \)

hold for \( z \to 0, z \in \Sigma_\delta \).

**Proof.** Because of the Helmholtz decomposition in weighted \( L_q \)-Spaces (see [5: Theorem 5]) we can assume without loss of generality that \( f \in J_q(\Omega; \omega_n^{s,q}) \). Therefore the asymptotic expansion for \( R_+(z) \) simply follows from the expansion of \( R_0(z) \), equations (13) - (14), the continuity of the Riesz transforms \( S_j \) and \( R_j \) in \( L_q(\mathbb{R}^n_+; \omega_n^{s,q}) \) and \( L_q(\mathbb{R}^n_+; \omega_n^{s,q}) \) if \( -\frac{n}{q} < s < \frac{n}{q} \) and the fact

\[ R_+(z)f = \int_0^\infty e^{-tz} WE(t)V f \, dt. \]
In order to get the result for $D^\alpha R_+(z)$ (|\alpha| \leq 2) we use the relations
\[
\partial_i U = (I - U)|\nabla| = -(I - U)\sum_{i=1}^{n-1} S_i \partial_i \\
\partial_i S = S \partial_i \quad (i = 1, \ldots, n) \\
\partial_i U = U \partial_i \quad (i = 1, \ldots, n - 1)
\]
and prove the expansion in the same way as in the case $\alpha = 0$. We note that
the first equation is a consequence of
\[
F_{x'\rightarrow \xi'} [Uf](\xi', x_n) = |\xi'| \int_0^{x_n} e^{-|\xi|(x_n-y_n)} \tilde{f}(\xi', x_n) dy_n
\]
(see the proof of [12: Theorem 1.1]); the other equations are obvious. Finally,
we get the expansion of $\nabla P_+(z)$ in the same way using $|\nabla|D\gamma = \partial_n U - U \partial_n$.

Because of estimate (12) and Ukai’s formula we also easily get

Lemma 3.4. Let $u(t) = WE(t)Vu_0$ with $u_0 \in J_q(\mathbb{R}^n_+; \omega^{s'q}_n)$ denote the
solution of the homogeneous non-stationary Stokes equations (1) – (3), (5) for $\Omega = \mathbb{R}^n_+$ and $f = 0$. Then
\[
\|u(t)\|_{L_q(\mathbb{R}^n_+; \omega^{s'q}_n)} \leq C(1 + t)^{-\sigma}\|u_0\|_{L_q(\mathbb{R}^n_+; \omega^{s'q}_n)}
\]
with $1 < q < \infty$, $-\frac{n}{q} < s' \leq 0 \leq s < \frac{n}{q'}$, $s' = s - 2\sigma$ and $t \geq 0$.

8. Resolvent expansions in aperture domains

We consider the resolvent equations system
\[
(z - \Delta)u + \nabla p = f \quad \text{in } \Omega \quad (16) \\
\text{div} u = 0 \quad \text{in } \Omega \quad (17) \\
u|_{\partial \Omega} = 0 \quad \text{on } \partial \Omega \quad (18) \\
\Phi(u) = 0
\]
for an aperture domain $\Omega$.

Theorem 4.1. Let $1 < q < \infty$, $0 < \delta < \pi$, $n \geq 3$, $1 < \sigma < \frac{n}{2}$, $\sigma \notin \mathbb{Z}$,
$-\frac{n}{q} < s' \leq 0 \leq s < \frac{n}{q'}$ and $s' := s - 2\sigma$. Then there are an $\varepsilon > 0$ and operators
\[
R(z) \in \mathcal{L}(L_q(\Omega; \omega^{s'q}_n), W^2_q(\Omega; \omega^{s'q}_n)) \\
P(z) \in \mathcal{L}(L_q(\Omega; \omega^{s'q}_n), W^1_q(\Omega; \omega^{s'q}_n))
\]
depending continuously on $z \in \Sigma_{\delta,\varepsilon} \cup \{0\}$ with the following properties:

1. The pair $u = R(z)f$ and $p = P(z)f$ is a solution of problem (16) – (19).

2. $R(z) \in \mathcal{L}(L_q(\Omega; \omega_n^{s_q}), W^2_q(\Omega))$ for every $z \in \Sigma_{\delta,\varepsilon}$.

3. The operator-valued function $R(z)$ ($z \in \Sigma_{\delta,\varepsilon,0}$) has an expansion

$$R(z) = \sum_{j=0}^{[\sigma]-1} z^j G_j + G_r(z)$$

in $\mathcal{L}(L_q(\Omega; \omega_n^{s_q}), W^2_q(\Omega; \omega_n^{s_q}))$ where $G_r(z) = O(z^{\sigma-1})$ for $z \to 0$.

**Proof.** We use the technique used in the proof of [8: Theorem 3.1]. Let $\Omega \cup B_r(0) = \mathbb{R}^n_+ \cup \mathbb{R}^n_-$ and denote $\mathbb{R}^n_+ = \mathbb{R}^n_+ \cup \mathbb{R}^n_-$, $\Omega_\pm = \Omega \cap \mathbb{R}^n_\pm$ and $\Omega_b = \Omega \cap B_b(0)$. Let $\varphi, \psi \in C^\infty(\Omega)$ be cut-off functions with

$$\varphi(x) = \begin{cases} 1 & \text{for } |x| > R \\ 0 & \text{for } |x| < R - 1 \end{cases}$$

and

$$\psi(x) = \begin{cases} 1 & \text{for } |x| > R - 2 \\ 0 & \text{for } |x| < R - 3. \end{cases}$$

We identify $\psi f$ with its extension by 0 to $\mathbb{R}^n_\pm$. Moreover, we define

$$R_\pm(z) : L_q(\mathbb{R}^n_\pm; \omega_n^{s_q}) \to W^2_q(\mathbb{R}^n_\pm; \omega_n^{s_q})$$

by

$$R_\pm(z)g(x) = \begin{cases} R_+(z)(g|_{\mathbb{R}^n_+})(x) & \text{if } x \in \mathbb{R}^n_+ \\ R_-(z)(g|_{\mathbb{R}^n_-})(x) & \text{if } x \in \mathbb{R}^n_- \end{cases}$$

The operator

$$P_\pm(z) : L_q(\mathbb{R}^n_\pm; \omega_n^{s_q}) \to \dot{W}^1_q(\mathbb{R}^n_\pm; \omega_n^{s_q})$$

is defined analogously. Let $f_b := f|_{\Omega_b}$ and

$$(L, P) : L_q(\Omega_b)^n \to W^2_q(\Omega_b)^n \times \dot{W}^1_q(\Omega_b)$$

be the solution operator of the Stokes equation in the bounded domain $\Omega_b$.

Define

$$R_1(z) \in \mathcal{L}(L_q(\Omega; \omega_n^{s_q}), W^2_q(\Omega; \omega_n^{s_q}))$$

by

$$R_1(z)f = \varphi R_\pm(z)(\psi f) + (1 - \varphi)Lf_b.$$
by
\[ \Pi(z)f = \varphi P_\pm(z)(\psi f) + (1 - \varphi) Pf_b. \]

Obviously, the operator \( R_1(z) \) has the same type of expansion as \( R_\pm(z) \). Let
\[ P_\pm(z) = \sum_{j=0}^{[\sigma]-1} z^j P_{\pm,j} + P_{\pm,r}(z) \]
with
\[ P_{\pm,r}(z) = O(z^{\sigma-1}) \text{ in } L_q\left(\mathbb{R}_\pm^n; \omega_n^{s'q}\right) \]
be the expansion for \( P_\pm(z) \). We choose \( P_{\pm,j}f, P_{\pm,r}f \in \hat{W}_q^1(\mathbb{R}_\pm^n) \) such that
\[
\int_{D_R \cap \Omega} P_{\pm,0}f \, dx = \int_{D_R \cap \Omega} Pf_b \, dx \\
\int_{D_R \cap \Omega} P_{\pm,r}(z) \, dx = 0, \quad \int_{D_R \cap \Omega} P_{\pm,j}f \, dx = 0 \quad (j = 1, \ldots, [\sigma] - 1)
\]
where \( D_R = \{x \in \Omega : R - 1 < |x| < R\} \). Applying Poincaré’s inequality
\[
\|f\|_q \leq C\left(\|\nabla f\|_q + \left| \int_D f(x) \, dx \right| \right)
\]
for a bounded domain \( D \) with \( C^0 \)-boundary (see [2: Chapter 5/Theorem 4.19]) it follows that
\[
\|P_{\pm,0}f - Pf_b\|_{L_q(\Omega \cap \Omega)} \leq C\left(\|\nabla P_{\pm,0}f\|_{L_q(\Omega \cap \Omega)} + \|\nabla Pf_b\|_{L_q(\Omega \cap \Omega)}\right) \leq C\|f\|_{L_q(\Omega; \omega_n^{s'q})} \\
\|P_{\pm,j}f\|_{L_q(\Omega \cap \Omega)} \leq C\|\nabla P_{\pm,j}f\|_{L_q(\Omega \cap \Omega)} \leq C\|f\|_{L_q(\Omega; \omega_n^{s'q})} \\
\|P_{\pm,r}(z)f\|_{L_q(\Omega \cap \Omega)} \leq C\|\nabla P_{\pm,r}(z)f\|_{L_q(\Omega \cap \Omega)} \leq C|z|^{\sigma-1}\|f\|_{L_q(\Omega; \omega_n^{s'q})}.
\]

Because of these inequalities and the identity
\[ \nabla \Pi(z)f = \varphi \nabla P_\pm(z)(\psi f) + (1 - \varphi) \nabla Pf_b + (\nabla \varphi)(P_\pm(z)(\psi f) - Pf) \]
the operator \( \Pi(z) \) has the same type of expansion as \( P_\pm(z) \).

It remains to correct the divergence of \( R_1(z)f \). For this we apply Bogovskii’s Theorem (see, e.g., [6: Theorem 3.2]) to \( \text{div}(R_1(z)f) = \nabla \varphi \cdot \{R_\pm(z)(\psi f) - Lf_b\} \), which has compact support in \( D_R \). We note that
\[
\int_{D_R} \text{div}(R_1(z)f) = -\int_{B_R \cap \mathbb{R}_\pm^n} \text{div}((1 - \varphi)R_\pm(z)(\psi f)) \, dx - \int_{\Omega_b} \text{div}(\varphi Lf_b) \, dx \\
= -\int_{\partial(B_R \cap \mathbb{R}_\pm^n)} N \cdot (1 - \varphi)R_\pm(z)(\psi f) \, d\sigma - \int_{\partial\Omega_b} N \cdot \varphi Lf_b \, d\sigma \\
= 0.
\]
Since \( \text{div} R_1(z)f \in W_0^2(D_R) \cap W_0^1(D_R) \), we get a compact operator \( Q(z) : L_q(\Omega; \omega_n) \rightarrow W_0^2(D_R) \) with \( \text{div} Q(z)f = \text{div} R_1(z)f \). The operator \( Q(z) \) depends continuously on \( z \in \Sigma_\delta \cup \{0\} \).

We identify \( Q(z)f \) with its extension by zero to a function \( Q(z)f \in W_0^2(\Omega; \omega_n) \). Now let

\[
R_2(z) := R_1(z) - Q(z) \in L(L_q(\Omega; \omega_n), W_q^2(\Omega; \omega_n)).
\]

Then \( R_2(z)f \) solves

\[
(z - \Delta) R_2(z)f + \nabla \Pi(z)f = f + S(z)f \quad \text{in} \Omega
\]
\[
\text{div} R_2(z)f = 0 \quad \text{in} \Omega
\]
\[
R_2(z)f = 0 \quad \text{on} \partial \Omega
\]

for all \( f \in L_q(\Omega; \omega_n) \), where

\[
S(z)f = -\left\{2(\nabla \varphi) \cdot \nabla + (\Delta \varphi)\right\}\{R_\pm(z)(\psi f) - Lf_b\}
+ z(1 - \varphi)Lf_b + (\Delta - z)Q(z)f + \nabla \varphi(P_\pm(z)(\psi f) - Pf_b).
\]

Since \( \text{supp} S(z)f \subseteq \overline{D_R} \), we conclude \( S(z) \in L(L_q(\Omega; \omega_n)) \). The term \((\Delta - z)Q(z) \in L(L_q(\Omega; \omega_n)) \) is a compact operator since \( Q(z) : L_q(\Omega; \omega_n) \rightarrow W_0^2(D_R) \) is compact. Furthermore, \( S(z) - (\Delta - z)Q(z) : L_q(\Omega; \omega_n) \rightarrow W_0^1(D_R) \) is continuous, so \( S(z) \in L(L_q(\Omega; \omega_n)) \) is a compact operator. Moreover, \( S(z) \) is continuous in \( z \in \Sigma_\delta \cup \{0\} \) and has the same type of expansion in \( L(L_q(\Omega; \omega_n)) \) as \( R_\pm(z) \) in \( L(L_q(\Omega; \omega_n), W_q^2(\Omega; \omega_n)) \).

In the following Lemma 4.2 we show that \( I + S(0) \) is injective. Since \( S(0) \) is compact, the Fredholm alternative yields that \( (I + S(0))^{-1} \in L(L_q(\Omega; \omega_n)) \) exists. Therefore \( (I + S(z))^{-1} \) exists for all \( z \in \Sigma_{\delta, \varepsilon} \) for some \( \varepsilon > 0 \). More precisely,

\[
(I + S(z))^{-1} = (I + S(0))^{-1} \sum_{k=0}^{\infty} \left\{(S(0) - S(z))(I + S(0))^{-1}\right\}^k
\]

for all \( z \in \Sigma_{\delta, \varepsilon}, \) where \( \varepsilon > 0 \) is chosen so small that

\[
\|S(z) - S(0)\| \leq \frac{1}{2\|(I + S(0))^{-1}\|} \quad (z \in \Sigma_{\delta, \varepsilon}).
\]

Since \( S(z) \) and therefore all powers \((S(0) - S(z))^k\) have an expansion in \( L(L_q(\Omega; \omega_n)) \) of the same type as \( R_\pm(z) \), the inverse \( (I + S(z))^{-1} \) has the same.

If we now set \( R(z) = R_2(z)(I + S(z))^{-1} \) and \( P(z) = \Pi(z)(I + S(z))^{-1} \), we get the solution operators of the resolvent problem with the desired expansion.
Lemma 4.2. Let $S(z)$ denote the same operator as in the proof of Theorem 4.1. Then $I + S(0) \in \mathcal{L}(L_q(\Omega; \omega_n^{s_p}))$ is injective.

Proof. It is known [3, 4] that the Stokes equations in an aperture domain have a unique solution $(u, \tilde{p}) \in [\tilde{W}^2_p(\Omega) \cap \tilde{W}^1_r(\Omega)]^n \times \tilde{W}^1_r(\Omega)$ for arbitrary $1 < p < n$ and prescribed flux $\Phi(u) = \alpha \in \mathbb{R}$.

We calculate the flux of $R_2(0)$. Since $M \subset B_r$, the identity $R_2(0)f(x) = Lf_b(x)$ holds for all $x \in M$. Denote by $B_r^+$ the connected component of $B_r(0) \setminus M$ “above” $M$. Then we conclude that

$$0 = \int_{B_r^+} \text{div} Lf_b \, dx = \int_{\partial B_r^+} Lf_b \cdot N \, d\sigma = \int_M Lf_b \cdot N \, d\sigma = \int_M R_2(0)f \cdot N \, d\sigma.$$ 

Therefore we get $R_2(0)f = 0$ and $\Pi(0) = \text{const}$ if we show that $R_2(0)f \in [\tilde{W}^2_p(\Omega) \cap \tilde{W}^1_r(\Omega)]^n$ and $\Pi(0)f \in \tilde{W}^1_r(\Omega)$.

Let $(I + S(0))f = 0$. That means $f = -S(0)f$, and therefore the support of $f$ is contained in $\overline{\Omega}_b$. This implies $f \in L^q_p(\Omega; \omega_n^{s_p})$ for all $s \in \mathbb{R}$ and $1 \leq p \leq q$.

Claim. $\nabla^2 R_2(0)f, \nabla \Pi(0)f \in L^q_p(\Omega)$ for all $1 < p \leq q$ and $\nabla R_2(0)f \in L^q_p(\Omega)$ with $\frac{1}{p'} = \frac{1}{p} - \frac{1}{n}$ and $1 < p < \min\{q, n\}$.

Proof of claim. For $i, j \in \{1, \ldots, n\}$ there holds

$$\partial_i \partial_j R_2(0)f = \varphi \partial_i \partial_j R_\pm(0)(\psi f) + \partial_i \partial_j [(1 - \varphi)Lf_b](\psi f) + (\partial_i \varphi) \partial_j R_\pm(0)(\psi f) + (\partial_j \varphi) \partial_i R_\pm(0)(\psi f) - \partial_i \partial_j Q(0)f.$$ 

The support of every term except the first one is contained in $\overline{\Omega}_b$. Therefore each of these function is an element of $L^q_p(\Omega)$ for every $1 \leq p \leq q$.

Considering the first term, Theorem 3.3 tells us that

$$\partial_i \partial_j R_\pm(0) \in \mathcal{L}(L^q_p(\mathbb{R}^n_\pm; \omega_n^{s_p}), L^q_p(\Omega, \omega_n^{s_p}))$$

for all $-\frac{n}{p} < s' \leq 0 \leq s < \frac{n}{p'}$, $s' = s - 2\sigma + 2$ and $1 < \sigma < \frac{n}{2}$. Since $f \in L^s_p(\Omega)$ for arbitrary $s \in \mathbb{R}$ and $1 \leq p \leq q$, we can apply Theorem 3.3 for $s' = 0$ and $s = 2\sigma - 2$. Therefore we choose $1 < \sigma < \frac{n}{2}$ such that $\frac{n}{n-2\sigma+2} < p$ which is equivalent to $2\sigma - 2 < \frac{n}{p'}$. Thus we get $\partial_i \partial_j R_\pm(0)(\psi f) \in L^q_p(\Omega)$ for every $1 < p \leq q$. With the same choice of $s$ and $s'$ we see that $\nabla \Pi(0)f \in L^q_p(\Omega)$ for all $1 < p \leq q$.

The same argumentation can be applied to

$$\partial_i R_2(0)f = \varphi \partial_i R_\pm(0)(\psi f) + \partial_i [(1 - \varphi)Lf_b](\psi f) + (\partial_i \varphi) R_\pm(0)(\psi f) - \partial_i Q(0)f.$$ 

In this case

$$\partial_i R_\pm(0) \in \mathcal{L}(L^q_p(\Omega; \omega_n^{s_p}), L^q_p(\Omega; \omega_n^{s'_p}))$$
holds for all \(-\frac{n}{r} < s' \leq 0 \leq s < \frac{n}{q}\), \(s' := s - 2\sigma + 1, 1 < \sigma < \frac{n}{2}\). The choice of \(s' = 0\) and \(s = 2\sigma - 1\) yields the condition \(2\sigma - 1 < \frac{n}{r}\). Since \(\frac{1}{r} + \frac{1}{n} = \frac{1}{p}\), this condition is equivalent to \(2\sigma - 2 < \frac{n}{p}\) which is equivalent to \(p > \frac{n}{n-2\sigma+2}\). This proves the claim.

Thus \(R_2(0)f = 0\) and \(\nabla \Pi(0)f = 0\). Since \(\text{supp}Q(0) \subseteq \{x : |x| \leq R\}\), it is obvious that for \(x \in \Omega\)

\[
R_2(0)f(x) = \begin{cases} 
R_{\pm}(0)(\psi f)(x) = 0 & \text{if } |x| \geq R \\
\nabla P_{\pm}(0)(\psi f)(x) = 0 & \text{if } |x| \leq R - 1 \\
\end{cases}
\]

This implies \(f = 0\) for \(|x| \geq R\) since

\[
\Delta R_{\pm}(0)(\psi f) + \nabla P_{\pm}(0)(\psi f) = \psi f \quad \text{in } \mathbb{R}^n.
\]

Similarly we get \(f = 0\) for \(x \in \Omega\) with \(|x| \leq R - 1\) since \(-\Delta Lf_b + \nabla Pf_b = f_b\) in \(\Omega_b\). The support of \((R_{\pm}(0)(\psi f), P_{\pm}(0)(\psi f))\) and of \(Lf_b, Pf_b\) is contained in \(\tilde{D} = \{x \in \Omega : R - 1 < |x| < b\}\). Therefore both terms solve the Stokes problem

\[
-\Delta u + \nabla p = f \quad \text{in } \tilde{D} \\
div u = 0 \quad \text{in } \tilde{D} \\
u = 0 \quad \text{on } \partial \tilde{D}.
\]

This implies that \(R_{\pm}(0)(\psi f) = Lf_b\) and \(\nabla P_{\pm}(0)(\psi f) = \nabla Pf_b\) in \(\tilde{D}\) because of the unique solvability of the Stokes equations in a bounded domain. Hence \(Q(z)f = 0, Lf_b = R_2(0)f = 0\) and \(\nabla Pf_b = \nabla \Pi(0)f = 0\) in \(\tilde{D}\) and finally \(f = 0\) in the whole domain \(\Box\).

5. Decay of the semigroup in weighted spaces

Let \(A_q = -P_q\Delta\) denote the Stokes operator for an aperture domain \(\Omega\).

**Theorem 5.1.** Let \(n \geq 3, 1 < \sigma < \frac{n}{2}, 1 < q < \infty, -\frac{n}{q} < s' \leq 0 \leq s < \frac{n}{q}\) and \(s' = s - 2\sigma\). Then there exists a constant \(C = C(q, s, s')\) such that

\[
\|e^{-tA_q}f\|_{L_q(\Omega; \omega_n^{s'q})} \leq C(1 + t)^{-\sigma} \|f\|_{L_q(\Omega; \omega_n^{sq})} \quad (t \geq 0)
\]

for all \(f \in J_q(\Omega) \cap L_q(\Omega; \omega_n^{sq})\). Furthermore,

\[
\|e^{-tA_q}f\|_{W^{q}_q(\Omega; \omega_n^{s'q})} \leq C(1 + t)^{-\sigma} \max \{\|f\|_{W^{q}_q(\Omega)}, \|f\|_{L_q(\Omega; \omega_n^{sq})}\} \quad (t \geq 0)
\]
for all \( f \in \mathcal{D}(A_q) \cap L_q(\Omega; \omega_n^sq) \).

**Proof.** The proof of the inequalities is nearly the same as the proof of [8: Theorem 1.1]. So we give only a sketch.

Since the semigroup \( e^{-tA_q} \) is bounded in \( J_q(\Omega) \), the first estimate is satisfied for \( 0 < t < 1 \). The second estimate holds for \( 0 < t < 1 \) because of the estimates
\[
\|f\|_{W^2_q(\Omega)} \leq c \|(I + A_q)f\|_{L_q(\Omega)} \leq C \|f\|_{W^2_q(\Omega)}
\]
for all \( f \in \mathcal{D}(A_q) \) (the first inequality is a consequence of [4: Theorem 2.1], the second inequality is obvious). For \( t \geq 1 \) consider the representation of the semigroup
\[
e^{-tA_q} = \frac{1}{2\pi i} \int_{\Gamma} e^{tz}(z + A_q)^{-1} dz
\]
where the curve \( \Gamma \) coincides outside a ball \( B_{\varepsilon}(0) \) (\( 0 < \varepsilon < \varepsilon_0 \)) with the rays \( e^{\pm\phi i\tilde{t}} \) (\( \tilde{t} > 0 \)) with \( \pi/2 < \phi < \delta \) (\( \delta \) and \( \varepsilon_0 \) are the same numbers as in Theorem 4.1). We split the curve \( \Gamma \) into two parts
\[
\Gamma_1 = \{ z \in \Gamma : 0 < |z| < \varepsilon \}
\]
\[
\Gamma_2 = \{ z \in \Gamma : \varepsilon \leq |z| \}.
\]
So we get
\[
e^{-tA_q} f = \frac{1}{2\pi i} \int_{\Gamma_1} e^{tz} R(z) f dz + \frac{1}{2\pi i} \int_{\Gamma_2} e^{tz} (z + A_q)^{-1} f dz
\]
for all \( f \in J_q(\Omega) \cap L_q(\Omega; \omega_n^sq) \) since \( R(z) f = (z + A_q)^{-1} f \) for \( z \in \Sigma_{\delta,\varepsilon} \). Using the resolvent estimate \( \|(z + A_q)^{-1} f\|_q \leq C|z|^{-1}\|f\|_q \) we easily get
\[
\left\| \frac{1}{2\pi i} \int_{\Gamma_2} e^{tz} (z + A_q)^{-1} f dz \right\|_{L_q(\Omega; \omega_n^sq)} \leq C \int_\varepsilon^\infty \frac{e^{ts \cos \phi}}{s} ds \|f\|_{L_q(\Omega)}
\]
\[
\leq C(\varepsilon, \phi) \frac{e^{-ct}}{t} \|f\|_{L_q(\Omega; \omega_n^sq)}
\]
with some constant \( C = C(\varepsilon, \phi) > 0 \). Analogously we get
\[
\left\| \frac{1}{2\pi i} \int_{\Gamma_2} e^{tz} (z + A_q)^{-1} f dz \right\|_{W^2_q(\Omega; \omega_n^sq)} \leq C \int_\varepsilon^\infty \frac{e^{ts \cos \phi}}{s} ds \|f\|_{W^2_q(\Omega)}
\]
\[
\leq C(\varepsilon, \phi) \frac{e^{-ct}}{t} \|f\|_{W^2_q(\Omega)}
\]
if we use (20) for \( f \in \mathcal{D}(A_q) \).
We use the resolvent expansion of Theorem 4.1 to estimate the first integral. Since \( \sum_{j=0}^{[\sigma]-1} z^j G_j \) is holomorphic in \( \mathbb{C} \), there holds
\[
\left\| \sum_{j=0}^{[\sigma]-1} \int_{\Gamma_1} e^{tz} z^j G_j \, dz \right\|_{L(L_q(\omega_n^{s''}), W^2_q(\omega_n^{s''})))} \leq C e^{\epsilon t \cos(\phi)} = Ce^{-ct}
\]
with \( C > 0. \) In order to estimate the remainder term we deform the curve \( \Gamma_1 \) to a curve \( \Gamma^* \) which coincides with \( z = e^{\pm \phi i \tilde{t}} \) \( (\tilde{t} \in [0, \epsilon]) \). Therefore
\[
\left\| \frac{1}{2\pi i} \int_{\Gamma_1} e^{tz} G_r(z) \, dz \right\|_{L(L_q(\omega_n^{s''}), W^2_q(\omega_n^{s''})))} \leq C \int_0^\infty e^{\lambda t \cos(\phi)} \lambda^{\sigma-1} \, d\lambda = C' t^{-\sigma}.
\]
Collecting all estimates we proved the theorem \( \blacksquare \)

6. The \( L_q-L_r \)-estimate

In order to get an estimate of \( \|e^{-tA_q}f\|_{L_q(\Omega_b)} \) where \( \Omega_b = \Omega \cap B_b(0) \), we need the following

**Lemma 6.1.** Let \( 1 < q < \infty \) and \( -\frac{n}{q} < s' < 0. \) Then
\[
\|e^{-tA_q}f\|_{L_q(\Omega; \omega_n^{s''}q)} \leq C (1 + t)^{\frac{s'}{2'}} \|f\|_{L_q(\Omega)}
\]
for all \( f \in J_q(\Omega) \) and
\[
\|e^{-tA_q}f\|_{W^2_q(\Omega; \omega_n^{s''}q)} \leq C (1 + t)^{\frac{s'}{2'}} \|f\|_{W^2_q(\Omega)}
\]
for all \( f \in \mathcal{D}(A_q) \).

**Corollary 6.2.** Let \( 1 < q < \infty \). Then for every \( 0 \leq s < \frac{n}{2q} \) there is a constant \( C = C(s, q, \Omega) \) with
\[
\|e^{-tA_q}f\|_{L_q(\Omega_b)} \leq C (1 + t)^{-s} \|f\|_{L_q(\Omega)}
\]
for all \( f \in J_q(\Omega) \) and
\[
\|e^{-tA_q}f\|_{W^2_q(\Omega_b)} \leq C (1 + t)^{-s} \|f\|_{W^2_q(\Omega)}
\]
for all \( f \in \mathcal{D}(A_q) \).
Proof of Lemma 6.1. If $1 < p < \frac{n}{2}$, then $\frac{n}{p} > 2$. So we can apply Theorem 5.1 with $s = 0$. Therefore we get

$$
\|e^{-tA_p}f\|_{W^m_p(\Omega;\omega_n^{\frac{n}{p}})} \leq C(1 + t)^{\frac{\theta}{r}}\|f\|_{W^m_p(\Omega)}
$$

(21)

for $m = 0, 2$, $f \in J_p(\Omega)$ resp. $f \in \mathcal{D}(A_p)$ and $-\frac{n}{p} < \tilde{s}' < -2$. In order to get the statement of the lemma we interpolate estimates (21) and

$$
\|e^{-tA_r}f\|_{W^m_p(\Omega)} \leq C\|f\|_{W^m_p(\Omega)} \quad (m = 0, 2; f \in J_r(\Omega) \text{ resp. } \mathcal{D}(A_r))
$$

(22)

for suitable $p$ close to 1 and large $r$. For this we need the statement about complex interpolation

$$
(L_p(\Omega;\omega_n^{\frac{n}{p}}), L_r(\Omega))_{[\theta]} = L_q(\Omega;\omega_n^{\frac{n}{p}(1-\theta)})
$$

with $0 < \theta < 1$ and $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{r}$ (see, for example, [1: Theorem 5.5.3]).

Now let $1 < q < \infty$ and $-\frac{n}{q} < s' < 0$ be given as in the assumptions. We set $\tilde{s}' = \frac{s'}{1-\theta}$ and $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{r}$ for $0 < \theta < 1$. Then we choose $0 < \theta < 1$ such that

$$
-\frac{n}{p}(1-\theta) < s' < -2(1-\theta) \iff -\frac{n}{p} < s' < -2
$$

which exists if $1 < p < \min\{\frac{n}{2}, q\}$. If we furthermore use $(J_p(\Omega), J_r(\Omega))_{[\theta]} = J_q(\Omega)$ (see Appendix), we get with these chosen $\theta$ and $p$ and the corresponding $r$ that

$$
\|e^{-tA_q}f\|_{L_q(\Omega;\omega_n^{\frac{n}{p}(1-\theta)})} \leq C[(1 + t)^{\frac{\theta}{r}}]^{1-\theta}\|f\|_{L_q(\Omega)} = C(1 + t)^{\frac{\theta}{r}}\|f\|_{L_q(\Omega)}
$$

for $f \in J_q(\Omega)$. Complex interpolation with the same parameters yields the estimate for $f \in \mathcal{D}(A_q)$. For this we use the second estimate of Theorem 5.1 and $(\mathcal{D}(A_p), \mathcal{D}(A_r))_{[\theta]} = \mathcal{D}(A_q)$. The latter equation will be proved in Appendix.

Proof of Theorem 1.1. The proof is similar to that of [8: Theorem 1.2] but a little bit shorter. It is sufficient to show the statement for $0 < \sigma < \frac{1}{2}$ since we can reduce the general case to this statement (choose $q = q_0 < q_1 < \ldots < q_k = r$ such that $\sigma_i := \frac{n}{2}(\frac{1}{q_i} - \frac{1}{q_{i+1}}) < \frac{1}{2}$ and apply the statement to $q_i$ and $q_{i+1}$).

Step 1: The inequality holds for $t \geq 2$. Let $\tilde{u}_0 := e^{-A_q}u_0$. Then $\tilde{u}_0 \in \mathcal{D}(A_q)$ and $\|\tilde{u}_0\|_{W^s_q(\Omega)} \leq C\|u_0\|_{L_q(\Omega)}$. Moreover, let $\tilde{u}(t) := e^{-tA_q}\tilde{u}_0$ and $\tilde{p}(t) \in \tilde{W}^1_q(\Omega)$ be the pressure corresponding to $\tilde{u}(t)$. Let $\Omega \cup B_r(0) = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B_r(0)$ and $b > r + 1$. We choose a cut-off function $\psi \in C^\infty(\Omega)$ with
\[ \psi(x) = 1 \text{ for } |x| \geq b \text{ and } \psi(x) = 0 \text{ for } |x| \leq b-1. \] Then \( \text{div}(\psi \tilde{u}(t)) = \nabla \psi \cdot \tilde{u}(t) \in W^1_{0,q}(D_b) \) with \( D_b = \{ x \in \Omega : b - 1 < |x| < b \} \) and \( \int_{D_b} \nabla \psi \cdot \tilde{u}(t) \, dx = 0. \) Applying Bogovskii’s theorem \([6: \text{Theorem 3.2}]\) we know that there exists a \( v_0(t) \in W^2_{0,q}(D_b) \) with \( \text{div}v_0(t) = \text{div}(\psi \tilde{u}(t)) \) and

\[ \| v_0(t) \|_{W^2_q(D_b)} \leq C \| \tilde{u}(t) \|_{W^1_q(D_b)}. \quad (23) \]

Therefore we have

\[ \| \partial_t v_0(t) \|_{W^1_q(D_b)} \leq C \| e^{-tA_q} A_q \tilde{u}_0 \|_{L_q(D_b)} \leq C(1 + t)^{-\frac{n}{2}} \| \tilde{u}_0 \|_{W^2_q(\Omega)} \quad (24) \]

with an arbitrary \( 0 \leq \tilde{s} < \frac{n}{2q}. \) If we define \( v_1(t) = \psi \tilde{u}(t) - v_0(t), \) it solves the equations

\[ \partial_t v_1(t) - \Delta v_1(t) + \nabla(\psi \tilde{p}(t)) = h(t) \quad \text{in } (0, \infty) \times \mathbb{R}^n_+ \quad (25) \]
\[ \text{div}v_1(t) = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^n_+ \quad (26) \]
\[ v_1(t)|_{\partial \Omega} = 0 \quad \text{in } (0, \infty) \quad (27) \]
\[ v_1(0) = v_1 \quad (28) \]

with \( v_1 = \psi \tilde{u}_0 - v_0(0) \) and

\[ h(t) = -\{2(\nabla \psi) \cdot \nabla + (\Delta \psi)\} \tilde{u}(t) - (\partial_t - \Delta)v_0(t) + (\nabla \psi) \tilde{p}(t). \]

Moreover, \( \text{supp} h(t) \subseteq \overline{D_b}. \) We choose the pressure \( \tilde{p}(t) \) such that \( \int_{D_b} \tilde{p}(t) \, dx = 0. \) If we now apply (23) - (24), Poincaré’s inequality \([2: \text{Theorem 4.19}]\) and Corollary 6.2, we get

\[ \| h(t) \|_{L_q(D_b)} \leq C \left( \| \tilde{u}(t) \|_{W^1_q(D_b)} + \| v_0(t) \|_{W^2_q(D_b)} + \| \partial_t v_0(t) \|_{L_q(D_b)} + \| \tilde{p}(t) \|_{L_q(D_b)} \right) \]
\[ \leq C \left( (1 + t)^{-\frac{\tilde{s}}{2}} \| \tilde{u}_0 \|_{W^2_q(\Omega)} + \| \nabla \tilde{p}(t) \|_{L_q(\Omega)} \right) \]
\[ \leq C \left( (1 + t)^{-\frac{\tilde{s}}{2}} \| \tilde{u}_0 \|_{W^2_q(\Omega)} + \| \partial_t \tilde{u}(t) \|_{L_q(D_b)} + \| \tilde{u}(t) \|_{W^2_q(D_b)} \right) \]
\[ \leq C(1 + t)^{-\frac{\tilde{s}}{2}} \| \tilde{u}_0 \|_{W^2_q(\Omega)} \]

with an arbitrary \( \tilde{s} \) such that \( 0 \leq \tilde{s} < \frac{n}{q}. \)

Let \( E_\pm(t) \) denote the semigroup of the Stokes operator in \( \mathbb{R}^n_+ \) and \( P_\pm \) denote the Helmholtz projection in \( L_q(\mathbb{R}^n_+; \omega_{n}^{\infty}). \) Since \( v_1(t) \) solves (25) - (28), the identity

\[ v_1(t) = E_\pm(t)v_1 + \int_0^t E_\pm(t - \tau)P_\pm h(\tau) \, d\tau \]
holds. Because of Corollary 3.4 and the $L_q - L_r$-estimate in the half space [12: Theorem 3.1] the semigroup $E_{\pm}(t)$ satisfies
\[ \|E_{\pm}(t)f\|_{L_{r}(\mathbb{R}^{n}_{+})} \leq Ct^{-\sigma}\|f\|_{L_{q}(\mathbb{R}^{n}_{+})} \]
\[ \|E_{\pm}(t)f\|_{L_{q}(\mathbb{R}^{n}_{+})} \leq C(1 + t)^{-\frac{s}{2}}\|f\|_{L_{q}(\mathbb{R}^{n}_{+};\omega_{n}^{\alpha})} \]
with $1 < q \leq r < \infty$, $0 \leq s < \frac{n}{q'}$ and $\sigma = \frac{n}{2}(\frac{1}{q} - \frac{1}{r})$ for all $t > 0$ and $f \in J_q(\mathbb{R}^{n}_{+})$ resp. $f \in J_q(\mathbb{R}^{n}_{+};\omega_{n}^{\alpha})$. Using both inequalities we get
\[ \|E_{\pm}(t)f\|_{L_{r}(\mathbb{R}^{n}_{+})} \leq Ct^{-\sigma}\|E_{\pm}\left(\frac{t}{2}\right)f\|_{L_{q}(\mathbb{R}^{n}_{+})} \leq C(1 + t)^{-\frac{s}{2}}\|f\|_{L_{q}(\mathbb{R}^{n}_{+};\omega_{n}^{\alpha})} \]
for $f \in J_q(\mathbb{R}^{n}_{+};\omega_{n}^{\alpha})$ and $t > 0$. Therefore we conclude
\[ \|E_{\pm}(t)v_{1}\|_{L_{r}(\mathbb{R}^{n}_{+})} \leq Ct^{-\sigma}\|v_{1}\|_{L_{q}(\mathbb{R}^{n}_{+})} \leq Ct^{-\sigma}\|\tilde{u}_{0}\|_{L_{q}(\Omega)} \]
and
\[ \left\| \int_{0}^{t} E_{\pm}(t - \tau)P_{\pm}h(\tau) \, d\tau \right\|_{L_{r}(\mathbb{R}^{n}_{+})} \]
\[ \leq C \int_{0}^{t} (t - \tau)^{-\sigma}(1 + t - \tau)^{-\frac{s}{2}}\|P_{\pm}h(\tau)\|_{L_{q}(\mathbb{R}^{n}_{+};\omega_{n}^{\alpha})} \, d\tau \]
\[ \leq C \int_{0}^{t} (t - \tau)^{-\sigma}(1 + t - \tau)^{-\frac{s}{2}}\|h(\tau)\|_{L_{q}(D_{b})} \, d\tau \]
\[ \leq C \int_{0}^{t} (t - \tau)^{-\sigma}(1 + t - \tau)^{-\frac{s}{2}}(1 + \tau)^{-\frac{\tilde{s}}{2}} \, d\tau \|\tilde{u}_{0}\|_{W_{q}^{2}(\Omega)}. \]
We now choose $0 \leq s < \frac{n}{q'}$ and $\sigma \leq \frac{\tilde{s}}{2} < \frac{n}{2q}$ such that $\frac{s}{2} + \frac{\tilde{s}}{2} > 1$, $\frac{s}{2} + \sigma \neq 1$ and $\frac{\tilde{s}}{2} \neq 1$ (this is possible since $\frac{n}{2q} + \frac{n}{2q'} = \frac{n}{2} > 1$). If we apply Lemma A.2 (see Appendix) with this choice of $s$ and $\tilde{s}$, we get
\[ \left\| \int_{0}^{t} E_{\pm}(t - \tau)P_{\pm}h(\tau) \, d\tau \right\|_{L_{r}(\mathbb{R}^{n}_{+})} \leq Ct^{-\sigma}\|\tilde{u}_{0}\|_{W_{q}^{2}(\Omega)} \]
and therefore
\[ \|v_{1}(t)\|_{L_{r}(\mathbb{R}^{n}_{+})} \leq Ct^{-\sigma}\|\tilde{u}_{0}\|_{W_{q}^{2}(\Omega)}. \]
Since $u(t, x) = v_{1}(t, x)$ for all $x \in \Omega \setminus \Omega_{b}$, the previous estimates, Corollary 6.2 and Sobolev’s embedding theorem imply that
\[ \|\tilde{u}(t)\|_{L_{r}(\Omega)} \leq \|\tilde{u}(t)\|_{L_{r}(\Omega_{b})} + \|v_{1}(t)\|_{L_{r}(\Omega \setminus \Omega_{b})} \]
\[ \leq C\left(\|\tilde{u}(t)\|_{W_{q}^{2}(\Omega_{b})} + \|v_{1}(t)\|_{L_{r}(\Omega \setminus \Omega_{b})}\right) \]
\[ \leq Ct^{-\sigma}\|\tilde{u}_{0}\|_{W_{q}^{2}(\Omega)} \]
\[ \leq Ct^{-\sigma}\|f\|_{L_{q}(\Omega)}. \]
Since $\tilde{u}(t) = e^{-(t+1)A_q}u_0$, we have proved the theorem for $t \geq 2$.

**Step 2:** The inequality holds for $t < 2$. The case $t < 2$ is proved in the same way as in the proof of [8: Theorem 1.2] using Sobolev’s embedding theorem and an interpolation method.

**Proof of Theorem 1.2.** Because of the semigroup property of $e^{-tA_q}$ and Theorem 1.1 it suffices to prove the statement for $\sigma = 0$, i.e. $1 < q = r < n$. The proof for the case $t < 2$ uses the same interpolation method as in the proof of Theorem 1.2.

So let $t \geq 2$ and $v_1(t)$, $v_0(t)$, $h(t)$ be the functions used in the proof of Theorem 1.1. Then

$$\nabla v_1(t) = \nabla E_\pm(t)v_1 + \int_0^t \nabla E_\pm(t-\tau)P_\pm h(\tau) \, d\tau.$$ 

The estimate for the Stokes semigroup in $\mathbb{R}^n_\pm$ yields

$$\|\nabla E_\pm(t)v_1\|_{L_q(\mathbb{R}^n_\pm)} \leq Ct^{-\frac{n}{2}}\|v_1\|_{L_q(\mathbb{R}^n_\pm)}.$$ 

Now we choose $0 \leq s < \frac{n}{q}$ and $1 \leq \tilde{s} < \frac{n}{q}$ with $\frac{s}{2} + \frac{\tilde{s}}{2} > 1$, $\frac{\tilde{s}}{2} \neq 1$ and $\frac{1}{2} + \frac{s}{2} \neq 1$. So we get because of Corollary 6.2 and Lemma A.2 (see Appendix)

$$\left\|\int_0^t \nabla E_\pm(t-\tau)P_\pm h(\tau) \, d\tau\right\|_{L_q(\mathbb{R}^n_\pm)} 
\leq C \int_0^t (t-\tau)^{-\frac{1}{2}}(1 + t-\tau)^{-\frac{s}{2}} \|P_\pm h(\tau)\|_{L_q(\mathbb{R}^n_\pm; \omega^{s,q})} d\tau 
\leq C \int_0^t (t-\tau)^{-\frac{1}{2}}(1 + t-\tau)^{-\frac{s}{2}} \|h(\tau)\|_{L_q(\Omega_b)} d\tau 
\leq C \int_0^t (t-\tau)^{-\frac{1}{2}}(1 + t-\tau)^{-\frac{s}{2}}(1 + \tau)^{-\frac{\tilde{s}}{2}} d\tau \|\tilde{u}_0\|_{W^{2,q}_q(\Omega)} 
\leq Ct^{-\frac{1}{2}}\|\tilde{u}_0\|_{W^{2,q}_q(\Omega)}.$$ 

Moreover, let $\tilde{s} = 1 < \frac{n}{q}$. Therefore we get for $t \geq 1$

$$\|\nabla e^{-(t+1)A_q}f\|_{L_q(\Omega)} \leq C \left(\|\nabla \tilde{u}(t)\|_{L_q(\Omega_b)} + \|\nabla v_1(t)\|_{L_q(\mathbb{R}^n_\pm)}\right) 
\leq C \left((1 + t)^{-\frac{s}{2}} + t^{-\frac{1}{2}}\right)\|\tilde{u}_0\|_{W^{2,q}_q(\Omega)} 
\leq Ct^{-\frac{1}{2}}\|f\|_{L_q(\Omega)}.$$ 

Thus the theorem is also true for $t \geq 2$. □
A. Appendix

It remains to prove the necessary technical lemma used in the last section.

**Lemma A.1.** Let \( 1 < p, q, r < \infty \), \( \theta \in (0, 1) \) with \( \frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{p} \) and let \( \Omega \) be an aperture domain. Then

\[
(D(A_r), D(A_p))[\theta] = D(A_q)
\]

\[
(J_r(\Omega), J_p(\Omega))[\theta] = J_q(\Omega).
\]

**Proof.** To prove the first equality we define a continuous projection \( P_q: \mathcal{W}_2^q(\Omega)^n \to D(A_q) \) for arbitrary \( 1 < q < \infty \). For a function \( u \in \mathcal{W}_2^q(\Omega)^n \) let \( (v, p) \in \mathcal{W}_2^q(\Omega)^n \times \mathcal{W}_1^q(\Omega) \) denote the unique solution of the resolvent equations (16) - (19) with right-hand side \( f = (z - \Delta)u \) for some fixed \( z \in \Sigma_\delta \) (see [9: Theorem 2.1]). We set \( P_q u = v \). Then

\[
\|v\|_{\mathcal{W}_2^q(\Omega)} \leq C \|(z - \Delta)u\|_{L_q(\Omega)} \leq C \|u\|_{\mathcal{W}_2^q(\Omega)}.
\]

If \( u \in D(A_q) \), \((u, 0)\) is the unique solution of these equations. Therefore \( P_q \) is a continuous projection on \( D(A_q) \).

If \( u \in \mathcal{W}_r^2(\Omega)^n \cap \mathcal{W}_q^2(\Omega)^n \), the corresponding solutions in \( \mathcal{W}_r^2(\Omega)^n \) and \( \mathcal{W}_q^2(\Omega)^n \) coincide (see [3: Lemma 3.2]). Therefore we can extend \( P_q \) and \( P_r \) to a well-defined projection \( P(u_r + u_q) = P_r u_r + P_q u_q \) on \( \mathcal{W}_r^2(\Omega)^n + \mathcal{W}_q^2(\Omega)^n \) with \( P|_{\mathcal{W}_r^2(\Omega)^n} = P_r \) and \( P|_{\mathcal{W}_q^2(\Omega)^n} = P_q \). Therefore we conclude

\[
D(A_q) = P(\mathcal{W}_r^2(\Omega)^n, \mathcal{W}_q^2(\Omega)^n)[\theta] = (\mathcal{P}_{\mathcal{W}_r^2(\Omega)^n}, \mathcal{P}_{\mathcal{W}_q^2(\Omega)^n})[\theta] = (D(A_r), D(A_p))[\theta].
\]

The second equality immediately follows from the fact that \( P_q = P_r \) on \( J_q(\Omega) \cap J_r(\Omega) \) (see [4: Lemma 3.2]).

**Lemma A.2.** Let \( 0 \leq \alpha < 1 \), \( \beta \geq 0 \), \( \alpha \leq \gamma \), \( \beta + \gamma > 1 \), \( \alpha + \beta \neq 1 \) and \( \gamma \neq 1 \). Then

\[
\int_{0}^{t} (t-s)^{-\alpha} (1+t-s)^{-\beta} (1+s)^{-\gamma} ds \leq Ct^{-\alpha}.
\]

**Proof.** The case \( t \in (0, 1) \) is trivial. For \( t > 1 \) we simply estimate

\[
\int_{0}^{\frac{t}{2}} (t-s)^{-\alpha} (1+t-s)^{-\beta} (1+s)^{-\gamma} ds \leq Ct^{-\alpha - \beta} \int_{0}^{\frac{t}{2}} (1+s)^{-\gamma} ds
\]

\[
\leq Ct^{-\alpha - \beta} \begin{cases} t^{1-\gamma} & \text{if } \gamma < 1 \\ 1 & \text{if } \gamma > 1 \end{cases}
\]

\[
\leq Ct^{-\alpha}.
\]
Similarly we get
\[
\int_{\frac{t}{2}}^{t} (t-s)^{-\alpha}(1+t-s)^{-\beta}(1+s)^{-\gamma}ds \leq Ct^{-\gamma} \left\{ \begin{array}{ll}
t^{1-\alpha-\beta} & \text{if } \alpha + \beta < 1 \\
1 & \text{if } \alpha + \beta > 1
\end{array} \right.
\]
\[ \leq Ct^{-\alpha} \]
and the proof is finished \( \Box \)

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References


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