On the Fredholm Property of the Stokes Operator in a Layer-Like Domain

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Abstract. The Stokes problem is studied in the domain $\Omega \subset \mathbb{R}^3$ coinciding with the layer $\Pi = \{ x = (y, z) : y = (y_1, y_2) \in \mathbb{R}^2, z \in (0, 1) \}$ outside some ball. It is shown that the operator of such problem is of Fredholm type; this operator is defined on a certain weighted function space $D^l_\beta (\Omega)$ with norm determined by a stepwise anisotropic distribution of weight factors (the direction of $z$ is distinguished). The smoothness exponent $l$ is allowed to be a positive integer, and the weight exponent $\beta$ is an arbitrary real number except for the integer set $\mathbb{Z}$ where the Fredholm property is lost. Dimensions of the kernel and cokernel of the operator are calculated in dependence of $\beta$. It turns out that, at any admissible $\beta$, the operator index does not vanish. Based on the generalized Green formula, asymptotic conditions at infinity are imposed to provide the problem with index zero.

Keywords: Stokes equations, layer-like domains, Fredholm property, weighted spaces

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a domain coinciding outside the ball $B_{R_0} = \{ x \in \mathbb{R}^3 : |x| < R_0 \}$ with the infinite layer

$$\Pi = \left\{ x = (y, z) : y = (y_1, y_2) \in \mathbb{R}^2, z \in (0, 1) \right\}. \quad (1.1)$$

For simplicity we assume the boundary $\partial \Omega$ to be smooth. Without loss of generality we also fix $R_0 = 1$. The set $\partial \Omega \setminus B_1$ contains infinite parts of two planes

$$S^{(0)} = \{ x : y \in \mathbb{R}^2, z = 0 \}$$
$$S^{(1)} = \{ x : y \in \mathbb{R}^2, z = 1 \}$$

which form the boundary $\partial \Pi$ of the layer $\Pi$. We consider the Stokes system

$$\begin{aligned}
-\nu \Delta u + \nabla p &= f \\
-\text{div} \ u &= g
\end{aligned} \quad \text{(in } \Omega) \quad (1.2)$$
with the boundary conditions

\[ \mathbf{u} = \mathbf{h} \quad \text{(on } \partial \Omega) \quad (1.3) \]

where

- \( \mathbf{u} = (u_1, u_2, u_3) \) is the velocity field
- \( p \) is the pressure in the fluid
- \( \mathbf{f} = (f_1, f_2, f_3) \) is an external force
- \( g \) is a given scalar-valued function in \( \Omega \)
- \( \mathbf{h} \) is a given vector-valued function on \( \partial \Omega \)
- \( \nu \) is the constant coefficient of viscosity
- \( \nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \)
- \( \Delta = \nabla \cdot \nabla \)
- \( \text{div } \mathbf{u} = \nabla \cdot \mathbf{u} \)
- "·" means the scalar product in \( \mathbb{R}^3 \).

In the previous paper [15] we have studied the properties of solutions \((\mathbf{u}, p)\) to problem (1.2) - (1.3) in a two-parametric scale of weighted function spaces \( \mathcal{D}_{l\beta}^l(\Omega) \) and \( \mathcal{R}_{l\beta}^l(\Omega; \partial \Omega) \) such that the mapping

\[ \mathcal{D}_{l\beta}^l(\Omega) \ni (\mathbf{u}, p) \mapsto \mathcal{S}_{l\beta}(\mathbf{u}, p) = (f, g, h) \in \mathcal{R}_{l\beta}^l(\Omega; \partial \Omega), \quad (1.4) \]

where \( \mathcal{S}_{l\beta} \) is the operator of the Stokes problem (1.2) - (1.3), becomes continuous. In (1.4) \( l \) is a regularity index and \( \beta \) a weight index. The exact definitions of these spaces and their properties are presented in Section 2. In terms of these spaces we have proved (see [15]) regularity results and a coercive estimate for the solution \((\mathbf{u}, p) \in \mathcal{L}_{\beta}^2(\Omega) \times \mathcal{L}_{\beta}^2(\Omega)\) where the latter space consists of functions with finite norm

\[ \| (\mathbf{u}, p); \mathcal{L}_{\beta}^2(\Omega) \times \mathcal{L}_{\beta}^2(\Omega) \| = \left( \int_{\Omega} (1 + |y|^2)\beta (|\mathbf{u}|^2 + |p|^2) \, dx \right)^{\frac{1}{2}}. \]

Moreover, in [15] the asymptotic representation of the solution \((\mathbf{u}, p) \in \mathcal{L}_{\beta}^2(\Omega) \times \mathcal{L}_{\beta}^2(\Omega)\) is constructed.

In this paper we prove the Fredholm property of mapping (1.4), calculate the dimensions of the kernel and cokernel and therefore the index of the operator \( \mathcal{S}_{l\beta} \) in (1.4). Moreover, we derive integral formulae for the coefficients in the asymptotic representation of the solution, which lead to a generalized Green formula. This formula, in particular, furnishes asymptotic conditions at infinity (in the same way as in the paper [16] where the Stokes operator was studied in domains with cylindrical outlets to infinity). Note also that the Fredholm property of the Neumann problem operator for a second order elliptic equation in a layer-like domain was proved in [13].

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2. Weighted function spaces and preliminary results

2.1 Function spaces. Let $G$ be an arbitrary domain in $\mathbb{R}^n$ ($n \geq 2$). As usual, denote by $C^\infty(G)$ the set of all indefinitely differentiable functions in $G$ and let $C_0^\infty(G)$ be a subset of functions from $C^\infty(G)$ with compact supports in $G$. Further, $W^{l,2}(G)$ ($l \geq 0$) indicates the Sobolev space and $W^{l-\frac{1}{2},2}(\partial G)$ ($l \geq 1$) the space of traces on the boundary $\partial G$ of functions from $W^{l,2}(G)$. Besides, $W^{0,2}(G) = L^2(G)$ and $W^{l,2}_{loc}(G)$ consists of functions which belong to $W^{l,2}(K)$ for every compact $K \subset \overline{G}$. The spaces of scalar- and vector-valued functions are not distinguished in notations. The norm of an element $u$ in the function space $X$ is denoted by $\|u; X\|$.

Let $\Omega \subset \mathbb{R}^3$ be a layer-like domain. Denote by $C_0^\infty(\overline{\Omega})$ the subset of functions from $C^\infty(\Omega)$ with compact supports in $\overline{\Omega}$ (functions from $C_0^\infty(\overline{\Omega})$ are equal to zero for large $|x|$, but not necessarily on $\partial \Omega$). We define the norm

$$
\|u; V^l_{\beta}(\Omega)\| = \left( \int_{\Omega} \sum_{|\mu|=0}^l (1 + r^2)^{\beta_l + |\mu|} |\nabla^\mu u(x)|^2 \, dx \right)^{\frac{1}{2}}
$$

with homogeneous isotropic weight distribution. In (2.1) $r = |y|$ ($y \in \mathbb{R}^2$), $x = (y, z) \in \mathbb{R}^3$, $\mu = (\mu_1, \mu_2, \mu_3)$ with $\mu_1, \mu_2, \mu_3 \geq 0$ is a multi-index, and

$$
\nabla^\mu u = \frac{\partial^{|\mu|} u}{\partial x_1^{\mu_1} \partial x_2^{\mu_2} \partial x_3^{\mu_3}} \quad (|\mu| = \mu_1 + \mu_2 + \mu_3).
$$

Analogously,

$$
\|u; V^l_{\beta}(\mathbb{R}^2)\| = \left( \int_{\mathbb{R}^2} \sum_{|\gamma|=0}^l (1 + r^2)^{\beta_l + |\gamma|} |\nabla^\gamma u(y)|^2 \, dy \right)^{\frac{1}{2}}
$$

for functions $u$ depending on $y \in \mathbb{R}^2$ only where $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1, \gamma_2 \geq 0$. The spaces $V^l_{\beta}(\Omega)$ and $V^l_{\beta}(\mathbb{R}^2)$ are the closures of $C_0^\infty(\overline{\Omega})$ and $C_0^\infty(\mathbb{R}^2)$ in norms (2.1) and (2.2), respectively. The spaces $V^l_{\beta}(G)$ with norm (2.1) or (2.2) were first employed by V. A. Kondratiev [1] (Kondratiev spaces) while treating solutions of elliptic boundary value problems in domains $G \subset \mathbb{R}^n$ ($n \geq 2$) with conical outlets to infinity (in this case the weight in (2.1) should be changed to $(1 + |x|^2)$).

Let $\beta \in \mathbb{R}$ and let $l, \kappa$ denote integers such that $l \geq 0$ and $0 \leq \kappa \leq l$. We introduce the space $V^l_{\beta,\kappa}(\Omega)$ as the closure of $C_0^\infty(\overline{\Omega})$ in the norm

$$
\|v; V^l_{\beta,\kappa}(\Omega)\| = \left( \sum_{\alpha + |\gamma| \leq l} \int_{\Omega} (1 + r^2)^{\beta + |\gamma| - (|\gamma| - \kappa)^+} |\partial^\alpha z^{\gamma} v(y, z)|^2 \, dy dz \right)^{\frac{1}{2}}
$$

(2.3)

where $\alpha \geq 0$, $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1, \gamma_2 \geq 0$, $|\gamma| = \gamma_1 + \gamma_2$, $\partial^\alpha z^{\gamma} = \frac{\partial^\alpha}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2}}$, $\partial^\gamma y = \frac{\partial^{|\gamma|}}{\partial y_1^{\gamma_1} \partial y_2^{\gamma_2}}$, and $(t)_+ = \frac{t + |t|}{2}$ is the positive part of $t \in \mathbb{R}$. On the Fredholm Property
As it can be observed in (2.3), differentiation in $z$ does not change the weight multiplier. Differentiation in $y$ of order $|\gamma| \leq \kappa$ increases the weight exponent by $|\gamma|$ (i.e. reflects the Kondratiev distribution of weights [1]). At $|\gamma| = \kappa$ the weight distribution function has a step. Namely, the subtrahend $(|\gamma| - \kappa)_+$ compensates the growth of the weight exponent provided $|\gamma| > \kappa$. In the case of a cone where all directions are equivalent such step-weighted spaces were introduced and investigated in [4, 5].

It is easy to see that $$V^0_{\beta}(\Omega) = V^0_{\beta,0}(\Omega) = L^2_{\beta}(\Omega)$$
while
$$\|v; L^2_{\beta}(\Omega)\| = \left( \int_{\Omega} (1 + r^2)^\beta |v(x)|^2 dx \right)^{\frac{1}{2}}.$$

Finally, for $l \geq 1$ we introduce the trace space $V^{l-\frac{1}{2}}_{\beta,\kappa}(\partial\Omega)$ of functions $v \in V^l_{\beta,\kappa}(\Omega)$ supplied with the norm
$$\|w; V^{l-\frac{1}{2}}_{\beta,\kappa}(\partial\Omega)\| = \inf \{ \|v; V^l_{\beta,\kappa}(\Omega)\| : v = w \text{ on } \partial\Omega \}. \quad (2.4)$$

The trace $w$ on $\partial\Omega$ of $v \in V^l_{\beta,\kappa}(\Omega)$ is forgetting the normal direction $z$ and the weight distribution in the norm of $V^{l-\frac{1}{2}}_{\beta,\kappa}(\partial\Omega)$ turns into an isotropic one while preserving the step property. This becomes evident after using an equivalent norm in $V^{l-\frac{1}{2}}_{\beta,\kappa}(\partial\Omega)$.

**Lemma 2.1** (see [15]). The norm $\|\zeta; V^{l-\frac{1}{2}}_{\beta,\kappa}(\partial\Omega)\| \ (\kappa \leq l)$ is equivalent to
$$\|\zeta\| = \left\{ \|\zeta; W^{l-\frac{1}{2},2}(\partial\Omega \cap B_2)\|^2 + \sum_{j=0}^{1} \left( \sum_{0 \leq |\gamma| \leq l-1} \int_{S^{(j)} \setminus B_1} (1 + r^2)^{\beta + |\gamma| - (|\gamma| - \kappa)_+} |\partial_y^\gamma \zeta(y)|^2 dy \right) + \sum_{|\gamma| = l-1} \int_{S^{(j)} \setminus B_1} \int_{S^{(j)} \setminus B_1} |\partial_y^\gamma ((1 + |\tilde{y}|^2)^{\beta + \kappa} \zeta(\tilde{y})) |^2 |y - \tilde{y}|^{-3} d\tilde{y} dy \right\}^{\frac{1}{2}}. \quad (2.5)$$

In (2.5) integration over $S_0$ and $S_1$ is performed separately in order to avoid confusion. The reason is that for large $r$ the boundary $\partial\Omega$ consists of two non-intersecting parts and the distance in $\mathbb{R}^3$ between two points $y$ and $\tilde{y}$ located one above the other on $S_0$ and $S_1$ is equal to 1, while the distance between them on $\partial\Omega$ is $O(|y|)$. Interpreting the symbol $|y - \tilde{y}|$ appropriately one can delete the first sum over $j$ in (2.5) and replace $S_j \setminus B_1$ by $\partial\Omega \setminus B_1$.

**2.2 Auxiliary propositions.** Below we make use of basic properties of the spaces $V^l_{\beta,\kappa}(\Omega)$ which we collect in this section.
Lemma 2.2 (see [15]). Let \( v \in \mathcal{V}^{l}_{\beta,\kappa}(\Omega) \) \((l \geq 1, 0 \leq \kappa \leq l - 1, \beta \in \mathbb{R})\). Then \( \partial_y v \in \mathcal{V}^{l-1}_{\beta+1,\kappa-1}(\Omega) \) and \( \partial_z v \in \mathcal{V}^{l-1}_{\beta,\kappa}(\Omega) \). There holds the inequality
\[
\| \partial_y v; \mathcal{V}^{l-1}_{\beta+1,\kappa-1}(\Omega) \| + \| \partial_z v; \mathcal{V}^{l-1}_{\beta,\kappa}(\Omega) \| \leq c \| v; \mathcal{V}^{l}_{\beta,\kappa}(\Omega) \|.
\]

Lemma 2.3.

(i) The embeddings
\[
\mathcal{V}^{l}_{\beta,\kappa}(\Omega) \hookrightarrow \mathcal{V}^{l-1}_{\beta,\kappa}(\Omega) \quad (l \geq 1, 0 \leq \kappa \leq l - 1) \tag{2.6}
\]
\[
\mathcal{V}^{l}_{\beta_1,\kappa}(\Omega) \hookrightarrow \mathcal{V}^{l}_{\beta,\kappa}(\Omega) \quad (l \geq 0, 0 \leq \kappa \leq l, \beta_1 > \beta) \tag{2.7}
\]
are continuous.

(ii) If \( l \geq 1, 0 \leq \kappa \leq l - 1 \) and \( \varepsilon > 0 \), then the embedding
\[
\mathcal{V}^{l}_{\beta,\kappa}(\Omega) \hookrightarrow \mathcal{V}^{l-1}_{\beta-\varepsilon,\kappa}(\Omega) \tag{2.8}
\]
is compact.

Proof. Continuity of the embeddings (2.6) - (2.7) follows from the definition of the norm (2.1). Moreover,
\[
\| u; \mathcal{V}^{l-1}_{\beta-\varepsilon,\kappa}(\Omega \setminus B_{2R}) \| \leq c R^{-\varepsilon} \| u; \mathcal{V}^{l}_{\beta,\kappa}(\Omega \setminus B_{R}) \|.
\]
Since \( \mathcal{V}^{l}_{\beta,\kappa}(\Omega \cap B_{2R}) \) coincides with \( W^{l,2}(\Omega \cap B_{2R}) \) algebraically and topologically, well known properties of Sobolev spaces show that the embedding operator (2.8) can be represented as sum of a small operator (as \( R \to \infty \)) and a compact one. Thus (2.8) is compact.

Let us prove one simple interpolation result.

Lemma 2.4. Let \( v \in [\mathcal{V}^{1}_{\beta,0}(\Omega)]^* \), where \([\mathcal{V}^{1}_{\beta,0}(\Omega)]^* \) is the dual space to \( \mathcal{V}^{1}_{\beta,0}(\Omega) \) with respect to the scalar product in \( L^2(\Omega) \). Suppose that \( \nabla v \in L^2_{-\beta}(\Omega) \). Then \( v \in L^2_{-\beta}(\Omega) \) and
\[
\| v; L^2_{-\beta}(\Omega) \|^2 \leq c \left( \| v; [\mathcal{V}^{1}_{\beta,0}(\Omega)]^* \|^2 + \| \nabla v; L^2_{-\beta}(\Omega) \|^2 \right).
\]

Proof. Let us cover the domain \( \Omega \) by the infinite union of "cubes"
\[
Q_{s,k} = \{ x \in \Omega : |x_1 - s|, |x_2 - k| \leq \frac{1}{2} \} \quad (s, k \in \mathbb{Z}).
\]
By [17 : Chapter 3/Lemma 7.1], for any function \( v \in W^{-1,2}(Q_{s,k}) \) with \( \nabla v \in L^2(Q_{s,k}) \) there holds the inclusion \( v \in L^2(Q_{s,k}) \) and the estimate
\[
\| v; L^2(Q_{s,k}) \|^2 \leq c \left( \| v; W^{-1,2}(Q_{s,k}) \|^2 + \| \nabla v; L^2(Q_{s,k}) \|^2 \right)
\]
with constant \( c \) independent of \( s, k \in \mathbb{Z} \). Let us multiply the last inequalities by \( (1 + (s^2 + k^2))^{-\beta} \) and sum them over all \( s, k \in \mathbb{Z} \). Taking into account that \( (1 + r^2) \) is equivalent to \( (1 + (s^2 + k^2)) \) in \( Q_{s,k} \), we obtain
\[
\| v; L^2_{-\beta}(\Omega) \|^2 \leq c \left( \sum_{k,s \in \mathbb{Z}} (1 + (s^2 + k^2))^{-\beta} \| v; W^{-1,2}(Q_{s,k}) \|^2 + \| \nabla v; L^2_{-\beta}(\Omega) \|^2 \right).
\]
Further, the equivalency of the norms \( \| \eta(1 + r^2)^{\beta/2}; W^{1,2}(\Omega) \| \) and \( \| \eta; V_{\beta,0}^1(\Omega) \| \) gives the inequality

\[
\sum_{k,s \in \mathbb{Z}} (1 + (s^2 + k^2))^{-\beta} \| v; W^{-1,2}(Q_{s,k}) \|^2 \leq c \| v; [V_{\beta,0}^1(\Omega)]^* \|^2
\]

which competes the proof of the lemma.

### 2.3 Space \( D_{\beta}^l(\Omega) \) - the domain of the Stokes operator.

We fix some weight and regularity indeces, i.e. numbers \( \beta \in \mathbb{R} \) and \( l \in \mathbb{N}_0 \) and denote by \( D_{\beta}^l(\Omega) \) the space of vector functions \((u, p)\) satisfying the inclusions

\[
\begin{align*}
    u' &\in V_{\beta+1,l}^l(\Omega) & u_3 &\in V_{\beta+2,l-1}^l(\Omega) \quad (2.9) \\
p &\in V_{\beta,l}^l(\Omega) & \partial_z p &\in V_{\beta+2,l-1}^{l-1}(\Omega). \quad (2.10)
\end{align*}
\]

The norm in \( D_{\beta}^l(\Omega) \) is given by the formula

\[
\|(u, p); D_{\beta}^l(\Omega)\| = \|u'; V_{\beta+1,l}^l(\Omega)\| + \|u_3; V_{\beta+2,l-1}^l(\Omega)\| + \|p; V_{\beta,l}^l(\Omega)\| + \|\partial_z p; V_{\beta+2,l-1}^{l-1}(\Omega)\|. \quad (2.11)
\]

Such definition of the space \( D_{\beta}^l(\Omega) \) has been used in the paper [15]. For purposes of this paper it is more convenient to employ the following equivalent definition. Let us represent the pressure function \( p \) as sum

\[
p(x) = p_\perp(y, z) + \overline{p}(y)
\]

where

\[
\overline{p}(y) = \int_0^1 p(y, z) \, dz
\]

is the mean value of \( p \) with respect to \( z \in (0, 1) \). The projection \( p_\perp \) obviously has zero mean value:

\[
p_\perp(y, z) = \overline{p}(y, z) - \overline{p}(y) = \overline{p}(y) - \overline{p} = 0.
\]

Moreover,

\[
\partial_y p_\perp(y, z) = \partial_y p(y, z) - \partial_y \overline{p}(y) = \partial_y \overline{p}(y) - \partial_y \overline{p}(y) = 0.
\]

Hence by the one-dimensional Poincare inequality we obtain \( p_\perp \in L_{\beta+2}^2(\Omega), \partial_y p_\perp \in L_{\beta+3}^2(\Omega) \) and

\[
\|p_\perp; L_{\beta+2}^2(\Omega)\| \leq c \|\partial_z p_\perp; L_{\beta+2}^2(\Omega)\| = c \|\partial_z p; L_{\beta+2}^2(\Omega)\|
\]

\[
\|\partial_y p_\perp; L_{\beta+3}^2(\Omega)\| \leq c \|\partial_z \partial_y p_\perp; L_{\beta+3}^2(\Omega)\|.
\]

Thus \( p_\perp \in V_{\beta+2,l}^l(\Omega) \) and

\[
\|p_\perp; V_{\beta+2,l}^l(\Omega)\| \leq c \|\partial_z p; V_{\beta+2,l-1}^{l-1}(\Omega)\|.
\]
For the mean value $\overline{p}$ we get the inclusion $\overline{p} \in V_{\beta+1,l}^l(\mathbb{R}^2)$ and the estimate

$$\|\overline{p}; V_{\beta+1,l}^l(\mathbb{R}^2)\| \leq c \|p; V_{\beta+l}^l(\Omega)\|.$$ 

Therefore the space $D_{\beta}^l(\Omega)$ may be redefined as space of all vector functions $(u, p)$ such that $u$ satisfies inclusions (2.9) and $p$ admits representation (2.12) with $p \perp \in V_{\beta+2,l}^l(\Omega)$.

An equivalent norm in $D_{\beta}^l(\Omega)$ is given by the formula

$$\|(u, p); D_{\beta}^l(\Omega)\| = \|u'; V_{\beta+1,l}^l(\Omega)\| + \|u_3; V_{\beta+2,l-1}^l(\Omega)\| + \|p'; V_{\beta+l}^l(\Omega)\| + \|\overline{p}; V_{\beta+l}^l(\mathbb{R}^2)\|.$$ 

2.4 Space $R_{\beta}^l(\Omega; \partial \Omega)$ – the range of the Stokes operator. The space $R_{\beta}^l(\Omega; \partial \Omega)$ ($l \geq 1$) consists of triples $(f, g, h)$ such that

$$g \in V_{\beta+2,l-1}^l(\Omega)$$

$$h' \in V_{\beta+1,l}^{l+\frac{1}{2}}(\partial \Omega)$$

$$h_3 \in V_{\beta+2,l-1}^{l+\frac{1}{2}}(\partial \Omega)$$

while $f$ admits the representation

$$f = f_0 + \partial_z f_1 + \nabla \psi$$

with

$$f_0 \in V_{\beta+2,l-1}^{l-1}(\Omega)$$

$$f_1' \in V_{\beta+1,l}^{l}(\Omega)$$

$$f_{13} \in V_{\beta+2,l-1}^{l}(\Omega)$$

$$\psi_\perp \in V_{\beta+2,l}^{l}(\Omega)$$

$$\overline{\psi} \in V_{\beta+l}^{l}(\mathbb{R}^2)$$

The norm in $R_{\beta}^l(\Omega; \partial \Omega)$ is given by

$$\|(f, g, h); R_{\beta}^l(\Omega; \partial \Omega)\| = \inf \left\{ \|f_0; V_{\beta+2,l-1}^{l-1}(\Omega)\| + \|f_1'; V_{\beta+1,l}^{l}(\Omega)\| + \|f_{13}; V_{\beta+2,l-1}^{l}(\Omega)\| + \|\psi_\perp; V_{\beta+2,l}^{l}(\Omega)\| + \|\overline{\psi}; V_{\beta+l}^{l}(\mathbb{R}^2)\| \right\}$$

where the infimum is taken over all representations (2.16). From Lemmata 2.2 and 2.3 we derive the following assertions.
Lemma 2.5. The embeddings
\[
\mathcal{R}_\beta^l(\Omega; \partial\Omega) \hookrightarrow \mathcal{R}_\beta^{l-1}(\Omega; \partial\Omega) \\
\mathcal{R}_\beta^l(\Omega; \partial\Omega) \hookrightarrow \mathcal{R}_\beta^l(\Omega; \partial\Omega)
\]
are continuous.

Theorem 2.1. The operator \( S_{\beta}^l \) of problem (1.2) – (1.3),
\[
\mathcal{D}_{\beta}^l(\Omega) \ni (u, p) \mapsto S_{\beta}^l(u, p) = (f, g, h) \in \mathcal{R}_\beta^l(\Omega; \partial\Omega)
\]
is continuous.

2.5 Coercive estimate for the solution of problem (1.2) - (1.3). The following result is proved in [15].

Theorem 2.2. Let \((u, p) \in L_\beta^2(\Omega) \times L_\beta^2(\Omega)\) be the solution of problem (1.2) – (1.3) with right-hand side \((f, g) \in \mathcal{R}_\beta^l(\Omega; \partial\Omega)\) \((l \geq 1, \beta \in \mathbb{R})\). Then \((u, p) \in \mathcal{D}_{\beta}^l(\Omega)\) and

\[
\| (u, p); \mathcal{D}_{\beta}^l(\Omega) \| \\
\leq c \left( \| (f, g, h); \mathcal{R}_{\beta}^l(\Omega; \partial\Omega) \| + \| u; L_\beta^2(\Omega) \| + \| p_\perp; L_\beta^2(\mathbb{R}^2) \| \right).
\]

In order to prove the Fredholm property of mapping (2.19) we need to transform estimate (2.20) into

\[
\| (u, p); \mathcal{D}_{\beta}^l(\Omega) \| \leq c \left( \| (f, g, h); \mathcal{R}_{\beta}^l(\Omega; \partial\Omega) \| + \| K(u, p); \mathcal{D}_{\beta}^l(\Omega) \| \right)
\]

where \( K \) is a compact operator in \( \mathcal{D}_{\beta}^l(\Omega) \). As shown in [15], the function \( \bar{p} \in L_\beta^2(\mathbb{R}^2) \cap W_{loc}^{1,2}(\mathbb{R}^2) \) satisfies the Poisson equation

\[
-\frac{1}{6} \Delta_y \bar{p}(y) = \mathcal{F}(y) \quad (y \in \mathbb{R}^2)
\]

where
\[
\mathcal{F}(y) = \mathcal{F}^{(1)}(y) + \text{div}' \mathcal{F}^{(2)}(y) + \Delta_y \mathcal{F}^{(3)}(y) + \Delta_y \mathcal{F}^{(4)}(y)
\]
\[
\mathcal{F}^{(0)}(y) = \int_0^1 \partial_z p(y, z) (\frac{1}{6} z - \frac{1}{2} z^2 + \frac{1}{3} z^3) dz
\]
\[
\mathcal{F}^{(1)}(y) = 2\nu \int_0^1 g(y, z) dz
\]
\[
\mathcal{F}^{(2)}(y) = -\nu \int_0^1 f'(y, z) z(z - 1) dz
\]
\[
\mathcal{F}^{(3)}(y) = -\nu \int_0^1 \text{div}' y u'(y, z) z(z - 1) dz.
\]
The inclusion \((f, g, h) \in \mathcal{R}_2^l(\Omega; \partial\Omega)\) furnishes \(f' \in L_{2;+}^{\lambda+1}(\Omega), \text{ div}_y' f' \in L_{2;+}^{\lambda+2}(\Omega)\) and \(g \in L_{2;+}^{\lambda+2}(\Omega)\). Hence, \(\mathcal{F}(1) \in L_{2;+}^{\lambda+2}(\mathbb{R}^2)\), \(\text{ div}_y \mathcal{F}(2) \in L_{2;+}^{\lambda+2}(\mathbb{R}^2)\) and

\[
\|\mathcal{F}(1); L_{2;+}^{\lambda+2}(\mathbb{R}^2)\| + \|\text{ div}_y \mathcal{F}(2); L_{2;+}^{\lambda+2}(\mathbb{R}^2)\| \leq c\|f, g, h; \mathcal{R}_2^l(\Omega; \partial\Omega)\|.
\]

Further, \((u, p) \in \mathcal{D}_2^l(\Omega)\) so that

\[
\begin{align*}
\Delta_y' \text{ div}_y u' \in L_{2;+}^{\lambda+3}(\Omega) & \subset L_{2;+}^{\lambda+1}(\Omega) \\
\delta_z p \in L_{2;+}^{\lambda+2}(\Omega) & \subset L_{2;+}^{\lambda+4}(\Omega) \subset L_{2;+}^{\lambda+2}(\Omega).
\end{align*}
\]

This implies \(\Delta_y' \mathcal{F}(0) \in L_{2;+}^{\lambda+2}(\mathbb{R}^2)\), \(\Delta_y' \mathcal{F}(3) \in L_{2;+}^{\lambda+2}(\mathbb{R}^2)\) and

\[
\begin{align*}
\|\Delta_y' \mathcal{F}(0); L_{2;+}^{\lambda+2}(\mathbb{R}^2)\| + \|\Delta_y' \mathcal{F}(3); L_{2;+}^{\lambda+2}(\mathbb{R}^2)\| & \\
\leq c\left(\|\Delta_y' \text{ div}_y' u'; L_{2;+}^{\lambda+2}(\Omega)\| + \|\Delta_y' (\delta_z p); L_{2;+}^{\lambda+2}(\Omega)\| \right).
\end{align*}
\]

Thus,

\[
\mathcal{F} = \mathcal{F}(1) + \text{ div}_y \mathcal{F}(2) + \Delta_y' (\mathcal{F}(0) + \mathcal{F}(3)) \in L_{2;+}^{\lambda+2}(\mathbb{R}^2)
\]

and

\[
\begin{align*}
\|\mathcal{F}; L_{2;+}^{\lambda+2}(\mathbb{R}^2)\| & \\
\leq c\left(\|f, g, h; \mathcal{R}_2^l(\Omega)\| + \|\Delta_y' \text{ div}_y' u'; L_{2;+}^{\lambda+2}(\Omega)\| + \|\Delta_y' (\delta_z p); L_{2;+}^{\lambda+2}(\Omega)\| \right). \quad (2.23)
\end{align*}
\]

The punctured space \(\mathbb{R}^2 \setminus \{0\}\) might be interpreted as two-dimensional cone (a complete one) in \(\mathbb{R}^2\) so that \(\mathbb{R}^2\) is a domain with conical outlet to infinity. Therefore general theorems on elliptic problems in such domains can be applied while treating the solution \(\overline{p}\) of equation (2.22). It is known (see [1, 2, 12]) that such problems have the Fredholm property in the scale of Kondratie spaces \(V_\gamma^l(\mathbb{R}^2)\) if and only if every power solution \(w(y) = r^{-\lambda}\Psi(\varphi)\) of the corresponding homogeneous problem is trivial, provided that \(\lambda\) lies on the line \(\lambda \in \mathbb{C} : \text{ Re } \lambda = \gamma - l + 1\) \(\{(r, \varphi)\) are polar coordinates in \(\mathbb{R}^2\). For the Laplace operator (2.22) all power solutions consist of harmonic polynomials of orders \(m \in \mathbb{N}_0\) and derivatives of the fundamental solution \(\Gamma(y) = -\frac{1}{2\pi} \ln |y|\). This information together with the general results (see [1, 2, 12]) and estimate (2.23) gives

**Lemma 2.6.** Let \(\overline{p} \in L_{2;+}^2(\mathbb{R}^2) \cap W_{\text{loc}}^{l,2}(\mathbb{R}^2)\) \((l \geq 2, \beta \neq \pm 1)\) be the solution of the Poisson equation (2.22). Then \(\overline{p} \in V_{2;+}^2(\mathbb{R}^2)\) and there holds the inequality

\[
\begin{align*}
\|\overline{p}; V_{2;+}^2(\mathbb{R}^2)\| & \\
\leq c\left(\|\mathcal{F}; L_{2;+}^2(\mathbb{R}^2)\| + \|\mathcal{K}_1\overline{p}; V_{2;+}^2(\mathbb{R}^2)\| \right) \\
\leq c\left(\|f, g, h; \mathcal{R}_2^l(\Omega; \partial\Omega)\| + \|\Delta_y' \text{ div}_y' u'; L_{2;+}^2(\Omega)\| \right. \\
& \left. + \|\Delta_y' (\delta_z p); L_{2;+}^2(\Omega)\| + \|\mathcal{K}_1\overline{p}; V_{2;+}^2(\mathbb{R}^2)\| \right).
\end{align*}
\]

where \(\mathcal{K}_1\) is a compact operator in \(V_{2;+}^2(\mathbb{R}^2)\).
Remark 2.1. Lemma 2.6 remains valid also for \( l = 1 \) and \( l = 0 \). However, because of the shortage of the regularity in these cases the Poisson equation (2.22) for \( \overline{p} \) should be understood in the sense of distributions, i.e. the solution \( \overline{p} \in L^2_\beta(\mathbb{R}^2) \) satisfies the integral identity

\[
-\frac{1}{6} \int_{\mathbb{R}^2} \overline{p}(y) \Delta'_y \eta(y) \, dy = \int_{\mathbb{R}^2} \left( \mathcal{F}^{(1)}(y) \eta(y) - \mathcal{F}^{(2)}(y) \cdot \nabla'_y \eta(y) + (\mathcal{F}^{(0)}(y) + \mathcal{F}^{(3)}(y)) \Delta'_y \eta(y) \right) \, dy
\]

for all \( \eta \in C^\infty_0(\mathbb{R}^2) \) where

\[
\mathcal{F}^{(0)} \in L^2_{\beta+2}(\mathbb{R}^2) \subset L^2_{\beta+1}(\mathbb{R}^2) \\
\mathcal{F}^{(1)} \in L^2_{\beta+2}(\mathbb{R}^2) \subset L^2_{\beta+1}(\mathbb{R}^2) \\
\mathcal{F}^{(2)} \in L^2_{\beta+1}(\mathbb{R}^2) \\
\mathcal{F}^{(3)} \in L^2_{\beta+2}(\mathbb{R}^2) \subset L^2_{\beta+1}(\mathbb{R}^2).
\]

Since results analogous to Lemma 2.6 are true for the solution \( \overline{p} \in L^2_\beta(\mathbb{R}^2) \) of the Poisson identity (2.25) (e.g. [2]: Section 6.3] and [12: Theorems 3.5.7 and 4.2.4]), we conclude the estimate

\[
\| \overline{p} L^2_\beta(\mathbb{R}^2) \| \leq c \left( \| (f, g, h); R^l_\beta(\Omega; \partial \Omega) \| + \| \text{div}'_y u'; L^2_{\beta+1}(\Omega) \| \\
+ \| \partial_z \overline{p}; L^2_{\beta+1}(\Omega) \| + \| \tilde{K}_1 \overline{p}; L^2_\beta(\mathbb{R}^2) \| \right)
\]

(2.26)

where \( \tilde{K}_1 \) is a compact operator in \( L^2_\beta(\mathbb{R}^2) \).

First, let \( l \geq 2 \) and \( \beta \not\in \pm N_0 \). Using inequality (2.24) we can rewrite estimate (2.20) in the form

\[
\|(u, p); D^{l}_\beta(\Omega)\| \leq c \left( \| (f, g, h); R^l_\beta(\Omega; \partial \Omega) \| + \| u; L^2_\beta(\Omega) \| \\
+ \| p_\perp; L^2_\beta(\Omega) \| + \| \Delta'_y \text{div}'_y u'; L^2_{\beta+2}(\Omega) \| \\
+ \| \Delta'_y (\partial_z p); L^2_{\beta+2}(\Omega) \| + \| K_1 \overline{p}; V^2_\beta(\mathbb{R}^2) \| \right)
\]

(2.27)

By Lemma 2.2, \( \Delta'_y \text{div}'_y u' \in V^{l-2}_{\beta+4,l-3}(\Omega) \) and \( \Delta'_y (\partial_z p) \in V^{l-3}_{\beta+4,l-3}(\Omega) \). Moreover, by virtue of Lemma 2.3 the embeddings

\[
V^{l-2}_{\beta+4,l-3}(\Omega) \hookrightarrow L^2_{\beta+2}(\Omega) \\
V^{l-3}_{\beta+4,l-3}(\Omega) \hookrightarrow L^2_{\beta+2}(\Omega) \\
V^{l+1}_{\beta+1,l}(\Omega) \hookrightarrow L^2_\beta(\Omega) \\
V^{l+1}_{\beta+2,l-2}(\Omega) \hookrightarrow L^2_\beta(\Omega) \\
V^l_{\beta+2,l}(\Omega) \hookrightarrow L^2_\beta(\Omega)
\]
are compact. Hence, there hold the inequalities
\[
\|\Delta_y \text{div}' u' \|_{L^2(\Omega)} \leq c \|K_2 u' \|_{V_{\beta+1,t}(\Omega)}
\]
\[
\|\Delta_y (\partial_2 p) \|_{L^2(\Omega)} \leq c \|K_3 p_\perp \|_{V_{\beta+2,t}(\Omega)}
\]
\[
\|(u', u_3) \|_{L^2(\Omega)} \times L^2(\Omega)} \leq c \|K_4 (u', u_3) \|_{V_{\beta+1,t}(\Omega)} \times V_{\beta+2,I}(\Omega)}
\]
\[
\|p_\perp \|_{L^2(\Omega)} \leq c \|K_5 p_\perp \|_{V_{\beta+2,t}(\Omega)}
\]

where \(K_i \ (i = 2, 3, 4, 5)\) are compact operators. Therefore from (2.27) estimate (2.21) follows. In the cases \(l = 0\) and \(l = 1\) we analogously get estimate (2.21) using inequality (2.26) instead of (2.24). Thus, we have proved

**Theorem 2.3.** Let \((u, p) \in D_{\beta}^l(\Omega)\) be the solution of problem (1.2) – (1.3) with right-hand side \(f, g, h) \in R_{\beta}^l(\Omega; \partial\Omega) \ (l \geq 1, \beta \in \mathbb{R} \setminus \{\pm N_0\})\). Then estimate (2.21) holds with \(K\) being a compact operator in \(D_{\beta}^l(\Omega)\).

### 2.6 Asymptotic representation of the solution.

Let us formulate a result concerning the asymptotic behavior of the solution \((u, p)\) of problem (1.2) - (1.3).

**Theorem 2.4 (see [15]).** Assume that
\[
(f, g, h) \in R_{\beta+k}^l(\Omega; \partial\Omega) \ (l \geq 1, \beta \notin \pm N_0, k \in \mathbb{N}).
\]

Then the solution
\[
(u, p) \in L_{\beta}^2(\Omega) \times L_{\beta}^2(\Omega)
\]
of problem (1.2) – (1.3) admits the asymptotic representation
\[
\begin{pmatrix}
    u \\
    p
\end{pmatrix} = \chi(r) \sum_{-\beta-k-1 < m < -\beta-1} \begin{pmatrix}
    c_m^+ u_m^+(y, z) + c_m^- u_m^-(y, z) \\
    c_m^+ p_m^+(y) + c_m^- p_m^-(y)
\end{pmatrix} + \begin{pmatrix}
    \tilde{u} \\
    \tilde{p}
\end{pmatrix}
\]

where \(\chi\) is a smooth cut-off function with \(\chi(r) = 1\) for \(r \geq 2\) and \(\chi(r) = 0\) for \(r \leq 1\),
\[
\begin{aligned}
    u_m^+(y, z) &= \begin{cases}
        \frac{1}{2^{\beta}} z(z - 1) \nabla'_y p_m^+(y), & m \geq 0, \\
        0, & m < 0
    \end{cases} \\
    u_m^-(y, z) &= 0, \\
    p_0^+(y) &= 1, \\
    p_0^-(y) &= -\frac{1}{2\pi} \ln r
    \end{aligned}
\]
\[
\begin{aligned}
    p_m^+(y) &= (2\pi|m|)^{-\frac{1}{2}} r^m \cos(m\varphi) \\
    p_m^-(y) &= (2\pi|m|)^{-\frac{1}{2}} r^m \sin(|m|\varphi)
\end{aligned}
\]
\[
c_m^\pm \ (m \in \pm N_0) \text{ are constants and } (\tilde{u}, \tilde{p}) \in D_{\beta+k}^l(\Omega). \text{ There holds the estimate}
\]
\[
\|\begin{pmatrix}
    \tilde{u} \\
    \tilde{p}
\end{pmatrix} \|_{D_{\beta+k}^l(\Omega)} + \sum_{-\beta-k-1 < m < -\beta-1} (|c_m^+| + |c_m^-|)
\leq c \left( \|f, g, h) \|_{R_{\beta+k}^l(\Omega; \partial\Omega)} + \|u \|_{L_{\beta}^2(\Omega)} + \|p_\perp \|_{L_{\beta}^2(\Omega)} + \|\tilde{p} \|_{L_{\beta}^2(\mathbb{R}^2)} \right)
\]

**Remark 2.2.** Analogous asymptotic formulae were obtained also for second order scalar elliptic operators (see [9, 11]) and for the Lame operator (see [6 - 8, 10]).
2.7 Green’ formula. Let \((u, p) \in D^l_\beta(\Omega)\) and \((v, q) \in C_0^\infty(\Omega)\). Then for the Stokes problem (1.2) – (1.3) there holds Green’ formula

\[
\int_\Omega (-\nu \Delta u + \nabla p) \cdot v \, dx - \int_\Omega q \, \text{div} u \, dx + \int_{\partial \Omega} u \cdot (nq - \nu \partial_n v) \, ds = \int_\Omega (-\nu \Delta v + \nabla q) \cdot u \, dx - \int_\Omega p \, \text{div} v \, dx + \int_{\partial \Omega} v \cdot (np - \nu \partial_n u) \, ds.
\]

(2.33)

Here \(n\) is the unit vector of the outward normal to \(\partial \Omega\) and \(\partial_n = \frac{\partial}{\partial n}\) denotes the derivative with respect to \(n\). Note that all integrals in (2.33) are finite since \((v, q)\) is identically zero for large \(|x|\). It is not difficult to verify that the integrals in (2.33) remain finite if \((v, q) \in D^l_{-\beta - 2}(\Omega)\). Therefore by continuity we conclude the following assertion.

**Lemma 2.7.** Green’ formula (2.33) holds true for any pairs \((u, p) \in D^l_\beta(\Omega)\) and \((v, q) \in D^l_{-\beta - 2}(\Omega)\).

3. The Fredholm property

In this section we prove the main result of the paper: the Fredholm property of the Stokes operator \(S^l_\beta\), i.e. we prove that the range \(S^l_\beta D^l_\beta(\Omega)\) is a closed subspace of \(R^l_\beta(\Omega; \partial \Omega)\) and that

\[
\dim \ker S^l_\beta < \infty
\]

\[
\dim \text{coker } S^l_\beta < \infty.
\]

**Theorem 3.1.** The operator \(S^l_\beta\) \((l \geq 1)\) of the Stokes problem (1.2) – (1.3) is of Fredholm type, if \(\beta \notin \mathbb{Z}\). If \(\beta \in \mathbb{Z}\), then the range of \(S^l_\beta\) is not closed.

**Proof.** The finite-dimensionality of \(\ker S^l_\beta\) and the closedness of the range \(S^l_\beta D^l_\beta(\Omega)\) follow from estimate (2.21) (see Theorem 2.3) and a lemma by J. Peetre (see [18] or [3: Lemma 2.5.1]).

Let us prove the finite-dimensionality of \(\text{coker } S^l_\beta\). We show that the subspace \(\ker (S^l_\beta)^* = \text{coker } S^l_\beta\) admits the representation

\[
\text{coker } S^l_\beta = \left\{ (v, q, (nq - \nu \partial_n v)|_{\partial \Omega}) : (v, q) \in \ker S^l_{-\beta - 2} \right\}.
\]

(3.1)

Let us consider the bounded linear functional \(F_{(v, q)}\) given on \(R^l_\beta(\Omega; \partial \Omega)\) by the formula

\[
F_{(v, q)}(f, g, h) = \int_\Omega f \cdot v \, dx - \int_\Omega g q \, dx + \int_{\partial \Omega} h \cdot (nq - \nu \partial_n v) \, ds
\]

(3.2)

\((v, q) \in D^l_{-\beta - 2}(\Omega)\).

If \((f, g, h) \in S^l_\beta D^l_\beta(\Omega)\) and \((v, q) \in \ker S^l_{-\beta - 2}\), then from Green’s formula (2.33) it follows that \(F_{(v, q)}(f, g, h) = 0\). Thus \(F_{(v, q)}\) is orthogonal to \(S^l_\beta D^l_\beta(\Omega)\) and therefore \(F_{(v, q)} \in \ker (S^l_\beta)^*\). Hence we have proved the inclusion

\[
\left\{ (v, q, (nq - \nu \partial_n v)|_{\partial \Omega}) : (v, q) \in \ker S^l_{-\beta - 2} \right\} \subset \ker (S^l_\beta)^*.
\]

(3.3)
In order to prove the inverse inclusion we first consider the case \( l = 1 \) and introduce the operator \( S^*_{\beta} \) adjoint to \( S_{\beta} \) (with respect to the scalar product in \( L^2(\Omega)^4 \times L^2(\partial \Omega)^3 \)).

For brevity we write \( S_{\beta}, D_{\beta}(\Omega) \) etc., omitting the regularity index \( l = 1 \). We mention as well known fact (see, e.g., [3, 19]) that the operator \( S^*_{\beta} \) acts on the space of distributions by the formula

\[
R_{\beta}(\Omega; \partial \Omega)^* \ni (v, q, w) \mapsto S^*_{\beta}(v, q, w) = S(\pi_{\Omega}v, \pi_{\Omega}q) + w \otimes \delta_{\partial \Omega}.
\]

Here \( \pi_{\Omega}v \) and \( \pi_{\Omega}q \) are the extensions of \( v \) and \( q \), respectively, by zero from \( \Omega \) to the entire \( \mathbb{R}^3 \); \( \delta_{\partial \Omega} \) is the Dirac function concentrated on \( \partial \Omega \) so that \( w \otimes \delta_{\partial \Omega} \) is the distribution defined by the formula

\[
(w \otimes \delta_{\partial \Omega}, \varphi)_{\mathbb{R}^3} = (w, \varphi)_{\partial \Omega} \quad (\varphi \in C_0^\infty(\mathbb{R}^3))
\]

where \((\cdot, \cdot)_{\partial \Omega}\) denotes the scalar product in \( L^2(\partial \Omega) \), and

\[
S(\pi_{\Omega}v, \pi_{\Omega}q) = (-\nu \Delta \pi_{\Omega}v + \nabla \pi_{\Omega}q; -\text{div} \pi_{\Omega}v)
\]

is the Stokes operator (1.2). Note that due to Green’s formula (2.33) this operator is formally self-adjoint.

Let \( \omega, \tilde{\omega} \) be two neighbourhoods of a point in \( \overline{\Omega} \) and \( \tilde{\omega} \subset \tilde{\omega}. \) If the right-hand side \( U = (U_1, U_2, U_3, U_4) \) of the equation

\[
S^*_{\beta}(v, q, w) = U \in D_{\beta}(\Omega)^*
\]

belongs to \( H^s(\Omega \cap \tilde{\omega})^3 \times H^{s+1}(\Omega \cap \tilde{\omega}) \), then first \((v, q) \) belongs to \( H^{s+2}(\Omega \cap \omega)^3 \times H^{s+1}(\Omega \cap \omega) \), second it satisfies the relations \( S(v, q) = U \) in \( \Omega \cap \omega \) and \( v = 0 \) on \( \partial \Omega \cap \omega \), and third \( w \) coincides with the trace of \((nq - \nu \partial_n v)\) on \( \partial \Omega \cap \omega \) (see [19] and [3: Chapter 2.5.3]). Since \( \ker S^*_{\beta} \) contains the solutions \((v, q, w) \in R_{\beta}(\Omega; \partial \Omega)^* \) of the homogeneous equation (3.4) (i.e. \( U = 0 \)), we conclude that \((v, q) \in C_{loc}^\infty(\Omega) \) solves the homogeneous Stokes problem (1.2) - (1.3) and \( w \) is the trace of \((nq - \nu \partial_n v)\) on \( \partial \Omega \). Further, by definition \( R_{\beta}(\Omega; \partial \Omega) \) contains the subspace

\[
R = L^2_{\beta+2}(\Omega)^3 \times [V^1_{\beta+2,0}(\Omega) \times V^3_{\beta+1,1}(\partial \Omega)^2 \times V^3_{\beta+2,0}(\partial \Omega)]^*
\]

(we assume that \( f_1 = 0 \) and \( \psi = 0 \) in representation (2.16) for \( f \), i.e. \( f = f_0 \)). Consequently, \( R_{\beta}(\Omega; \partial \Omega)^* \subset R^* \). The first two factors in \( R^* \) coincide with \( L^2_{\beta-2}(\Omega)^3 \times [V^1_{\beta+2,0}(\Omega)]^* \) and hence we have \( v \in L^2_{\beta-2}(\Omega)^3 \) and \( q \in [V^1_{\beta+2,0}(\Omega)]^* \).

Let us show that \( q \) belongs to \( L^2_{\beta-2}(\Omega) \). Denote by \( \zeta_\rho \) the smooth cut-off function with \( \zeta_\rho(r) = 1 \) for \( r \leq \rho \), \( \zeta_\rho(r) = 0 \) for \( r \geq 2\rho \) and

\[
\begin{align*}
|\nabla \zeta_\rho(r)| & \leq c(1 + r^2)^{-\frac{1}{2}} \\
|\nabla \nabla \zeta_\rho(r)| & \leq c(1 + r^2)^{-1}
\end{align*}
\]

(3.5)
with constant $c$ independent of $\rho$ and $r$. We multiply the homogeneous Stokes equations (1.2) by $\zeta_\rho(r)^2(1 + r^2)^{-\beta-1} \mathbf{v}(x)$ and integrate by parts in $\Omega$:

$$
\nu \int_{\Omega} \zeta_\rho(r)^2(1 + r^2)^{-\beta-1} |\nabla \mathbf{v}(x)|^2 \, dx
= \int_{\Omega} q \mathbf{v}(x) \cdot \nabla [\zeta_\rho(r)^2(1 + r^2)^{-\beta-1}] \, dx
- \nu \int_{\Omega} \nabla \mathbf{v}(x) \cdot \mathbf{v}(x) \nabla [\zeta_\rho(r)^2(1 + r^2)^{-\beta-1}] \, dx
= I_1 + I_2.
$$

Using (3.5) it is easy to show that

$$
|I_2| \leq \frac{\nu}{4} \int_{\Omega} \zeta_\rho(r)^2(1 + r^2)^{-\beta-1} |\nabla \mathbf{v}(x)|^2 \, dx + c(\nu) \int_{\Omega} (1 + r^2)^{-\beta-2} |\mathbf{v}(x)|^2 \, dx. \quad (3.7)
$$

For the first summand $I_1$ we get

$$
|I_1| \leq \|q; [V_{\beta+2,0}^1(\Omega)]^*\| \|\nabla [\zeta_\rho(r)^2(1 + r^2)^{-\beta-1}]; V_{\beta+2,0}^1(\Omega)\|
\leq c \|q; [V_{\beta+2,0}^1(\Omega)]^*\|
\times \left( \int_{\Omega} (1 + r^2)^{-\beta-2} |\mathbf{v}|^2 \, dx + \nu \int_{\Omega} \zeta_\rho^2(1 + r^2)^{-\beta-1} |\nabla \mathbf{v}|^2 \, dx \right)^{\frac{1}{2}}
\leq \frac{\nu}{4} \int_{\Omega} \zeta_\rho^2(1 + r^2)^{-\beta-1} |\nabla \mathbf{v}|^2 \, dx
+ c(\nu) \left( \|q; [V_{\beta+2,0}^1(\Omega)]^*\|^2 + \int_{\Omega} (1 + r^2)^{-\beta-2} |\mathbf{v}|^2 \, dx \right). \quad (3.8)
$$

Substituting (3.7), (3.8) into (3.6) we derive the estimate

$$
\int_{\Omega} \zeta_\rho^2(1 + r^2)^{-\beta-1} |\nabla \mathbf{v}|^2 \, dx \leq c \left( \|q; [V_{\beta+2,0}^1(\Omega)]^*\|^2 + \int_{\Omega} (1 + r^2)^{-\beta-2} |\mathbf{v}|^2 \, dx \right)
< \infty \quad (3.9)
$$

with constant $c$ independent of $\rho$. Passing in (3.9) $\rho \to \infty$, we get $\nabla \mathbf{v} \in L^2_{-\beta-1}(\Omega)$. Since the solution $(\mathbf{v}, p)$ is smooth, from local estimates it follows (see [15: Proof of Lemma 3.1]) that $\nabla q \in L^2_{-\beta}(\Omega) \subset L^2_{-\beta-2}(\Omega)$ and

$$
\|\nabla q; L^2_{-\beta}(\Omega)\| \leq c \|\nabla \mathbf{v}; L^2_{-\beta-1}(\Omega)\|.
$$

By Lemma 2.4 we conclude that $q \in L^2_{-\beta-2}(\Omega)$ and

$$
\|q; L^2_{-\beta-2}(\Omega)\| \leq c \left( \|q; [V_{\beta+2,0}^1(\Omega)]^*\| + \|\nabla q; L^2_{-\beta-2}(\Omega)\| \right) < \infty.
$$
Thus the solution \( (v, p) \) of the homogeneous Stokes problem (1.2) - (1.3) belongs to \( L^2_{-\beta-2}(\Omega)^3 \times L^2_{-\beta-2}(\Omega) \). By Theorem 2.2, \( (v, p) \) belongs to \( \mathcal{D}_{-\beta-2}(\Omega) \) and hence

\[
\ker S^l_\beta \subset \left\{ (v, q, (nq - \nu \partial_n v)|_{\partial \Omega}) : (v, q) \in \ker S_{-\beta-2} \right\}.
\] (3.10)

Formulae (3.3) and (3.10) prove representation (3.1) of \( \ker S_\beta \). Since the numbers \( \beta \) and \( -\beta - 2 \) belong to the prohibited set \( \mathbb{Z} \) simultaneously, \( \dim \ker S_{-\beta-2} < \infty \) and the finite-dimensionality of \( \ker S_\beta \) is proved. Moreover, from (3.2) and Green’s formula (2.33) we derive the following compatibility conditions for the Stokes problem (1.2) - (1.3):

\[
\int_{\Omega} f \cdot v \, dx - \int_{\Omega} g \, q \, dx + \int_{\partial \Omega} h \cdot (nq - \nu \partial_n v) \, ds = 0
\] (3.11)

for all \( (v, p) \in \ker S_{-\beta-2} \).

Let us consider the case \( l > 1 \). Assume that \( (f, g, h) \in \mathcal{R}^l_\beta(\Omega; \partial \Omega) \subset \mathcal{R}^1_\beta(\Omega; \partial \Omega) \) with \( \beta \notin \mathbb{Z} \). If the right-hand side \( (f, g, h) \) satisfies the compatibility conditions (3.11), then there exists a solution \( (u, p) \in \mathcal{D}^l_\beta(\Omega) \) of problem (1.2) - (1.3). By virtue of Theorem 2.2 we get \( (u, p) \in \mathcal{D}^l_\beta(\Omega) \). This means that \( (f, g, h) \) is orthogonal to \( \ker [S^l_\beta]^* \). By the Hahn-Banach theorem this gives

\[
\ker [S^l_\beta]^* \subset \left\{ (v, q, (nq - \nu \partial_n v)|_{\partial \Omega}) : (v, q) \in \ker S^l_{-\beta-2} \right\}.
\]

Since by Theorem 2.2 \( \ker S^l_{-\beta-2} = \ker S^l_{-\beta-2} \), the last relation together with (3.3) furnishes

\[
\ker [S^l_\beta]^* = \left\{ (v, q, (nq - \nu \partial_n v)|_{\partial \Omega}) : (v, q) \in \ker S^l_{-\beta-2} \right\}.
\] (3.12)

Thus in the case \( \beta \notin \mathbb{Z} \)

\[
\dim \ker [S^l_\beta]^* = \dim \ker S^l_{-\beta-2} < \infty.
\]

This proves the Fredholm property for \( S^l_\beta \) with \( l > 1 \) and \( \beta \notin \mathbb{Z} \).

Consider now the case \( \beta \in \mathbb{Z} \). Since \( \mathcal{D}^l_\beta(\Omega) \subset \mathcal{D}^l_{\beta-\varepsilon}(\Omega) \) and \( \mathcal{R}^l_\beta(\Omega; \partial \Omega) \subset \mathcal{R}^l_{\beta-\varepsilon}(\Omega; \partial \Omega) \) for all \( \varepsilon > 0 \), it follows that

\[
\ker S^l_\beta \subset \ker S^l_{\beta-\varepsilon}
\]

\[
\text{coker } S^l_\beta \subset \text{coker } S^l_{\beta+\varepsilon}.
\]

Consequently, the subspaces \( \ker S^l_\beta \) and \( \text{coker } S^l_\beta \) are finite-dimensional for all \( \beta \in \mathbb{R} \). We shall show that for \( \beta \in \mathbb{Z} \) the range \( \text{Im } S^l_\beta \) is not closed and hence \( S^l_\beta \) looses the Fredholm property.

Let \( \beta = -m - 1 \) \( (m \in \mathbb{Z}) \). Denote by \( \chi \) the smooth cut-off function with \( \chi(r) = 1 \) for \( r < 1 \) and \( \chi(r) = 0 \) for \( r > 2 \) and let \( \chi_R(r) = \chi(\frac{r}{R}) \) \( (R \geq 2) \). We take

\[
p_0(y) = -(2\pi)^{-1} \ln r
\]

\[
p_m(y) = (2\pi|m|)^{-\frac{1}{2}} r^m \cos(m\varphi) \quad (m \neq 0)
\]

\[
u_m(y, z) = \frac{1}{2\pi} \varepsilon(z - 1) \nabla p_m(y)
\]
and put
\[(\hat{u}_m, \hat{p}_m) = (1 - \chi(r))\chi_R(u_m, p_m).\]
It is easy to compute that
\[
\|D^l_{-m-1}(\Omega)\|_2^2 \\
\geq \|D^l_{-m-1}(\Omega)\|_2^2 \\
\geq c\left(1 + \int^R_{\frac{2R}{2}} r^{-2m}r^{2(m-1)} + r^{-2(m+1)}r^{2m}r \, dr\right) \\
\geq c\left(1 + \ln \frac{R}{2}\right).
\]
(3.13)

On the other hand, \((u_m, p_m)\) satisfies the homogeneous Stokes problem (1.2) - (1.3) in \(\Omega \setminus \{x : r = 0\}\). Therefore
\[-\nu \Delta \hat{u}_m + \nabla \hat{p}_m = [-\nu \Delta + \nabla, (1 - \chi)\chi_R](u_m, p_m) \equiv f_m \quad (x \in \Omega) \\
\text{div} \hat{u}_m = [\text{div}, (1 - \chi)\chi_R]u_m \equiv g_m \quad (x \in \Omega) \\
\hat{u}_m = 0 \quad (x \in \partial \Omega)
\]
where \([A, B]\) stands for the commutator of the operators \(A\) and \(B\). The right-hand side \((f_m, g_m)\) has a compact support lying in the union of the annuli \(\{x \in \Omega : 1 < r < 2\}\) and \(\{x \in \Omega : R < r < 2R\}\). Calculating the norm \(\|(f_m, g_m)\|_2^2\), \(\mathcal{R}_{-m-1}(\Omega; \partial \Omega)\|_2^2\), we find that it is bounded by the expression
\[
c\left(1 + \int^R_{\frac{2R}{2}} R^{-2}r^{-2m}r^{2m}r \, dr\right) \leq \text{const}
\]
(3.14)
where \(c\) is independent of \(R \in (2, \infty)\). The range \(\text{Im} \mathcal{S}_{-m-1}^l\) is closed if and only if for every \((v, q) \in D^l_{-m-1}(\Omega) \odot \ker \mathcal{S}_{-m-1}^l\) the estimate
\[
\|D^l_{-m-1}(\Omega)\|_2 \leq c_\ast \|\mathcal{S}_{-m-1}^l(v, q); \mathcal{R}_{-m-1}^l(\Omega; \partial \Omega)\|_2
\]
holds true with constant \(c_\ast\) independent of \((v, q)\). Letting \(R \to \infty\) in formulae (3.14) and (3.13) we see that for \((\hat{u}_m, \hat{p}_m)\) the last estimate fails, i.e. \(\text{Im} \mathcal{S}_{-m-1}^l\) is not closed. The theorem is proved \(\blacksquare\)

Lemma 3.1. If \(\beta \geq -1\), then \(\mathcal{S}_{\beta}^l\) is a monomorphism, and if \(\beta < -1\), then \(\mathcal{S}_{\beta}^l\) is an epimorphism.

Proof. Let \(\beta \geq -1\) and \((u, p) \in \ker \mathcal{S}_{\beta}^l\). Multiplying the homogeneous equations (1.2) by \(u\) and integrating by parts in \(\Omega\), we derive
\[
\nu \int_\Omega |\nabla u(x)|^2dx = 0.
\]
(Note that by definition of the space \(D_{\beta}^l(\Omega)\) all the integrals involved converge for \(\beta \geq -1\).) From (3.15) it follows \(|\nabla u(x)| = 0\) and hence \(u(x) = 0\). The Stokes equations (1.2) imply \(\nabla p = 0\) in \(\Omega\), i.e. \(p(x) = c\). If \(c \neq 0\), then the integral \(\int_\Omega (1 + r^2)\beta |c|^2dx\) diverges (recall that \(\beta \geq -1\)) what contradicts with the condition \(p \in L^2_\beta(\Omega)\). Thus \(c = 0\) and \(\ker \mathcal{S}_{\beta}^l = 0\) for \(\beta \geq -1\). For \(\beta < -1\) the relation \(\dim \ker \mathcal{S}_{\beta}^l = 0\) follows from (3.12), since in this case \(-2 - \beta > -1\) and \(\ker \mathcal{S}_{-2-\beta}^l = 0\) \(\blacksquare\)
4. Coefficients in the asymptotics and computation of the index

Let \((u, p) \in \mathcal{D}'_\beta(\Omega) \ (\beta > -1)\) be a solution of the Stokes problem (1.2) - (1.3) with right-hand side \((f, g, h) \in \mathcal{R}^l_{\beta+k}(\Omega; \partial\Omega) \ (k \in \mathbb{N})\). From Theorem 2.4 it follows that the solution \((u, p)\) admits the asymptotic representation (2.30) - (2.31). On the other hand, by Lemma 3.1 we know that the operator \(S^l_{\beta}\) with \(\beta > -1\) is a monomorphism, i.e. the solution is unique. Therefore, the coefficients \(c^\pm_m \ (m \in \mathbb{N})\) in the asymptotic formulae (2.30) - (2.31) are uniquely determined by the right-hand side \((f, g, h)\). In this section we find integral formulae for the coefficients \(c^\pm_0\) and \(c^\pm_m \ (m \in \mathbb{N})\).

We start with the computation of \(c^+_0\).

**Lemma 4.1.** Let \((u, p) \in \mathcal{D}'_\beta(\Omega), \beta \in (-2, -1)\), be a solution of problem (1.2) - (1.3) with right-hand side \((f, g, h) \in \mathcal{R}^l_{\beta+1}(\Omega; \partial\Omega)\). Then the coefficient \(c^+_0\) in the asymptotic formula
\[
\begin{pmatrix} u(x) \\ p(x) \end{pmatrix} = \chi(r) \begin{pmatrix} c^+_0 u^+_0(y, z) + c^-_0 u^-_0(y, z) \\ c^+_0 p^+_0(y) + c^-_0 p^-_0(y) \end{pmatrix} + \begin{pmatrix} \hat{u}(x) \\ \hat{p}(x) \end{pmatrix}
\]
where \((\hat{u}, \hat{p}) \in \mathcal{D}'_{\beta+1}(\Omega)\) (see (2.30)) admits the integral representations
\[
c^-_0 = -12\nu \left( \int_{\partial\Omega} h \cdot n \, ds - \int_\Omega g \, dx \right).
\]

**Proof.** Let us apply to the solutions \((u, p)\) and \((u^+_0, p^+_0)\) Green’s formula in the domain \(\Omega_R = \{x \in \Omega : r < R \ (R > 2)\}\):
\[
\begin{align*}
&\int_{\Omega_R} (-\nu \Delta u + \nabla p) \cdot 0 \, dx - \int_{\partial\Omega_R \cup S_R} \text{div} u \, dx + \int_{\partial\Omega_R \cup S_R} u \cdot n \, ds = 0,
\end{align*}
\]
where \(\partial\Omega_R = \partial\Omega \cap \Omega_R\) and \(S_R = \{x \in \Omega : r = R\}\). This furnishes
\[
- \int_{\Omega_R} g \, dx + \int_{\partial\Omega_R} h \cdot n \, ds + \int_{S_R} u \cdot n \, ds = 0.
\]
Taking into account representation (4.1) for \(u\), we compute
\[
\begin{align*}
\int_{S_R} u \cdot n \, ds &= c^-_0 \int_{S_R} u^-_0 \cdot n \, ds + \int_{S_R} \hat{u} \cdot n \, ds \\
&= -\frac{c^-_0}{4\nu \pi} \int_{S_R} z(z-1)\nabla \ln r \cdot \nabla r \, ds + \int_{S_R} \hat{u} \cdot n \, ds \\
&= \frac{c^-_0}{12\nu} + \int_{S_R} \hat{u} \cdot n \, ds.
\end{align*}
\]
Since \(\hat{u} \in L^2_{\beta+2}(\Omega), \beta \in (-2, -1)\), we get
\[
\left| \int_{S_R} \hat{u} \cdot n \, ds \right| \leq c \left( R^{-2(\beta+2)+1} \int_{S_R} (1+r)^{2(\beta+2)} |\hat{u}|^2 \, ds \right)^{\frac{1}{2}}
\leq c \left( R \int_{S_R} (1+r)^{2(\beta+2)} |\hat{u}|^2 \, ds \right)^{\frac{1}{2}}
\]
\[
= o(R^{-1}) \to 0 \quad \text{as} \ R \to \infty
\]
(at least for some subsequence \(R_l\)). Substituting the last two formulae into (4.3) and passing to the limit as \(R_l \to \infty\), we derive (4.2) \(\blacksquare\).
In the previous lemma we have already used a special solution of the homogeneous Stokes problem \( \zeta_0^+ (x) = (u_0^+(y, z), p_0^+(y))^T = (0, 1)^T \). Let us construct special solutions \( \zeta_m^+ = (\xi_m^+, \eta_m^+)^T \) for \( m \in \mathbb{N} \).

**Lemma 4.2.** For every \( m \in \mathbb{N} \) there exist solutions \( \zeta_m^+ \) of the homogeneous Stokes problem (1.2) - (1.3) which admit the asymptotic forms

\[
\zeta_m^+ = \left( \begin{array}{c} \xi_m^+ (x) \\ \eta_m^+ (x) \end{array} \right) = \left( \begin{array}{c} u_m^+ (y, z) \\ p_m^+ (y) \end{array} \right) + \left( \begin{array}{c} \xi_m^+ (x) \\ \eta_m^+ (x) \end{array} \right) \quad (m \in \mathbb{N})
\]

where \( (u_m^+(y, z), p_m^+(y)) \) are given by (2.31) and \( (\xi_m^+, \eta_m^+) \in \mathcal{D}_\gamma^l (\Omega) \) with arbitrary \( \gamma \) satisfying the relation

\[
-1 < \gamma < 0.
\]

**Proof.** We shall look for the solution \( (\xi_m^+, \eta_m^+) \) in form (4.4). Since \( (u_m^+, p_m^+) \) solve the homogeneous Stokes problem (1.2) - (1.3) in the layer \( \Pi \), we obtain for \( (\xi_m^+, \eta_m^+) \) the non-homogeneous problem (1.2) - (1.3) with right-hand side \( (0, 0, h_m^+) \) where \( h_m^+ = -u_m^+ |_{\partial \Omega} \) has compact support contained in \( \{ x \in \partial \Omega : |x| < 1 \} \). Thus, \( (0, 0, h_m^+) \in \mathcal{R}_\gamma^l (\Omega; \partial \Omega) \subset \mathcal{R}_{\gamma-1}^l (\Omega; \partial \Omega) \). Since \( (\gamma - 1) \in (-2, -1) \), the operator \( \mathcal{S}_{\gamma-1}^l \) is of Fredholm type (Theorem 3.1) and \( \dim \ker \mathcal{S}_{\gamma-1}^l = 0 \) (Lemma 3.1). Therefore, problem (1.2) - (1.3) is solvable in \( \mathcal{D}_{\gamma-1}^l (\Omega) \) for all right-hand sides from \( \mathcal{R}_{\gamma-1}^l (\Omega; \partial \Omega) \) and we find the remainder \( (\tilde{\xi}_m^+, \tilde{\eta}_m^+) \in \mathcal{D}_{\gamma-1}^l (\Omega) \). Moreover, \( (\tilde{\xi}_m^+, \tilde{\eta}_m^+) \) admits the asymptotic representation (4.1):

\[
\tilde{\xi}_m^+ (x) \in \mathcal{D}_\gamma^l (\Omega).
\]

We normalize \( (\tilde{\xi}_m^+, \tilde{\eta}_m^+) \) by the condition \( \lim_{|x| \to \infty} \tilde{\eta}_m^+(x) = 0 \), so that \( c_0^+ = 0 \). Since \( \tilde{\xi}_m^+ |_{\partial \Omega} = -u_m^+ |_{\partial \Omega} \) on \( \partial \Omega \), from (4.2) we get

\[
c_0^+ = 12 \nu \int_{\partial \Omega} h_m^+ \cdot n \, ds = 12 \nu \int_{\Omega} \div u_m^+ (y, z) \, dx = 0 \quad (m \in \mathbb{N}).
\]

Thus we obtain \( (\hat{\xi}_m^+, \hat{\eta}_m^+) = (\tilde{\xi}_m^+, \tilde{\eta}_m^+) \in \mathcal{D}_\gamma^l (\Omega) \) and this concludes the proof of the lemma.

Let us compute now the coefficients \( c_{-m}^\pm \) \( (m \in \mathbb{N}) \).

**Lemma 4.3.** Let \( (u, p) \in \mathcal{D}_{\beta}^l (\Omega) \) \( (\beta > -1) \) be a solution of problem (1.2) - (1.3) with right-hand side \( (f, g, h) \in \mathcal{R}_{\beta+k}^l (\Omega; \partial \Omega) \) \( (k \in \mathbb{N}) \). Then the coefficients \( c_{-m}^\pm \) in the asymptotic formulae (2.30) - (2.31) admit the integral representations

\[
c_{-m}^\pm = -12 \nu \left( \int_{\Omega} f \cdot \xi_m^\pm \, dx - \int_{\Omega} g \eta_m^\pm \, dx + \int_{\partial \Omega} (\eta_m^\pm n - \nu \partial_n \xi_m^\pm) \, ds \right)
\]

\[
(-\beta - k - 1 < -m < -\beta - 1)
\]

\(^1\) Note that for \( m \in \mathbb{N} \) the functions \( p_m^\pm \) are harmonic polynomials and therefore \( (u_m^\pm, p_m^\pm) \in C^\infty (\Omega) \).
where \((\xi_m^\pm, \eta_m^\pm)\) are the solutions of the homogeneous problem (1.2) – (1.3) constructed in Lemma 4.2.

**Proof.** Let us apply to \((u, p)\) and \((\xi_m^\pm, \eta_m^\pm)\) Green’s formula in the domain \(\Omega_R = \{x \in \Omega : r < R (R > 2)\}:

\[
\int_{\Omega_R} (-\nu \Delta u + \nabla p) \cdot \xi_m^\pm dx - \int_{\Omega_R} \div u \eta_m^\pm dx + \int_{\partial \Omega_R \cup S_R} u \cdot (n \eta_m^\pm - \nu \partial_n \xi_m^\pm) ds = (4.7)
\]

\[
= \int_{\Omega_R} (-\nu \Delta \xi_m^\pm + \nabla \eta_m^\pm) \cdot u dx - \int_{\Omega_R} \div \xi_m^\pm p dx + \int_{\partial \Omega_R \cup S_R} \xi_m^\pm \cdot (np - \nu \partial_n u) ds.
\]

Since \((\xi_m^\pm, \eta_m^\pm)\) fulfills the homogeneous equations (1.2) – (1.3), from (4.7) we derive

\[
\int_{\Omega_R} f \cdot \xi_m^\pm dx - \int_{\Omega_R} g \eta_m^\pm dx + \int_{\partial \Omega_R} h \cdot (n \eta_m^\pm - \nu \partial_n \xi_m^\pm) ds + \int_{S_R} u \cdot (n \eta_m^\pm - \nu \partial_n \xi_m^\pm) ds = \int_{S_R} \xi_m^\pm \cdot (np - \nu \partial_n u) ds.
\]

Let us calculate the right-hand side of (4.8). Taking account of the asymptotic representations (2.30) – (2.31) and (4.4) for \((u, p)\) and \((\xi_m^\pm, \eta_m^\pm)\), respectively, we get

\[
\int_{S_R} \xi_m^\pm \cdot (np - \nu \partial_n u) ds
\]

\[
= \int_{S_R} \xi_m^\pm \cdot (np - \nu \partial_n u) ds + \int_{S_R} u_m^\pm \cdot \sum_{-\beta - k - 1 < l < \beta - 1} [n(c_{l^-}^p p_{l^-}^+ + c_{l^-}^- p_{l^-}^-) - \nu(c_{l^-}^p \partial_n u_{l^-}^+ + c_{l^-}^- \partial_n u_{l^-}^-)] ds.
\]

The first integral in the right-hand side here can be majorated by

\[
\left( R \int_{S_R} |\xi_m^\pm|^2 (1 + r^2)^{\gamma + 1} ds \right)^{\frac{1}{2}} \left( R \int_{S_R} |p|^2 (1 + r^2)^{\beta} R^{-2(\beta + \gamma + 1) - 2} ds \right)^{\frac{1}{2}} + \left( R \int_{S_R} |u|^2 (1 + r^2)^{\beta + 1} R^{-2(\beta + \gamma + 1) - 4} ds \right)^{\frac{1}{2}} \leq c \left( R \int_{S_R} |\xi_m^\pm|^2 (1 + r^2)^{\gamma + 1} ds \right)^{\frac{1}{2}} \left( R \int_{S_R} |p|^2 (1 + r^2)^{\beta} ds \right)^{\frac{1}{2}} \left( R^{-1} \int_{S_R} |u|^2 (1 + r^2)^{\beta + 1} ds \right)^{\frac{1}{2}}.
\]

Since \(\xi_m^\pm \in L_{\gamma + 1}^2(\Omega), u \in L_{\beta + 1}^2(\Omega), p \in L_{\beta}^2(\Omega)\) (see the definition of the space \(D_{\beta}^l(\Omega)\)), expression (4.10) vanishes as \(R \to \infty\) (at least, for some subsequence \(R_j \to \infty\)). Further, using the relations

\[
\int_0^{2\pi} \cos(m \varphi) \sin(|l| \varphi) d \varphi = 0
\]

\[
\int_0^{2\pi} \sin(|m| \varphi) \sin(|l| \varphi) d \varphi = \int_0^{2\pi} \cos(m \varphi) \cos(l \varphi) d \varphi = \pi \delta_{m,l}
\]
we find that
\[
\int_{S_R} \mathbf{u}_m^\pm \cdot \sum_{-\beta < l < -\beta - 1} \left[ \mathbf{n}(c_{m}^+p_{m}^+ + c_{m}^-p_{m}^-) - \nu(c_{m}^+\partial_n\mathbf{u}_m^+ + c_{m}^-\partial_n\mathbf{u}_m^-) \right] ds \\
= \int_{S_R} \mathbf{u}_m^\pm \cdot \mathbf{n}(c_{m}^+p_{m}^+ + c_{m}^-p_{m}^-) ds \\
- \nu \int_{S_R} \mathbf{u}_m^\pm \cdot (c_{m}^+\partial_n\mathbf{u}_m^+ + c_{m}^-\partial_n\mathbf{u}_m^-) ds \\
= c_{m}^- \int_{S_R} (2\nu)^{-1}z(z-1)\partial_n p_{m}^\pm p_{m}^- ds + R^{-2}c(m) \\
= -\frac{1}{24\nu}c_{m}^- + o(R^{-1}).
\]
Analogously one can compute the integral
\[
\int_{S_R} \mathbf{u} \cdot (\mathbf{n}n_{m}^\pm - \nu\partial_n\mathbf{\xi}_{m}^\pm) ds = \frac{1}{24\nu}c_{m}^- + o(R^{-1}).
\]
Substituting formulae (4.9) - (4.12) into (4.8) and passing $R \to \infty$, we derive formula (4.6) \[\square\]

Now we are in a position to compute the dimensions of $\ker S_{\beta}^l$ and $\coker S_{\beta}^l$.

**Theorem 4.1.**

(i) If $\beta \in (k - 1, k)$ \(0 \leq k \in \mathbb{Z}\), then \(\dim \coker S_{\beta}^l = 2k + 1\).

(ii) If $\beta \in (q - 1, q)$ \( (\mathbb{Z} \ni q \leq -1)\), then \(\dim \ker S_{\beta}^l = -2q - 1\).

(iii) If $\beta \in (p, p + 1)$ \( (p \in \mathbb{Z})\), then \(\text{Ind} S_{\beta}^l = -2p - 1\).

**Proof.** Let \((f, g, h) \in \mathcal{R}_{\beta}^l(\Omega; \partial\Omega)\) \( (\beta \in (k - 1, k), k \geq 0)\). Then there exists a solution \((u, p) \in \mathcal{D}_{\beta}^l(\Omega)\) \((\beta_1 = \beta - k - 1 \in (-2, -1))\) of problem (1.2) - (1.3). (Note that $\mathcal{R}_{\beta}^l(\Omega; \partial\Omega) \subset \mathcal{R}_{\beta_1}^l(\Omega; \partial\Omega)$ and by Lemma 3.1 the operator $S_{\beta_1}^l$ \( (\beta_1 \in (-2, -1))\) is an epimorphism.) For \((u, p)\) there holds the asymptotic formula (2.30) where the constants $c_{0}^\pm$ and $c_{m}^\pm$ \( (m = 1, \ldots, k)\) admit the integral representations (4.2) and (4.6), respectively. Hence under $2k + 1$ compatibility conditions
\[
\int_{\partial\Omega} \mathbf{h} \cdot \mathbf{n} ds - \int_{\Omega} g dx = 0 \\
\int_{\Omega} \mathbf{f} \cdot \mathbf{\xi}_{m}^\pm dx - \int_{\Omega} g n_{m}^\pm dx + \int_{\partial\Omega} \mathbf{h} \cdot (n_{m}^\pm \mathbf{n} - \nu\partial_n\mathbf{\xi}_{m}^\pm) ds = 0 \ (m = 1, \ldots, k)
\]
we obtain
\[
\left( \begin{array}{c}
\mathbf{u}(x) \\
p(x)
\end{array} \right) = c_{0}^+ \left( \begin{array}{c}
0 \\
1
\end{array} \right) + \left( \begin{array}{c}
\tilde{\mathbf{u}}(x) \\
\tilde{p}(x)
\end{array} \right)
\]
where \((\tilde{\mathbf{u}}, \tilde{p}) \in \mathcal{D}_{\beta}^l(\Omega)\). Normalizing this solution by the condition $\lim_{|x| \to \infty} p(x) = 0$ we get \((u, p) = (\tilde{\mathbf{u}}, \tilde{p}) \in \mathcal{D}_{\beta}^l(\Omega)\). Thus assuming $2k + 1$ compatibility conditions to be valid, we have proved the existence of the solution \((u, p) \in \mathcal{D}_{\beta}^l(\Omega)\). Since for $\beta \in$
(k − 1, k) (k ≥ 0) the operator \( S^l_{\beta} \) is a Fredholm monomorphism (see Lemma 3.1), these conditions are necessary. Therefore, we conclude
\[
\dim \text{coker } S^l_{\beta} = 2k + 1.
\]
Statement (ii) follows now from the fact that
\[
\dim \ker S^l_{\beta} = \dim \text{coker } S^l_{-\beta - 2}.
\]
Statement (iii) has become evident \( \blacksquare \)

5. Asymptotic conditions at infinity

As follows from Lemma 3.1, there is no admissible \( \beta \) such that the operator \( S^l_{\beta} \) is of index zero. In order to compensate this lack we introduce function spaces with detached asymptotics and impose conditions at infinity. For \( \beta < -1 \) the operator \( S^l_{\beta} \) is an epimorphism, and for \( \beta > -1, \) \( S^l_{\beta} \) is a monomorphism (see Lemma 3.1). Let us take
\[
\beta_\pm = -1 \pm N \pm \delta \quad (N \in \mathbb{N}_0, \delta \in (0, 1)).
\]
For simplicity we fix the regularity index \( l \) and omit it in notations. Moreover, we denote
\[
S^l_{\beta_\pm} = S_\pm, \quad D^l_{\beta_\pm}(\Omega) = D_\pm(\Omega), \quad \mathcal{R}^l_{\beta_\pm}(\Omega; \partial \Omega) = \mathcal{R}_\pm(\Omega; \partial \Omega).
\]
Let us consider the mapping \( S_- : D_-(\Omega) \hookrightarrow \mathcal{R}_-(\Omega; \partial \Omega) \) and its preimage \( D_\pm(\Omega) \) of the linear \( \mathcal{R}_+(\Omega; \partial \Omega) \subset \mathcal{R}_- (\Omega; \partial \Omega) \) (since the preimage is related both to the indices " + " and " − ", we mark it by " ± "). Due to Theorem 2.4, \( D_\pm(\Omega) \) consists of vector functions \( U = (u, p) \) taking the asymptotic form
\[
U = \begin{pmatrix} u \\ p \end{pmatrix} = \sum_{-N \leq m \leq N} \chi \begin{pmatrix} c_m^+(u_m^+ / p_m^+) \\ c_m^-(u_m^- / p_m^-) \end{pmatrix} + \begin{pmatrix} \tilde{u} \\ \tilde{p} \end{pmatrix} \quad \text{(5.2)}
\]
where \( \tilde{U} = (\tilde{u}, \tilde{p}) \in D_+(\Omega) \) and \( (u_m^\pm, p_m^\pm) \) are given by (2.31). This means that \( D_\pm(\Omega) \) is formed by the sum of linear combinations of the special solutions \( (u_m^\pm, p_m^\pm) \) and the "rapidly" decaying remainder \( \tilde{U} = (\tilde{u}, \tilde{p}) \in D_+(\Omega) \). Furthermore, the quotient space \( D_\pm(\Omega) / D_+ (\Omega) \) can be identified with \( \mathbb{R}^{4N+2} \) and we introduce in \( D_\pm(\Omega) \) the norm induced by the asymptotic representation (5.2)
\[
\| U; D_\pm(\Omega) \| = \left( \| \tilde{U}; D_+(\Omega) \|^2 + \| a; \mathbb{R}^{2N+1} \|^2 + \| b; \mathbb{R}^{2N+1} \|^2 \right)^{\frac{1}{2}}
\]
where \( a \) and \( b \) are columns of height \( 2N + 1 \),
\[
a = (c_0^+, c_1^+, \ldots, c_N^+, c_{-N}^-)^T, \\
b = (c_0^+, c_1^+, \ldots, c_N^+, c_{-N}^-)^T.
\]
Let \( \mathcal{G}_\pm \) be the restriction of \( S_- \) on \( D_\pm(\Omega) \). Due to estimate (2.32),
\[
\| a; \mathbb{R}^{2N+1} \| + \| b; \mathbb{R}^{2N+1} \| \leq c \left( \| \mathcal{G}_\pm U; \mathcal{R}_+(\Omega; \partial \Omega) \| + \| (u, p); L^2_\pm (\Omega) \| \right).
\]
Therefore the operator
\[
\mathcal{G}_\pm : D_\pm (\Omega) \hookrightarrow \mathcal{R}_+(\Omega; \partial \Omega) \quad \text{(5.4)}
\]
of problem (1.2) - (1.3) is continuous. Moreover, in view of Theorems 3.1 and 4.1, it inherits properties of \( S_- \) and the following assertion is valid.
Theorem 5.1. The mapping (5.4) is a Fredholm epimorphism and
\[ \dim \ker \mathfrak{S}_\pm = \dim \ker \mathcal{S}_- = 2N + 1. \]  (5.5)

There appear the continuous projections
\[
\begin{align*}
\mathbb{D}_\pm(\Omega) &\ni U \mapsto \pi_1 U = a \in \mathbb{R}^{2N+1} \\
\mathbb{D}_\pm(\Omega) &\ni U \mapsto \pi_0 U = b \in \mathbb{R}^{2N+1}.
\end{align*}
\]  (5.6)

We also determine
\[ \pi = \begin{pmatrix} \pi_1 \\ \pi_0 \end{pmatrix} : \mathbb{D}_\pm(\Omega) \mapsto \mathbb{R}^{4N+2}. \]

We treat \( \pi_0 U, \pi_1 U \) and \( \pi U \) as columns in \( \mathbb{R}^{2N+1}, \mathbb{R}^{2N+1} \) and \( \mathbb{R}^{4N+2} \), respectively.

Let us connect with Green’s formula (2.33) the linear form
\[ Q_\Omega(U, V) = Q_\Omega(u, p; v, q) \]
defined by
\[ Q_\Omega(U, V) \equiv (-\nu \Delta u + \nabla p, v)_\Omega + (-\text{div } u, q)_\Omega + (u, q n - \nu \partial_n v)_{\partial \Omega} \]
\[ - (u, -\nu \Delta v + \nabla q)_{\partial \Omega} - (p - \text{div } v)_{\partial \Omega} - (p n - \nu \partial_n u, v)_{\partial \Omega} \]  (5.7)

where \((\cdot, \cdot)_\Omega\) and \((\cdot, \cdot)_{\partial \Omega}\) stand for extensions of the scalar products in \( L^2(\Omega) \) and \( L^2(\partial \Omega) \), respectively. Since \( (u^m_{\pm}, p^m_{\pm}) \) satisfy the homogeneous equations (1.2) - (1.3) in \( \Pi \setminus \{x \in \mathbb{R}^3 : r = 0\} \), for any \( U, V \in \mathbb{D}_\pm(\Omega) \) we get the inclusions (see (5.2))
\[
\begin{pmatrix}
  (-\nu \Delta u + \nabla p, u|_{\partial \Omega}) \\
  (-\nu \Delta v + \nabla q, v|_{\partial \Omega})
\end{pmatrix} \in \mathcal{R}_+ (\Omega, \partial \Omega)
\]
and therefore all integrals in the left-hand side of (5.7) converge. Hence \( Q_\Omega \) is a continuous antisymmetric form on \( \mathbb{D}_\pm(\Omega)^2 \),
\[ Q_\Omega(V; U) = -Q_\Omega(U; V). \]  (5.8)

Due to Lemma 2.7,
\[ Q_\Omega(V; U) = Q_\Omega(U; V) = 0 \]  (5.9)
for all \( V \in \mathbb{D}_+(\Omega) \subset \mathbb{D}_\pm(\Omega) \) and all \( U \in \mathbb{D}_\pm(\Omega) \). Thus \( Q_\Omega \) can be naturally treated as a form defined on the quotient space
\[ (\mathbb{D}_\pm(\Omega)/\mathbb{D}_+(\Omega))^2 \approx \mathbb{R}^{4N+2} \times \mathbb{R}^{4N+2}. \]

Lemma 5.1. If \( U, V \in \mathbb{D}_\pm(\Omega) \), then
\[ Q_\Omega(U; V) = \langle \pi_0 U, \pi_1 V \rangle_{2N+1} - \langle \pi_1 U, \pi_0 V \rangle_{2N+1} \]  (5.10)
where \( \langle \cdot, \cdot \rangle_K = 12\nu \langle \cdot, \cdot \rangle_K \) with \( \langle \cdot, \cdot \rangle_K \) being the scalar product in \( \mathbb{R}^K \).
Proof. According to the asymptotic form (5.2), we can represent $U$ as sum

$$U = \left( \begin{array}{c} u \\ p \end{array} \right) = \sum_{1 \leq m \leq N} \chi \left[ c_0^+ \left( \begin{array}{c} u_0^+ \\ p_0^+ \end{array} \right) + c_m^+ \left( \begin{array}{c} u_m^+ \\ p_m^+ \end{array} \right) \right]$$

$$+ \sum_{-N \leq m \leq -1} \chi \left[ c_0^- \left( \begin{array}{c} u_0^- \\ p_0^- \end{array} \right) + c_m^+ \left( \begin{array}{c} u_m^+ \\ p_m^+ \end{array} \right) \right] + \left( \tilde{u} \right) \left( \tilde{p} \right)$$

$$= U_N + U_{-N} + \tilde{U} \quad (\tilde{U} \in \mathcal{D}_+(\Omega)).$$

Analogously,

$$V = V_N + V_{-N} + \tilde{V} \quad (\tilde{V} \in \mathcal{D}_+(\Omega)).$$

By virtue of (5.9), $Q_\Omega(U, \tilde{V}) = Q_\Omega(\tilde{U}, V) = 0$ so that

$$Q_\Omega(U, V) - Q_\Omega(U_{-N}, V_N) - Q_\Omega(U_N, V_{-N}) - Q_\Omega(U_{-N}, V_{-N}) = Q_\Omega(U_N, V_N).$$

Arguing as in the proof of Lemmata 4.1 and 4.3 and applying Green’s formula in the truncated domain $\Omega_R$, we find that

$$\lim_{R \to \infty} \left( Q_{\Omega_R}(U_{-N}, V_N) + Q_\Omega(U_N, V_{-N}) \right) = \langle \pi_1 U, \pi_0 V \rangle_{2N+1} - \langle \pi_0 U, \pi_1 V \rangle_{2N+1}$$

$$\lim_{R \to \infty} Q_{\Omega_R}(U_{-N}, V_{-N}) = 0.$$  \hspace{1cm} (5.12)

Thus, the left-hand side of equality (5.11) is finite. The term $Q_{\Omega_R}(U_N, V_N)$ is equal to the sum $\sum_{j=1}^{2N} \alpha_j R^j$ where $\alpha_j$ are constants. Therefore, its limit as $R \to \infty$ can be finite only if $\alpha_j = 0$ ($j = 1, \ldots, 2N$; arguing as in the proof of Lemma 4.3, one can compute directly that $\alpha_j = 0$). Thus, we have got the equality $Q_\Omega(U_N, V_N) = 0$ which together with (5.11) - (5.12) implies (5.10).\[\blacksquare]\n
- We call (5.10) the generalized Green’s formula.

Lemma 5.2. Let

$$X = \begin{pmatrix} B \\ S \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} -T \\ Q \end{pmatrix}$$

where $B, T, S, Q$ are $(2N + 1) \times (4N + 2)$-matrices. Suppose that $X$ and $Y$ satisfy the relation

$$Y^* X = J = \begin{cases} 0 & I \\ -I & 0 \end{cases}.$$  \hspace{1cm} (5.14)

Then the generalized Green’s formula (5.10) may be rewritten as

$$(-\nu \Delta u + \nabla p, v)_\Omega + (-\text{div} u, q)_\Omega + (u, TV)_{\partial \Omega} + \langle B \pi U, T \pi V \rangle_{2N+1}$$

$$= (u, -\nu \Delta v + \nabla q)_\Omega + (p, -\text{div} v)_\Omega + (TU, v)_{\partial \Omega} + \langle S \pi U, Q \pi V \rangle_{2N+1}$$

where $TU = (p n - \nu \partial_n u)|_{\partial \Omega}$.\[\blacksquare]\n
Proof. Simple algebraic manipulations with matrices turn (5.10) into (5.15) (cf. [12: Section 6.2.2] and [16: Lemma 6.2]).\[\blacksquare\]
Remark 5.1.

1) From (5.14) it follows that \( \det \mathbf{X} \neq 0 \) and \( \mathbb{Y} = (\mathbb{J} \mathbf{X}^{-1})^* \). Therefore, for any \((2N+1) \times (4N+2)\)-matrix \( \mathbf{B} \), the rank of which is equal to \( 2N + 1 \), there exist matrices \( \mathbf{S}, \mathbf{T}, \mathbf{Q} \) such that (5.13) - (5.15) are fulfilled. If \( \mathbf{S} \) is also fixed and \( \det \left( \frac{\mathbf{B}}{\mathbf{S}} \right) \neq 0 \), then \( \mathbf{T} \) and \( \mathbf{Q} \) are uniquely defined.

2) If \( \mathbf{S} = \mathbf{T} \) and \( \mathbf{Q} = \mathbf{B} \), Green's formula (5.15) takes the form

\[
\begin{align*}
- \nu \Delta \mathbf{u} + \nabla p, \mathbf{v} & + (\mathbf{u}, T \mathbf{V})_{\partial \Omega} + \langle \mathbf{B} \pi \mathbf{U}, \mathbf{T} \mathbf{p} \mathbf{V} \rangle_{2N+1} \\
= (\mathbf{u}, - \nu \Delta \mathbf{v} + \nabla q)_{\Omega} + (p, - \text{div} \mathbf{v})_{\Omega} + (T \mathbf{U}, \mathbf{v})_{\partial \Omega} + \langle \mathbf{T} \mathbf{p} \mathbf{U}, \mathbf{B} \pi \mathbf{V} \rangle_{2N+1}.
\end{align*}
\]

(5.16)

\[
\begin{align*}
\bullet \text{ We call (5.16) the symmetric generalized Green's formula.}
\end{align*}
\]

Based on the generalized Green's formulæ (5.15) and (5.16) and arguing in the same way as in [12, 16], we provide problem (1.2) - (1.3) with the additional conditions

\[
\mathbf{B} \pi \mathbf{U} = \mathbf{H} \in \mathbb{R}^{2N+1}.
\]

(5.17)

\[
\bullet \text{ We call (5.17) the asymptotic conditions at infinity.}
\]

We connect problem (1.2) - (1.3), (5.17) with the mapping

\[
\mathbb{D}_\pm(\Omega) \ni \mathbf{U} \longmapsto \mathbf{A} \mathbf{U} = (\mathcal{G}_\pm \mathbf{U}, \mathbf{B} \pi \mathbf{U}) \in \mathbb{R}_\pm(\Omega; \partial \Omega)
\]

(5.18)

where \( \mathbb{R}_\pm(\Omega; \partial \Omega) = \mathcal{R}_+(\Omega; \partial \Omega) \times \mathbb{R}^{2N+1} \). It is clear that \( \mathbf{A} \) inherits the Fredholm property from \( \mathcal{G}_\pm \). Furthermore, in (5.18) we observe \( 2N + 1 \) additional conditions and therefore the difference of the indices of \( \mathcal{G}_\pm \) and \( \mathbf{A} \) is equal to \( 2N + 1 \), i.e. \( \text{Ind} \mathbf{A} = 0 \). Precisely, this equality follows from

\[
\text{Ind} \mathbf{A} = \text{Ind} (\mathcal{G}_\pm |_{\mathbf{U} \in \mathbb{D}_\pm(\Omega): \mathbf{B} \pi \mathbf{U} = 0}) = \text{Ind} \mathcal{G}_\pm - (2N + 1) = 0.
\]

Theorem 5.2.

1) \( \ker \mathbf{A} = \{ \mathbf{V} \in \ker \mathcal{G}_\pm : \mathbf{B} \pi \mathbf{V} = 0 \} \).

2) If the generalized Green's formula (5.15) is valid, then

\[
\text{coker} \mathbf{A} = \left\{ (\mathbf{V}, T \mathbf{V}|_{\partial \Omega}, \mathbf{T} \pi \mathbf{V}) : \mathbf{V} \in \ker \mathcal{G}_\pm, \mathbf{Q} \pi \mathbf{V} = 0 \right\}.
\]

(5.19)

Proof. The first assertion follows from the inclusion \( \ker \mathbf{A} \subset \ker \mathcal{G}_\pm \), the second one has been proved in [12: Proposition 6.2.5] (see also [16: Theorem 6.5]) \( \blacksquare \)

The subspace \( \text{dim} \ker \mathcal{G}_\pm \) contains the solution \( \zeta_0 = (0, 1) \) and the solutions \( \mathbf{\zeta}_m = (\xi_m^+, \eta_m^-) \) \( (m = 1, \ldots, N) \) of the homogeneous problem (1.2) - (1.3) (see Lemma 4.2). Since the dimension of \( \ker \mathcal{G}_\pm \) coincides with the number of linear independent solutions we have found that \( \ker \mathcal{G}_\pm \) becomes the linear hull of them:

\[
\ker \mathcal{G}_\pm = \mathcal{L} \{ \xi_0^+, \xi_1^+, \xi_1^-, \ldots, \xi_N^+, \xi_N^- \} = \{ \zeta = \mathfrak{Z} \mathbf{c} : \mathbf{c} \in \mathbb{R}^{2N+1} \}
\]

(5.20)

where \( \mathfrak{Z} = (\xi_0^+, \xi_1^+, \xi_1^-, \ldots, \xi_N^+, \xi_N^-) \) is a \( 4 \times (2N + 1) \)-matrix-function or, what is the same, a row of solutions. Due to Lemma 4.2, each element \( \zeta \in \ker \mathcal{G}_\pm \) can be represented in the form

\[
\zeta = \mathfrak{Z} \mathbf{c} = \chi \mathbb{Y} \mathbf{c} + \mathbf{\Theta} \mathbf{c}
\]

(5.21)
where the solution rows $\mathcal{X}$ and $\mathcal{Y}$ are defined by

$$
\mathcal{X} = \left( \begin{pmatrix} u_0^+ \\ p_0^+ \\ \vdots \\ u_N^+ \\ p_N^+ \end{pmatrix}, \begin{pmatrix} u_1^+ \\ p_1^+ \\ \vdots \\ u_N^+ \\ p_N^+ \end{pmatrix}, \begin{pmatrix} u_0^- \\ p_0^- \\ \vdots \\ u_N^- \\ p_N^- \end{pmatrix}, \begin{pmatrix} u_1^- \\ p_1^- \\ \vdots \\ u_N^- \\ p_N^- \end{pmatrix} \right),
$$

$$
\mathcal{Y} = \left( \begin{pmatrix} u_0^- \\ p_0^- \\ \vdots \\ u_N^- \\ p_N^- \end{pmatrix}, \begin{pmatrix} u_1^- \\ p_1^- \\ \vdots \\ u_N^- \\ p_N^- \end{pmatrix}, \begin{pmatrix} u_0^+ \\ p_0^+ \\ \vdots \\ u_N^+ \\ p_N^+ \end{pmatrix}, \begin{pmatrix} u_1^+ \\ p_1^+ \\ \vdots \\ u_N^+ \\ p_N^+ \end{pmatrix} \right),
$$

$M$ is a constant $(2N + 1) \times (2N + 1)$-matrix and $\tilde{\Pi} \in \mathcal{D}_+(\Omega)^{2N+1}$. Note that

$$
\begin{align*}
\pi_0 c &= c \\
\pi_1 c &= -Mc
\end{align*}
$$

(5.22)

- We call the matrix $M$ the augmented flow polarization matrix.

**Theorem 5.3.** $M$ is a symmetric matrix.

**Proof.** Let $c, C$ be arbitrary constant vectors in $\mathbb{R}^{2N+1}$. Since $3c$ and $3C$ are solutions of the homogeneous problem (1.2) - (1.3) we get $Q_{\Omega}(3c, 3C) = 0$. On the other hand, from the generalized Green’s formula (5.10) there follows that

$$
Q_{\Omega}(3c, 3C) = \langle \pi_0 3c, \pi_1 3C \rangle_{2N+1} - \langle \pi_1 3c, \pi_0 3C \rangle_{2N+1}
$$

$$
= \langle Mc, C \rangle_{2N+1} - \langle c, MC \rangle_{2N+1}
$$

$$
= \langle c, (M^* - M)C \rangle_{2N+1}
$$

$$
= 0.
$$

Thus, $M = M^*$. 

**Remark 5.2.** The matrix $M$ has the form $M = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}$ where $0 = (0, \ldots, 0)$ and $M$ is a symmetric $2N \times 2N$-matrix. This follows from the fact that the solution $\zeta_0^+$ has the form $\zeta_0^+ = (0, 1)^T$ and from the symmetry of $M$.

- We call the matrix $\Pi M$ the flow polarization matrix.

**Theorem 5.4.** Let $\mathcal{B} = \mathbb{B}(-M, I)^T$ where $I$ is the unit $(2N + 1) \times (2N + 1)$-matrix. Then

$$
\text{dim ker } A = 2N + 1 - \text{rank } \mathcal{B}.
$$

(5.23)

**Proof.** The elements $\zeta \in \text{ker } \mathcal{S}_\pm$ admit the representation $\zeta = 3c$ (c $\in \mathbb{R}^{2N+1}$; see (5.21)). Since $\pi_1 c = c$, $\pi_0 c = -Mc$ and due to the symmetry of $M$, $\mathbb{B} c = 0$ if and only if $\mathbb{B}(-M, I)^T c = 0$. Therefore, owing to Theorem 5.2/(1) we conclude (5.23).

**Remark 5.3.** In view of (5.19) the compatibility conditions for problem (1.2) - (1.3), (5.17) take the form

$$
(f, v)_{\Omega} + (g, q)_{\Omega} + (h, TU)_{\partial \Omega} + \langle H, \mathcal{T} \pi V \rangle_{2N+1} = 0
$$

(5.24)

for all $V = (v, q) \in \text{ker } \mathcal{S}_\pm$ with $Q\pi V = 0$.

In accordance with (5.19), (5.24) it is very natural to say that problems (1.2) - (1.3), (5.17) and (1.2) - (1.3) with additional conditions

$$
Q\pi V = K \in \mathbb{R}^{2N+1}
$$

(5.25)

are adjoint with respect to the generalized Green’s formula (5.15). In the case when the symmetric generalized Green’s formula (5.16) takes place, problem (1.2) - (1.3), (5.17) becomes formally self-adjoint.
Theorem 5.5.
1) If \( \Omega = \Pi \), then \( M = 0 \).
2) If \( \Omega \neq \Pi \) and \( \Omega \subset \Pi \), then the matrix \( M \) is positive definite.

Proof. Let \( c = (0, c') \) with \( c' \in \mathbb{R}^{2N} \setminus \{0\} \) be arbitrary. We take
\[
\mathbf{V} = (\mathbf{v}, q) = 3c = \mathbf{V}^0 + \mathbf{V}^# \in \ker \mathfrak{G}_\pm
\]
where
\[
\mathbf{V}^0 = (v^0, q^0) = \mathfrak{X}c
\]
\[
\mathbf{V}^# = (v^#, q^#) = -\lambda_2 \mathfrak{M}c + \tilde{\mathfrak{M}}c \in \mathcal{D}_\gamma^1(\Omega) \quad (\gamma \in (-1, 0))
\]
(see (5.21) and Lemma 4.2). By formula (4.6) and the definition of \( M \) we get
\[
\langle M\mathbf{c}', \mathbf{c}' \rangle_{2N} = \int_{\partial\Omega} \mathbf{v}^# \cdot T(\mathbf{V}) \, ds. \tag{5.26}
\]
(Note that \(-\nu \Delta v^# + \nabla q^# = 0 \) and \( \text{div} \, v^# = 0 \).) If \( \Omega = \Pi \), then \( \mathbf{V}^0 \) is the exact solution of the homogeneous problem (1.2) - (1.3). Hence \( \mathbf{V}^# = 0 \) and \( M = 0 \).

Since \( v^# = -v^0 \) on \( \partial\Omega \),
\[
\int_{\partial\Omega} \mathbf{v}^# \cdot T(\mathbf{V}) \, ds = \int_{\partial\Omega} \mathbf{v}^# \cdot T(\mathbf{V}^#) \, ds - \int_{\partial\Omega} \mathbf{v}^0 \cdot T(\mathbf{V}^0) \, ds. \tag{5.27}
\]
Integrating by parts in \( \Omega \) and \( \Pi \setminus \Omega \), we derive
\[
\int_{\partial\Omega} \mathbf{v}^# \cdot T(\mathbf{V}^#) \, ds = \int_{\Omega} |\nabla v^#|^2 \, dx \\
\int_{\partial\Omega} \mathbf{v}^0 \cdot T(\mathbf{V}^0) \, ds = -\int_{\Pi \setminus \Omega} |\nabla v^0|^2 \, dx. \tag{5.28}
\]
The sign ”\(-\)“ in the second equality of (5.28) appears because of the oposite direction of the outward normal \( n \). The Dirichlet integral of \( v^# \) is finite since \( \mathbf{V}^# \in \mathcal{D}_\gamma^1(\Omega) \) for \( \gamma \in (-1, 0) \). The formula
\[
\langle M\mathbf{c}', \mathbf{c}' \rangle_{2N} = \int_{\Omega} |\nabla v^#|^2 \, dx + \int_{\Pi \setminus \Omega} |\nabla v^0|^2 \, dx > 0
\]
follows from (5.26) - (5.28) and completes the proof.

Example 5.1. Let \( N = 0 \) and \( \mathbb{B} = (1, 0) \) is a matrix of size \( 1 \times 2 \). Then the condition \( \mathbb{B}\pi \mathbf{U} = \pi_1 \mathbf{U} = c_0^+ \) prescribes the total flux of the fluid over the surface \( S_R \). The matrix \( \tilde{3} \) consists of one solution \( \zeta_0^+ \). Hence \( \dim \ker \mathfrak{G}_\pm = 1 \), \( \pi_1 \tilde{3}c = 0 \) for all \( c \) and \( \mathfrak{M} = 0 \) (see (5.22)). We have \( \mathbb{B} = \mathbb{B}(-\mathfrak{M}, \mathbb{I})^T = 0 \) and, by Theorem 5.4, \( \dim \ker \mathfrak{A} = 1 - \operatorname{rank} \mathbb{B} = 1 \). Therefore the operator \( \mathfrak{A} \) is an epimorphism with one-dimentional kernel (constant pressure).
If $B = (0, 1)$, then $B\pi U = \pi_0 U = c_0^+$ prescribes the limit of the pressure component as $r \to \infty$. We get $\pi_0 \mathfrak{c} = 1$, $M = I$ and $B = B(-M, I)^T = I$. By Theorem 5.4, $\dim \ker A = 1 - \operatorname{rank} B = 0$ and the operator $A$ is an isomorphism.

**Example 5.2.** Let $N = 1$ and

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \alpha & \sin \alpha \\ 0 & 0 & 0 & -\sin \alpha & \cos \alpha \end{pmatrix}.$$

We consider the condition $B\pi U = (H_1, H_2, 0)^T$ which prescribes the total flux $H_1$ over $S_R$ and the linear flux $H_2$ of $u$ in the direction $e^\alpha = (\cos \alpha, \sin \alpha)$ (cf. [14]). We obtain $\mathfrak{S} = \{\zeta_0^+, \zeta_1^+, \zeta_1^-\}$, $\dim \ker \mathfrak{S}_\pm = 3$ and

$$B = B(-M, I)^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix}.$$

Hence $\dim \ker A = 3 - \operatorname{rank} B = 1$ and the operator $A$ is an epimorphism.

If we prescribe instead of the total flux the limit $H_1$ of the pressure component as $r \to \infty$, we shall take

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos \alpha & \sin \alpha \\ 0 & 0 & 0 & 0 & -\sin \alpha & \cos \alpha \end{pmatrix}$$

and consider the condition $B\pi U = (H_1, H_2, 0)^T$. In this case we get the unitary matrix

$$B = B(-M, I)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{pmatrix},$$

$\dim \ker A = 3 - \operatorname{rank} B = 0$ and the operator $A$ is an isomorphism.

**References**


