On Strong Unboundedness of Symmetric Operators

J. Friedrich

It will be shown that for each positive odd integer \( n \) there is a symmetric operator \( \mathcal{F} \) in a separable Hilbert space \( \mathcal{H} \) such that \( \mathcal{F}^1, \mathcal{F}^3, \ldots, \mathcal{F}^n \) are unbounded from below and \( \mathcal{F}^k \geq 0 \) for \( k > n \).

A symmetric operator \( \mathcal{F} \) with dense invariant domain \( D \subseteq D(\mathcal{F}) \) is said to be strongly unbounded from above (below), if

\[
\sup_{\varphi \in B_k(D)} \langle \mathcal{F}^k \varphi, \varphi \rangle = +\infty \quad (\inf_{\varphi \in B_k(D)} \langle \mathcal{F}^k \varphi, \varphi \rangle = -\infty),
\]

for \( k = 1, 3, 5, \ldots \), where

\[
B_k(D) = \{ \varphi \in D : |\langle \mathcal{F}^j \varphi, \varphi \rangle| \leq 1, j = 1, 2, \ldots, k - 1 \}.
\]

It was shown in [1] that two unbounded symmetric operators \( \mathcal{F}_1, \mathcal{F}_2 \) with dense invariant domains \( D_i = \bigcap\nolimits_{j=1}^\infty D(\mathcal{F}_j^i), j = 1, 2, \) possess dense invariant domains \( D_{ij} \subseteq D_i, j = 1, 2, \) such that \( \mathcal{F}_1 \upharpoonright D_{10} \) and \( \mathcal{F}_2 \upharpoonright D_{20} \) are unitarily equivalent if and only if they are both strongly unbounded from below or both strongly unbounded from above.

The latter result has applications to representations of unbounded operator algebras (see [3]). Clearly, a self-adjoint operator \( \mathcal{A} \) is strongly unbounded from below (above), if and only if it is unbounded from below (above) in the usual sense. The following theorem shows that the word "strongly" cannot be omitted in general. But the theorem seems to be of interest in itself.

**Theorem:** Let \( n \) be a given positive odd integer. There is a closed symmetric operator \( \mathcal{F} \) in a separable Hilbert space \( \mathcal{H} \) such that

(i) \( D = \bigcap\nolimits_{j=1}^\infty D(\mathcal{F}_j) \) is a core for each \( \mathcal{F}^k, k = 1, 2, \ldots, \) i.e., \( \mathcal{F}^k \upharpoonright D = \mathcal{F}^k \),

(ii) \( \inf_{\varphi \in B_k(D)} \langle \mathcal{F}^k \varphi, \varphi \rangle = -\infty \) if \( k \in \{1, 3, \ldots, n\} \),

(iii) \( \langle \mathcal{F}^k \varphi, \varphi \rangle \geq 0 \) if \( \varphi \in D \) and \( k > n \).

The idea of the proof is to construct \( \mathcal{F} \) as a restriction of a suitable self-adjoint operator. We start with an auxiliary construction of symmetric operators \( T_t, t > 0 \).
From now on, let $n$ be fixed. Consider a positive bounded operator $B$ in a separable Hilbert space $H$ with the properties that $\ker B = \{0\}$ and $BH \neq H$. Choose some vector $x \in H \setminus BH$. We define self-adjoint operators $A_t$, $t > 0$, in the Hilbert space $\mathbb{C} \oplus H =: \mathbb{H}$ by $D(A_t) := \mathbb{C} \oplus BH$ and $A_t(\lambda, Bf) = (-t^2 \lambda, f)$ for $(\lambda, Bf) \in D(A_t)$. The operators $T_t$ are defined by

$$D(T_t) := \{ (\lambda, f) \in D(A_t) : A_t(\lambda, f) \perp (t^{n-1}, x) \},$$

$$T_t := A_t \upharpoonright D(T_t).$$

Since $T_t$ is a densely defined closed symmetric operator and has deficiency indices $(1,1)$, [2: Theorem 1.9] yields

**Lemma 1:** $D_t := \bigcap_{j=1}^{\infty} D(T_t^j)$ is dense in $\mathbb{H}$ and

$$\overline{T_t^k \upharpoonright D_t} = T_t^k, \quad k = 1, 2, \ldots.$$ 

In the following we will need a suitable description of $D(T_t^k)$, which is given by

**Lemma 2:** Let $k$ be a positive odd integer. $(\lambda, f) \in \mathbb{H}$ belongs to $D(T_t^k)$ if and only if

$$(\lambda, f) = \lambda(1, B^k h_t^k) + (0, B^k g),$$

where $g \perp B^l x$, $l = 0, 1, \ldots, k - 1$, and

$$h_t^k := \begin{vmatrix}
0 & x & Bx & \cdots & B^{k-1}x \\
-t^{2k+n-1} & b_0 & b_1 & \cdots & b_{k-1} \\
-t^{k+n-3} & b_1 & b_2 & \cdots & b_k \\
& \vdots & \vdots & \ddots & \vdots \\
-t & b_{k-1} & b_k & \cdots & b_{2k-2} \\
-t & b_{k-1} & b_k & \cdots & b_{2k-2}
\end{vmatrix}^{-1}$$

$$b_j := \langle B^j x, x \rangle, \quad j = 0, 1, \ldots$$

**Proof:** The definition of $h_t^k$ is correct, since

$$\begin{vmatrix}
b_0 & \cdots & b_{k-1} \\
\vdots & \ddots & \vdots \\
b_{k-1} & \cdots & b_{2k-2}
\end{vmatrix}$$

is the Gramian determinant of the vectors $x, Bx, \ldots, B^{k-1}x$, which are linearly independent since $x \notin BH$.

Consider $(\lambda, f) \in D(T_t^k)$. Since $D(T_t^k) \subseteq D(A_t^k)$, $(\lambda, f) = \lambda(1, B^k h_t^k) + (0, B^k g)$ for some $g \in H$. By definition of $D(T_t^k)$,

$$A_t(\lambda, f) \perp (t^{n-1}, x), \quad j = 1, 2, \ldots, k,$$

i.e.

$$\langle \lambda(-t^j), B^{k-j} h_t^k \rangle + \langle 0, B^{k-j} g \rangle, (t^{n-1}, x) \rangle
= \lambda((t^{2j+n-1} + \langle h_t^k, B^{k-j} x \rangle) + \langle g, B^{k-j} x \rangle = 0$$

for $j = 1, 2, \ldots, k$. By definition of $h_t^k$, $\langle h_t^k, B^{k-j} x \rangle = (-1)^{k-j} t^{2j+n-1}$, which implies that $\langle g, B^{k-j} x \rangle = 0, j = 1, 2, \ldots, k$.

On the other hand, each vector of the described form belongs to $D(T_t^k)$.
Since we want to study the operators $T_t$ for $t \to 0$, the following statement will be useful.

**Corollary 3:** $h_t^k = t^{n+1}(h^k + O(t))$, where $h^k$ is a fixed non-zero vector and $O(t)$ tends to zero if $t$ tends to zero.

Next we express the positivity of $T_t^k$ in terms of $h_t^k$.

**Lemma 4:** Let $k$ be a positive odd integer. $\langle T_t^k \varphi, \varphi \rangle \geq 0$ for all $\varphi \in D_t$ if and only if

$$
1 - \sup_{0 \neq f, g \in \mathcal{D}} \frac{\langle B^k h_t^k, f \rangle^2}{\langle B^k h_t^k, h_t^k \rangle \langle B^k f, f \rangle} \geq 2^k \langle B^k h_t^k, h_t^k \rangle^{-1}.
$$

**Proof:** $\langle T_t^k \varphi, \varphi \rangle \geq 0$ for all $\varphi \in D_t$ iff $\langle T_t^k (0, B^g), (0, B^g) \rangle \geq 0$. Thus we conclude by Lemma 2 that $\langle T_t^k \varphi, \varphi \rangle \geq 0$ for all $\varphi \in D_t$ iff

$$
\langle T_t^k (1, B^k(h_t^k + g)), (1, B^k(h_t^k + g)) \rangle = -t^{2k} + \langle B^k(h_t^k + g), h_t^k + g \rangle \geq 0
$$

for all $g \perp B^x, l = 0, 1, \ldots, k - 1$. But

$$
\min_{t \in G} \langle B^k(h_t^k + \lambda g), h_t^k + \lambda g \rangle
$$

$$
= \langle B^k h_t^k, h_t^k \rangle \left(1 - |\langle B^k h_t^k, g \rangle|^2 (\langle B^k h_t^k, h_t^k \rangle (B^k g, g))^{-1}\right)
$$

for $g \neq 0$. Using this, (1) follows from the last inequality $\blacksquare$

Now we consider the left-hand side of inequality (1) for small $t$.

**Lemma 5:** It is

$$
\lim_{t \to 0} \inf \left(1 - \sup_{0 \neq f, g \in \mathcal{D}} \frac{\langle B^k f, f \rangle^2}{\langle B^k h_t^k, h_t^k \rangle \langle B^k f, f \rangle} \right) > 0
$$

for $k = 3, 5, \ldots$.

**Proof:** Let $H_\ast$ denote the completion of $H$ with respect to the inner product $\langle \cdot, \cdot \rangle_\ast := \langle B^k, \cdot, \cdot \rangle$. With some abuse of notation we denote the continuous extension of $B$ to $H_\ast$ again by $B$. Since $k \geq 3$,

$$
|\langle f, B^{k-1} x \rangle| \leq \langle B^k f, x \rangle
$$

for all $f \in \mathcal{H}$, i.e., the linear functional $f \mapsto \langle f, B^{k-1} x \rangle$ has a continuous extension to $H_\ast$. Thus, there is a vector $x_\ast \in H_\ast$ such that $\langle f, B^{k-1} x \rangle = \langle f, x_\ast \rangle$ for all $f \in H$. Obviously, $Bx_\ast = x$.

Consider

$$
a_t := \sup_{0 \neq f, \varphi \in H_\ast} \frac{|\langle h_t^k, f \rangle|_\ast^2}{\langle \varphi, \varphi \rangle_\ast \langle h_t^k, h_t^k \rangle_\ast}.
$$

If we denote the angle between $h_t^k$ and $x_\ast$ by $\alpha_t$, then $a_t$ is nothing but $\sin^2 \alpha_t$. Hence

$$
a_t = 1 - \cos^2 \alpha_t
$$

$$
= 1 - \frac{1}{\langle h_t^k, h_t^k \rangle} \langle \langle x_\ast, h_t^k \rangle \langle h_t^k, x_\ast \rangle \rangle^{-1}
$$

$$
= 1 - \frac{|\langle B^{k-1} h_t^k, x \rangle|_2^2 \langle \langle B^{k-2} x, h_t^k \rangle \langle B^{k-1} h_t^k, h_t^k \rangle \rangle^{-1}}{\langle \langle x_\ast, h_t^k \rangle \langle h_t^k, x_\ast \rangle \rangle^{-1}}.
$$
Since by definition of $h_i^k$,

$$|\langle B^{k-1}h_i^k, x \rangle| = |\langle h_i^k, B^{k-1}x \rangle| = t^{n+1},$$

we obtain

$$a_t = 1 - t^{2n+2}(\langle B^{k-2}x, x \rangle \langle B^kh_i^k, h_i^k \rangle)^{-1}.$$ 

Hence

$$1 - \sup_{t \in H, i=0,1,...,k-1} \frac{|\langle B^kh_i^k, f \rangle|^2}{\langle B^kh_i^k, h_i^k \rangle \langle B^kf, f \rangle} \geq 1 - \sup_{t \in H, i=0,1,...,k-1} \frac{|\langle B^kh_i^k, f \rangle|^2}{\langle B^kh_i^k, h_i^k \rangle \langle B^kf, f \rangle} \geq 1 - a_t = t^{2n+2}(\langle B^{k-2}x, x \rangle \langle B^kh_i^k, h_i^k \rangle)^{-1}.$$ 

Thus, by Corollary 3,

$$\lim \inf_{t \to +0} \left( 1 - \sup_{t \in H, i=0,1,...,k-1} \frac{|\langle B^kh_i^k, f \rangle|^2}{\langle B^kh_i^k, h_i^k \rangle \langle B^kf, f \rangle} \right) \geq \lim_{t \to +0} t^{2n+2}(\langle B^{k-2}x, x \rangle \langle B^kh_i^k, h_i^k \rangle)^{-1} = (\langle B^{k-2}x, x \rangle \langle B^kh_i^k, h_i^k \rangle)^{-1},$$

which proves our assertion.

**Proof of the Theorem:** Since

$$\lim_{t \to +0} t^{2n+4}(\langle B^{n+2}h_i^{n+2}, h_i^{n+2} \rangle)^{-1} = \lim_{t \to +0} t^2(\langle B^{n+2}x, x \rangle)^{-1} = 0$$

and by Lemma 4 and 5, there is an $\epsilon > 0$ such that $\langle T_t^{i+2}, \varphi \rangle \geq 0$ for all $t \in (0, \epsilon)$ and all $\varphi \in D_t$.

Since

$$\lim_{t \to +0} t^{2n}(\langle B^nh_i^n, h_i^n \rangle)^{-1} = \lim_{t \to +0} t^{-2}(\langle B^nh_i^n, h_i^n \rangle)^{-1} = \infty,$$

there is some $t_0 \in (0, \epsilon)$ such that $t_0^2(\langle B^nh_i^n, h_i^n \rangle) > 1$. Writing $T$ and $D$ instead of $T_t$ and $D_t$, this means by Lemma 4 that there is some $\varphi_0 \in D$ such that $\langle T^{n+2}\varphi_0, \varphi_0 \rangle < 0$. Obviously, $\langle T^{n-2-j}\varphi_0, T^{j}\varphi_0 \rangle < 0$ for $j = 0, 1, ..., (n-1)/2$ and, since $\langle T^{n+2}\varphi, \varphi \rangle \geq 0$ for all $\varphi \in D, \langle T^{k}\varphi, \varphi \rangle \geq 0$ for all $\varphi \in D$ and $k > n$.

Consider the closed symmetric operator $\mathcal{J} = \sum_{j=1}^{\infty} jT$ in the separable Hilbert space $\mathcal{H} = \sum_{j=1}^{\infty} H$. $\mathcal{J}$ satisfies the conditions in the theorem:

(i) is satisfied, since $D = \bigcap_{j=1}^{\infty} D(\mathcal{J}^j) = \sum_{j=1}^{\infty} D$ and since $D$ is a core for each $T^k$, $k = 1, 2, ...$.

(ii) is true, since

$$\inf_{\varphi \in \mathcal{B}_D} \langle \mathcal{J}^k\varphi, \varphi \rangle \leq j \left( \sum_{i=0}^{k-1} |\langle T^i\varphi_0, \varphi_0 \rangle| \right)^{-1} \langle T^k\varphi_0, \varphi_0 \rangle$$

for $j = 1, 2, ...$, and since $\langle T^k\varphi_0, \varphi_0 \rangle < 0$ for $k = 1, 3, ..., n$.

(iii) follows immediately from the fact that $\langle T^k\varphi, \varphi \rangle \geq 0$ for $k > n$, which completes the proof.
REFERENCES


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VERFASSER:

Dr. JÜRGEN FRIEDRICH
Sektion Mathematik der Karl-Marx-Universität
DDR-7010 Leipzig, Karl-Marx-Platz