On Nonlinear Volterra Integral Equations with State Dependent Delays in Several Variables

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Abstract. Lipschitz continuous solutions of nonlinear Volterra integral equations with state dependent delays in several variables are investigated. The results are based on a comparison method and the Banach fixed point principle.

Keywords. Initial problems, comparison method, neutral equations with state dependent delays, Volterra integral equations in several variables

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1. Introduction

Let $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}^n$ be the real $n$-dimensional Euclidean space with the Euclidean norm $|\cdot|$. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $a$, $r \in \mathbb{R}_+^n$. We denote by $C(U, V)$ the set of all continuous functions from $U$ to $V$. Let $G = \{x \in \mathbb{R}^n : 0 \leq x \leq a\}$, $\Omega = \{x \in \mathbb{R}^n : -r \leq x \leq a\}$ and $B = \Omega \setminus G$.

For $u : \Omega \to E$, where $E$ is a Banach space with the norm $\|\cdot\|$, we define the function $u_x(\tau) = u(x + \tau)$, $\tau \in B$, $x \in G$. There are given the functions $F \in C(G \times E^m \times C(B, E), E)$, $f_i \in C(G \times G \times C(B, E), E)$, $i = 1, \ldots, m$, $m \in \mathbb{N}$, $\theta, \Psi_i \in C(G \times C(B, E), E)$, $i = 1, \ldots, m$, $m \in \mathbb{N}$, $\beta, \alpha_i \in C(G, G)$, $i = 1, \ldots, m$, and $\varphi \in C(B, E)$.

Consider the problem

\begin{align*}
  u(x) &= F \left( x, \int_{H(x)} f \left( x, s, u\Psi(s, u_{\alpha_i}(s)) \right) ds, u\Theta(x, u_{\beta_i}(x)) \right), \quad x \in G \quad (1) \\
  u(x) &= \varphi(x), \quad x \in B, \quad (2)
\end{align*}

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where
\[ \int_{H(x)} f(x, s, u\psi(s, u_0(s))) ds = \left( \int_{H_1(x)} f_1(x, s, u\psi_1(s, u_0(s))) (ds)_{p_1}, \ldots, \int_{H_m(x)} f_m(x, s, u\psi_m(s, u_0(s))) (ds)_{p_m} \right), \]
and \( H_i(x) \subseteq \{ s \in \mathbb{R}^n : 0 \leq s \leq x \}, x \in G, i = 1, \ldots, m. \)

Assume that \( H_i(x) \) is contained in a \( p_i \)-dimensional hyperplane \( (1 \leq p_i \leq n) \), where \( p_i \) does not depend on \( x \), parallel to the coordinate axes, and that it is Lebesgue measurable, when considered as a \( p_i \)-dimensional set. Denote by \( L_{p_i}(H_i(x)) \) the \( p_i \)-dimensional measure of \( H_i(x) \) and let \( \Gamma_i, \tilde{\Gamma}_i \subseteq \{1, \ldots, n\}, i = 1, \ldots, m, \) be defined by
\[ \Gamma_i = \{ j : \text{the axis } O x_j \text{ is parallel to the hyperplane containing } H_i(x) \}, \]
\[ \text{and } \tilde{\Gamma}_i = \{1, \ldots, n\} \setminus \Gamma_i. \]

Put \( A = \{ i : p_i = n \}, A' = \{ i : 1 \leq p_i < n \}. \) Let a \( p_i \)-dimensional hyperplane containing \( H_i(x) \) be defined by \( x_{k_1} = x_{k_1}', x_{k_2} = x_{k_2}', \ldots, x_{k_l} = x_{k_l}', \)
\( l = n - p_i. \) Then \( \int_{H_i(x)} z(x, s) (ds)_{p_i}, s = (s_1, \ldots, s_n), \) denotes the \( p_i \)-dimensional Lebesgue integral with respect to the variables \( s_{k_j}, k_j \in \Gamma_i, \) and in the above integral we have \( s_{k_j} = x_{k_j} \) for \( k_j \in \tilde{\Gamma}_i. \)

We define
\[ G_i(x) = \left\{ s \in \mathbb{R}^n : s_{k_j} = x_{k_j}' \text{ for } k_j \in \tilde{\Gamma}_i, 0 \leq x_{k_j} \leq \phi^{(i)}_{k_j}(x) \text{ for } k_j \in \Gamma_i \right\}, \]
where \( \phi^{(i)}_{k_j} \in C(G, \mathbb{R}), k_j \in \Gamma_i, \) and \( H_i(x) \subseteq G_i(x) \subseteq \{ s \in \mathbb{R}^n : 0 \leq s \leq x \}. \)
Then we have \( L_{p_i}(G_i(x)) = \prod_{s \in \tilde{\Gamma}_i} \phi^{(i)}_{s}(x). \) To simplify we use the following notations:
\[ L(G(x)) = (L_{p_1}(G_1(x)), \ldots, L_{p_m}(G_m(x))), \]
and
\[ K(x) \int_{H(x)} f(x, s, u\psi(s, u_0(s))) ds = \sum_{j=1}^{m} K_j(x) \int_{H_j(x)} f_j(x, s, u\psi_j(s, u_0(s))) (ds)_{p_j}, \]
\[ K(x)L(G(x)) = \sum_{j=1}^{m} K_j(x)L_{p_j}(G_j(x)), \quad x \in G, \]
where \( x \in G, K = (K_1, \ldots, K_m) \in C(G, \mathbb{R}^m). \)

Ordinary functional differential equations with state dependent delays have attracted the attention of many authors \([1, 3, 5, 7–12, 20, 25], \) and [34]. The
particular case of equation (1) with $n$ partial differential equations with state dependent delays (see also [4, 6, 16]). A paper [19] initiated the study of the existence theory for first order functional differential equations of the form (1), (2) with $n$ to Volterra functional integral equations of type (1). One of the simplest problems for the ordinary functional differential equations of the neutral type: equations considered in [13, 17, 21, 26, 27, 29].

In case $r = 0$ the above problem leads to the single equation of type (1) without the additional condition (2). Therefore equation (1) is a generalization of equations investigated in [23, 24, 30, 31].

Various initial value problems for the hyperbolic functional differential equations of the neutral type with two independent variables

$$\begin{align*}
D_{xy} z(x, y) &= F(x, y, z(\alpha_1^0(x, y)), (\alpha_2^0(x, y))) D_x z(\alpha_1^0(x, y)), (\alpha_2^0(x, y)) D_y z(\beta_1(x, y)), (\beta_2(x, y))) \quad \text{for } (x, y) \in [0, \tilde{a}] \times [0, \tilde{b}],
\end{align*}$$

with the initial condition

$$z(x, y) = \varphi(x, y), \quad (x, y) \in [-r_1, \tilde{a}] \times [-r_2, \tilde{b}] \setminus [0, \tilde{a}] \times [0, \tilde{b}],$$

can also be transformed to the problem of type (1), (2). The Volterra functional integral equation corresponding to that problem takes the form

$$\begin{align*}
u(x, y) &= F(x, y, -\varphi(0, 0) + \varphi(\alpha_1^0(x, y), 0) + \varphi(0, \alpha_2^0(x, y)) + \int_{H_0(x, y)} u(s, t) \, ds \, dt, \\
D_x \varphi(\alpha_1^0(x, y), 0) + \int_{H_1(x, y)} u(s, t) \, ds, D_y \varphi(0, \alpha_2^0(x, y)) + \int_{H_2(x, y)} u(s, t) \, ds, u(\beta_1(x, y)), (\beta_2(x, y))) \quad \text{for } (x, y) \in [-r_1, 0] \times [-r_2, 0],
\end{align*}$$

where

$$\begin{align*}
H_0(x, y) &= \left\{(s, t) : s \in \left[0, \alpha_1^{(0)}(x, y)\right], t \in \left[0, \alpha_2^{(0)}(x, y)\right]\right\}, \\
H_1(x, y) &= \left\{(s, t) : s = \alpha_1^{(1)}(x, y), t \in \left[0, \alpha_2^{(1)}(x, y)\right]\right\}, \\
H_2(x, y) &= \left\{(s, t) : s \in \left[0, \alpha_1^{(2)}(x, y)\right], t = \alpha_2^{(2)}(x, y)\right\}.
\end{align*}$$

For this reason, in case $r_1 = r_2 = 0$ equation (1) is a generalization of the equations investigated in [23, 24, 30, 31].
The case where $\Psi(x, w) = \alpha(x)$, $\Theta(x, w) = \beta(x)$ was studied in [2] and [14]. The Cauchy problem and the Goursat problem for hyperbolic functional differential equations also lead to equations of type (1) (see [32]). Initial value problems for equations in more than two variables and problems for equations of higher order can be transformed in terms of Volterra functional integral equations. As a particular case of equation (1) we can obtain the system of Volterra integral equations considered in [18, 22, 28].

In this paper we prove a theorem of the existence and uniqueness of Lipschitz continuous solutions of the problem (1), (2). If we assume that the Lipschitz coefficient $l$ of the function $F$ with respect to the last variable satisfies the condition $l < 1$, then we have a theorem on the existence and uniqueness of solutions of (1), (2), which can be obtained by means of the Banach fixed point theorem. We relaxed this very restrictive condition. We proved that the integral operator defined by the right-hand side of (1) is a contraction with a weighted norm constructed with the help of a solution of a certain comparative integral equation.

2. Assumptions and lemmas

Suppose that for any $x \in G$ and $i \in A^\prime$ the set $H_i(x)$ is contained in a $p_i$-dimensional hyperplane $S_i(x)$, parallel to the $p_i$ coordinate axes, where $p_i = 1, \ldots , n - 1$. Then for any $y \in \mathbb{R}^n$, such that $x + y \in G$, there exists a vector $v_i(x, y) \in \mathbb{R}^n$ perpendicular to $S_i(x)$ and $-v_i(x, y) + H_i(x + y) \subseteq S_i(x)$.

**Assumption H1.** Suppose that

(i) there exist $\omega \in \mathbb{R}_+$: $L_n(H_i(x) \Delta H_i(\bar{x})) \leq \omega|x - \bar{x}|$, $i \in A$

(the sign $\Delta$ denotes the symmetric difference of two sets);

(ii) $L_{p_i}(H_i(x) \Delta (v_i(x, \bar{x} - x) + H_i(\bar{x}))) \leq \omega|x - \bar{x}|$,

$v_i(x, \bar{x}) \geq 0$, $\lim_{x \rightarrow \bar{x}} v_i(x, \bar{x} - x) = 0$, $i \in A^\prime$, $x, \bar{x} \in G$, $x \leq \bar{x}$;

(iii) $H_i(x) \subseteq H_i(\bar{x})$ for $x, \bar{x} \in G$, $x \leq \bar{x}$, and $i \in A$;

(iv) $H_i(x) + v_i(x, \bar{x} - x) \subseteq H_i(\bar{x})$ for $x, \bar{x} \in G$, $x \leq \bar{x}$, and $i \in A^\prime$.

**Assumption H2.** Suppose that

(i) $l, \bar{h} \in C(G, \mathbb{R}_+)$, $K \in C(G, \mathbb{R}_+)$, $\Theta, \Psi_i : G \times C(B, E) \rightarrow G$, $i = 1, \ldots , m$, are nondecreasing functions;

(ii) $\gamma_i, \zeta \in C(G, G)$, $i = 1, \ldots , m$, are nondecreasing functions, and $\gamma_i(x) \leq x$,

$\zeta(x) \leq x$ for $x \in G$, $i = 1, \ldots , m$;

(iii) the function $\bar{m} : G \rightarrow \mathbb{R}_+$ is defined by

$$\bar{m}(x) = \sum_{i=0}^{+\infty} l_i(x) h(\zeta_i(x)) < +\infty,$$
where \( \zeta_0(x) = x, \zeta_{i+1}(x) = \zeta_i(\zeta_i(x)), l_0(x) = 1, l_{i+1} = l(x)l_i(\zeta(x)), \) and \( i = 0, 1, \ldots, x \in G; \)
(iv) the function \( M : G \to \mathbb{R}_+ \) is given by
\[
M(x) = \sum_{i=0}^{+\infty} l_i(x)K(\zeta_i(x))L(G(\zeta_i(x))) < +\infty, \quad x \in G;
\]
(v) the function \( \bar{M} : G \to \mathbb{R}_+ \) given by
\[
\bar{M}(x) = \sum_{i=0}^{+\infty} l_i(x)K(\zeta_i(x))L(G(\zeta_i(x))) \left( \prod_{s \in \Gamma_i} x_s \right)^{-1}
\]
is bounded on \( G. \)

Further we will use the following notation:
\[
\tilde{m}(x) = \sum_{i=0}^{+\infty} l_i(x)\tilde{h}(\zeta_i(x))
\]
\[
(Vu)(x) = \sum_{i=0}^{+\infty} l_i(x)(K(\zeta_i(x)) \int_{H(\zeta_i(x))} u(\gamma_i(s))ds).
\]

**Remark 1.** Suppose that
(I) conditions (i)–(iv) of Assumption \( H_2 \) are satisfied;
(II) \( \tilde{h} \in C(G, \mathbb{R}_+) \) and \( \tilde{h}(x) \leq \bar{h}(x) \) for \( x \in G; \)
(III) \( g : G \to \mathbb{R}_+ \) is upper semicontinuous.

Then \( \tilde{m} \) and \( Vg \) are functions well defined for \( x \in G. \)

**Lemma 2.1.** Suppose that Assumptions \( H_1, H_2 \) are satisfied, \( \tilde{h} \in C(G, \mathbb{R}_+) \) is nondecreasing, and \( \tilde{h}(x) \leq \bar{h}(x) \) on \( G. \) Then
(I) there exists a nondecreasing solution \( \bar{g} \in C(G, \mathbb{R}_+) \) of the equation
\[
g(x) = \tilde{m}(x) + (Vg)(x), \quad x \in G, \tag{3}
\]
which is unique in the set \( P(G, \mathbb{R}_+) \) of upper semicontinuous functions from \( G \) to \( \mathbb{R}_+; \)
(II) the function \( \bar{g} \) is a solution of the equation
\[
g(x) = K(x) \int_{H(x)} g(\gamma(s))ds + l(x)g(\zeta(x)) + \tilde{h}(x), \quad x \in G, \tag{4}
\]
which is unique in the class \( P(G, \mathbb{R}_+, \bar{g}) \) of all functions from the class \( P(G, \mathbb{R}_+), \) such that \( \inf \{ \kappa \in \mathbb{R}_+ : g(x) \leq \kappa \bar{g}(x), x \in G \} < +\infty, \) where \( \bar{g} \) is a solution of (3) with \( \tilde{h} = \bar{h}. \)
(III) the function $\tilde{g}$ satisfies the condition

$$\lim_{i \to +\infty} l_i(x)\tilde{g}(\zeta_i(x)) = 0$$

uniformly on $G$.

Proof. First we show that equation (3) has a unique solution in the class $P(G, \mathbb{R}_+)$. We define the operator

$$(Tz)(x) = \tilde{m}(x) + (Vz)(x), \quad x \in G.$$ We prove that $T : P(G, \mathbb{R}_+) \to P(G, \mathbb{R}_+)$. Let $z \in P(G, \mathbb{R}_+)$ and

$$v_{ij}(x) = \int_{H_j(\zeta_i(x))} z(\gamma_j(s))(ds)_{p_j},$$

where $x \in G$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $m, n \in \mathbb{N}$. Because of $z \in P(G, \mathbb{R}_+)$, so there exists the sequence $\{z_k\}_{k \in \mathbb{N}}$, such that $z_k \in C(G, \mathbb{R}_+)$ and $z_{k+1} \leq z_k$, $z(x) = \lim_{k \to +\infty} z_k(x), \quad x \in G, k \in \mathbb{N}$. Let

$$v_{ij}^{(k)}(x) = \int_{H_j(\zeta_i(x))} z_k(\gamma_j(s))(ds)_{p_j},$$

where $x \in G$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k, m, n \in \mathbb{N}$. Functions $v_{ij}^{(k)}$ are continuous in $G$ (see [32]) and $v_{ij}^{(k+1)}(x) \leq v_{ij}^{(k)}(x)$. From Lebesgue’s theorem (about the integration of the sequence of nonincreasing functions) we have

$$v_{ij}(x) = \lim_{k \to +\infty} v_{ij}^{(k)}(x), \quad x \in G, i = 1, \ldots, n, j = 1, \ldots, m, k, m, n \in \mathbb{N}$$

and therefore $v_{ij} \in P(G, \mathbb{R}_+)$. From the Weierstrass criterium (elements of these series are continuous and nondecreasing) follows the uniform convergence of the series

$$\tilde{m}(x) = \sum_{i=0}^{+\infty} l_i(x)\tilde{h}(\zeta_i(x)), \quad M(x) = \sum_{i=0}^{+\infty} l_i(x)K(\zeta_i(x))L(G(\zeta_i(x))).$$

Now we have $l_i(x)\tilde{h}(\zeta_i(x)) \leq l_i(x)\tilde{h}(\zeta_i(x))$, and

$$l_i(x)K(\zeta_i(x))\int_{H(\zeta_i(x))} z(\gamma(s))(ds) \leq \left(\sup_{x \in G} z(x)\right)l_i(x)K(\zeta_i(x))L(G(\zeta_i(x))),$$

where $i \in \mathbb{N}$, $x \in G$. Hence we get that the series

$$\tilde{m} = \sum_{i=0}^{+\infty} l_i(x)\tilde{h}(\zeta_i(x)), \quad \sum_{i=0}^{+\infty} l_i(x)K(\zeta_i(x))\int_{H(\zeta_i(x))} z(\gamma(s))(ds)$$
are convergent. Now it is easily seen that \( \bar{m} \in C(G, \mathbb{R}_+) \), \( Vz \in P(G, \mathbb{R}_+) \) and consequently \( T : P(G, \mathbb{R}_+) \to P(G, \mathbb{R}_+) \).

Now we show that operator \( T \) is a contraction. We define the norm

\[
\|z\|_\lambda = \max_{x \in G} \left[ |z(x)| \cdot \exp \left( -\lambda \sum_{p=1}^{n} x_p \right) \right], \quad z \in P(G, \mathbb{R}_+),
\]

where \( \lambda > \Lambda = \max \{ 1, \sup_{x \in G} \bar{M}(x) \} \). For \( z, w \in P(G, \mathbb{R}_+) \) we get

\[
|\langle Tz \rangle(x) - \langle Tw \rangle(x) | \leq \sum_{i=0}^{+\infty} l_i(x) K(\zeta_i(x)) \int_{H(\zeta_i(x))} |z(\gamma(s)) - w(\gamma(s))|(ds)
\]
\[
\leq \|z - w\|_\lambda \sum_{i=0}^{+\infty} l_i(x) K(\zeta_i(x)) \int_{H(\zeta_i(x))} \exp \left( \lambda \sum_{p=1}^{n} s_p \right) ds.
\]

Using the estimation \( \exp(\epsilon t) - 1 \leq \epsilon \exp(t), \epsilon \in [0,1], t \geq 0 \), we have the following:

\[
\int_{H_j(\zeta_i(x))} \exp \left( \lambda \sum_{p=1}^{n} x_p \right)(ds)_{p_j} \leq \exp \left( \lambda \sum_{p \in \Gamma_j} x_p \right)(ds)_{p_j} \exp \left( \lambda \sum_{p \in \Gamma_j} s_p \right)(ds)_{p_j}
\]
\[
\leq \exp \left( \lambda \sum_{p \in \Gamma_j} x_p \right) \prod_{j=1}^{\infty} \left( \frac{1}{\lambda} \exp(\lambda \phi^{(j)}(\zeta_p(x)) - 1) \right)
\]
\[
\leq \frac{1}{\lambda} \exp \left( \lambda \sum_{p=1}^{n} x_p \right) L_{p_j} \left( G(\zeta_i(x)) \right) \left( \prod_{p \in \Gamma_j} x_p \right)^{-1}.
\]

Further we have

\[
|\langle Tz \rangle(x) - \langle Tw \rangle(x) |
\]
\[
\leq \frac{1}{\lambda} \|z - w\|_\lambda \sum_{i=0}^{+\infty} l_i(x) \sum_{j=1}^{m} K_j(\zeta_i(x)) L_{p_j} \left( G(\zeta_i(x)) \right) \left( \prod_{p \in \Gamma_j} x_p \right)^{-1} \exp \left( \lambda \sum_{p=1}^{n} x_p \right)
\]
\[
\leq \frac{\Lambda}{\lambda} \exp \left( \lambda \sum_{p=1}^{n} x_p \right) \|z - w\|_\lambda,
\]

and consequently \( \|Tz - Tw\|_\lambda \leq \frac{1}{\lambda} \|z - w\|_\lambda \). From the Banach theorem we get that for \( \lambda > \Lambda \) equation (3) has a unique solution \( \bar{z} \in P(G, \mathbb{R}_+) \).

Now we show that \( \bar{z} \in C(G, \mathbb{R}_+) \) and is nondecreasing. Indeed \( \bar{z}(x) = \lim_{n \to +\infty} z_n(x), \ x \in G \), where \( z_0 \in P(G, \mathbb{R}_+) \), \( z_0(x) = \text{const} \), and \( z_{n+1}(x) = (Tz_n)(x) = \bar{m}(x) + (Vz_n)(x), \ x \in G, n \in N \). The function \( z_0 \in C(G, \mathbb{R}_+) \) is nondecreasing. Therefore we easily get that \( z_n \in C(G, \mathbb{R}_+) \) for \( n \in N \), and functions \( z_n \) are nondecreasing for \( n \in N \). The point (1) is proved.
We prove point (II). Indeed, for $\tilde{z} \in C(G, \mathbb{R}_+)$ and $\tilde{h} = \tilde{h}$ we get from (3) the following:

$$l(x)\tilde{z}(\zeta_i(x)) = l_i(x)\sum_{j=0}^{\infty} l_j(\zeta_i(x))\tilde{h}(\zeta_{i+j}(x))$$

$$+ l_i(x)\sum_{j=0}^{\infty} l_j(\zeta_i(x))\left( K(\zeta_{i+j}(x)) \int_{H(\zeta_{i+j}(x))} \tilde{z}(\gamma(s))\,ds \right)$$

$$= \sum_{j=1}^{\infty} l_j(x)\tilde{h}(\zeta_j(x)) + \sum_{j=1}^{\infty} l_j(x)\left( K(\zeta_j(x)) \int_{H(\zeta_j(x))} \tilde{z}(\gamma(s))\,ds \right),$$

where $x \in G, i \in N$.

Now we show that an arbitrary solution of (3) denoted by $\tilde{z}$ is a solution of equation (4). If $\tilde{z}$ is a solution of (3), then we have

$$\tilde{z}(x) - K(x) \int_{H(x)} \tilde{z}(\gamma(s))\,ds - l(x)\tilde{z}(\zeta_i(x))$$

$$= \sum_{i=0}^{\infty} l_i(x)\tilde{h}(\zeta_i(x)) + \sum_{i=0}^{\infty} l_i(x)\left( K(\zeta_i(x)) \int_{H(\zeta_i(x))} \tilde{z}(\gamma(s))\,ds \right)$$

$$- K(x) \int_{H(x)} \tilde{z}(\gamma(s))\,ds - l(x)\sum_{i=0}^{\infty} l_i(\zeta_i(x))\tilde{h}(\zeta_i(\zeta_i(x)))$$

$$+ \sum_{i=0}^{\infty} l_i(\zeta_i(x))\left( K(\zeta_i(\zeta_i(x))) \int_{H(\zeta_i(\zeta_i(x)))} \tilde{z}(\gamma(s))\,ds \right)$$

$$= \tilde{h}(x).$$

Now we prove that $\tilde{z}$ is a unique solution of (4) in the class $P(G, \mathbb{R}_+, \tilde{g})$. Let $z \in P(G, \mathbb{R}_+, \tilde{g})$ be an arbitrary solution of (4). Then

$$z(x) = \sum_{i=0}^{n-1} l_i(x)K(\zeta_i(x)) \int_{H(\zeta_i(x))} z(\gamma(s))\,ds$$

$$+ \sum_{i=0}^{n-1} l_i(x)\tilde{h}(\zeta_i(x)) + l_i(x)z(\zeta_i(x)), \quad x \in G, n \in N.$$  (6)

Because $0 \leq z(x) \leq \kappa \tilde{z}(x)$ for a certain $\kappa \in \mathbb{R}_+$, then $\lim_{t \to +\infty} l_i(x)z(\zeta_i(x)) = 0$ uniformly on $G$. If in (6) $n \to +\infty$, then $z(x) = \tilde{m}(x) + (Vz)(x)$ for $x \in G$, and it means that it is a solution of (3). Because equation (3) has the only one solution, then $z(x) = \tilde{z}(x)$. The proof of Lemma 2.1 is finished. \qed
In the space $C(B, E)$ we define the norm
\[ \|u\|_0 = \sup_{\tau \in B} \|u(\tau)\|, \quad u \in C(B, E). \]

**Assumption $H_3$.** Suppose that there exist nondecreasing functions $\bar{p}_i$, $\bar{k}_i$, $\bar{\eta}_i \in C(G, G)$, such that
\[
\|f_i(x, t, w) - f_i(x, t, \bar{w})\| \leq \bar{p}_i(x)\|w - \bar{w}\|_0, \quad i = 1, \ldots, m
\]
\[
\|F(x, v, w) - F(x, \bar{v}, \bar{w})\| \leq \sum_{i=1}^{m} \bar{k}_i(x)\|v_i - \bar{v}_i\| + \bar{\eta}_i(x)\|w - \bar{w}\|_0
\]
where $t, x \in G, v, \bar{v} \in E^m, w, \bar{w} \in C(B, E)$, and for $x \in G$ we have $\eta(x) \leq x, \xi_i(x) \leq x$.

**Remark 2.** The consequence of Assumption $H_2$ is the fact that there exist functions $\delta_i, \Delta : G \to \mathbb{R}_+$, $i = 1, \ldots, m$, such that
\[
\|f_i(x, t, w)\| \leq \bar{p}_i(x)\|w\|_0 + \delta_i(x), \quad i = 1, \ldots, m
\]
\[
\|F(x, v, w)\| \leq \sum_{i=1}^{m} \bar{k}_i(x)\|v_i\| + \bar{\eta}_i(x)\|w\|_0 + \Delta(x),
\]
where $t, x \in G, \|w\|_0 \leq \bar{g}(a), \|v_i\|_0 \leq \bar{p}_i(a)\bar{g}(a)L_{p_i}(G_i(a))$, and
\[
\delta_i(x) = \max_{s \in [0, x]} \max_{t \in G} \|f_i(s, t, \theta)\|, \quad \Delta(x) = \max_{s \in [0, x]} \|F(s, \theta, \theta)\|.
\]

$\theta$ means the zero in the space $C(B, E)$.

**Lemma 2.2.** Suppose that the assumptions of Lemma 2.1 are satisfied with functions $\gamma_i(s) = \xi_i(x), i = 1, \ldots, m$, $\zeta(x) = \eta(x)$, $K(x) = \sum_{i=1}^{m} \bar{k}_i(x)\bar{p}_i(x)$, $\dot{h}(x) = \Delta(x) + \sum_{i=1}^{m} \bar{k}_i(x)\delta_i(x)L_{p_i}(G_i(x))$, $l(x) = \bar{l}(x)$, and Assumption $H_3$ holds. Then
\[
\mathcal{F} : B(\Omega, E, \bar{g}) \to B(\Omega, E, \bar{g}),
\]
where $B(\Omega, E, \bar{g}) = \{ u \in C(\Omega, E) : u|_B = \varphi, \|u(s)\| \leq \bar{g}(t), s \in [-r, t], t \in G \}$, and $\mathcal{F}$ is defined by right side of equation (1).
Proof. Let \( w \in B(\Omega, E, \bar{g}) \). Then for \( x \in G \) we have
\[
\|F[u](x)\| \leq \sum_{i=1}^{m} \tilde{k}_i(x) \int_{H_i(x)} \tilde{p}_i(x) \|u_{\Psi_i(s, u_\alpha(s))}\|_0 (ds) \rho_i \\
+ \tilde{l}(x)\bar{g}(\eta(x)) + \Delta(x) + \sum_{i=1}^{m} \tilde{k}_i(x)\delta_i(x)L_p(G_i(x)) \\
\leq \left[ \sum_{i=1}^{m} \tilde{k}_i(x)\tilde{p}_i(x) \right] \int_{H(x)} \bar{g}(\xi(s))ds + \tilde{l}(x)\bar{g}(\eta(x)) \\
+ \Delta(x) + \sum_{i=1}^{m} \tilde{k}_i(x)\delta_i(x)L_p(G_i(x)) \\
= \bar{g}(x).
\]

Therefore \( \|F[u](x)\| \leq \bar{g}(x) \) for \( x \in G \). Hence it follows that \( F[u] \in B(\Omega, E, \bar{g}) \). The lemma is proved. \( \square \)

**Assumption \( H_4 \).** Suppose that there exist nondecreasing functions \( \rho : G \to \mathbb{R}_+ \), \( \mu : G \to \mathbb{R}_+ \), and constants \( d, q, \nu, \sigma \in \mathbb{R}_+ \), such that

(i) \( \|\Theta(x, w) - \Theta(x, \bar{w})\| \leq \rho(x)\|w - \bar{w}\|_0 \)

(ii) \( \|\Theta(x, w) - \Theta(x, \bar{w})\| \leq d|x - \bar{x}| \)

(iii) \( \|f_i(x, t, w) - f_i(x, t, \bar{w})\| \leq q_i|x - \bar{x}| \)

(iv) \( \|F_i(x, v, w) - F_i(\bar{x}, v, \bar{w})\| \leq \nu|x - \bar{x}| \)

(v) \( \|\beta(x) - \beta(\bar{x})\| \leq \sigma|x - \bar{x}| \)

(vi) \( \|\Psi_i(x, w) - \Psi_i(x, \bar{w})\| \leq \mu_i(x)\|w - \bar{w}\|_0 \),

where \( (x, w), (x, \bar{w}) \in G \times C(B, E) \).

We define functions \( M_1, M_2, M_3 \in C(G, \mathbb{R}_+) \) as follows:

\[
M_1(x) = \omega \left[ \bar{g}(x) \sum_{i=1}^{m} p_i(x) + \sum_{i=1}^{m} \delta_i(x) \right] + \sum_{i=1}^{m} q_i L_p(G_i(x)) + \nu \tag{7}
\]

\[
M_2(x) = d\tilde{l}(x) \tag{8}
\]

\[
M_3(x) = \sigma\tilde{l}(x)\rho(x). \tag{9}
\]

Suppose that \( M_2(a) < 1 \) and \( [M_2(a) - 1]^2 - 4M_1(a)M_3(a) > 0 \). Let \( \lambda_1, \lambda_2 \) be two different positive roots of the equation \( M_3(a)\lambda^2 + [M_2(a) - 1]\lambda + M_1(a) = 0 \). Now we define the following class of functions:

\[
D([-r, a], E, \lambda) = \{ u \in B([-r, a], E, \bar{g}) : \|u(x) - u(\bar{x})\| \leq \lambda|x - \bar{x}|, x, \bar{x} \in G \},
\]

where \( \lambda \in [\lambda_1, \lambda_2] \), if \( M_3(a) \neq 0 \), and \( \lambda \geq M_1(a)[1 - M_2(a)]^{-1} \), if \( M_3(a) = 0 \).
Lemma 2.3. Suppose that the assumptions of Lemma 2.2 and Assumptions (i)–(v) of $H_4$ are satisfied and $M_2(a) < 1$, $[M_2(a) - 1]^2 - 4M_1(a)M_3(a) > 0$, where the functions $M_1$, $M_2$, $M_3$ are defined by (7)–(9). Then $\mathcal{F} : D([-r, a], E, \lambda) \to D([-r, a], E, \lambda)$.

Proof. From Assumptions $H_3$ and $H_4$ we have

$$
\|\mathcal{F}[u](x) - \mathcal{F}[u](\bar{x})\| \leq \nu|x - \bar{x}| + \bar{\nu}(x)\lambda[d|x - \bar{x}| + \rho(x)\|u_{\beta(x)} - u_{\beta(x)}\|_0]
$$

$$
+ \sum_{i=1}^{m} \omega(p_i(x)\bar{g}(\xi_i(x)) + \delta_i(x)) + q_iL_{p_i}(G_i(x))\|x - \bar{x}\|
$$

$$
\leq \nu|x - \bar{x}| + \bar{\nu}(x)\lambda[d|x - \bar{x}| + \rho(x)\lambda|\beta(x) - \beta(x)|]
$$

$$
+ \sum_{i=1}^{m} [\omega p_i(x)\bar{g}(x) + \omega \delta_i(x) + q_iL_{p_i}(G_i(x))]|x - \bar{x}|
$$

$$
\leq \left\{ [\sigma\bar{\nu}(x)\rho(x)]\lambda^2 + [\bar{d}(x)]\lambda + \left[ \nu + \bar{\nu}(x)\omega \sum_{i=1}^{m} p_i(x)
$$

$$
+ \omega \sum_{i=1}^{m} \delta_i(x) + \sum_{i=1}^{m} \delta_iL_{p_i}(G_i(x)) \right]\right\}|x - \bar{x}|
$$

$$
\leq \lambda|x - \bar{x}|,
$$

where $x, \bar{x} \in G$. This means that $\mathcal{F} : D([-r, a], E, \lambda) \to D([-r, a], E, \lambda)$. The lemma is proved.

3. The main theorem.

For $u \in D([-r, a], E, \lambda)$, where $\lambda$ is defined in Lemma 2.3, we define the norm

$$
\|u\|_x = \sup_{s \in [-r, x]} \|u(s)\|.
$$

Theorem 3.1. Let the assumptions of Lemma 2.3 and Assumption (vi) of $H_4$ hold. Then the Cauchy problem (1), (2) has the unique solution in the class $D([-r, a], E, \lambda)$.

Proof. Because of Assumptions $H_3$, $H_4$, $H_5$ for $u, \bar{u} \in D([-r, a], E, \lambda)$ we have

$$
\|\mathcal{F}[u](x) - \mathcal{F}[ar{u}](x)\| \leq \sum_{i=1}^{m} \bar{k}_i(x)p_i(x)(1 + \lambda \mu_i(x)) \int_{H_i(x)} \|w - \bar{w}\|_{\tilde{\alpha}_i(s)}(ds)_{p_i}
$$

$$
+ \bar{\nu}(x)(1 + \lambda \rho(x))\|u - \bar{u}\|_{\tilde{\beta}(x)},
$$

where $x \in G$, and $\tilde{\alpha}_i(s) = \max \{\zeta_i(s), \alpha_i(s)\}$, $\tilde{\beta}(x) = \max \{\eta(x), \beta(x)\}$. 

Let \( \tilde{z} \in C(G, E) \) be a solution of equation (4) with \( \tilde{h} = \hat{h} \). It is easily seen that \( \tilde{z}(x) \geq \hat{h}(x) \) for \( x \in G \). If \( \hat{h}(x) > 0 \) then \( \tilde{z} > 0 \). Suppose that \( \tilde{z} \) is any positive and nondecreasing extension of \( \tilde{z} \) onto the set \( \Omega \). For \( u \in D(\Omega, E, \lambda) \) we define the norm
\[
\|u\|_* = \max_{x \in G} \frac{1}{\tilde{z}(x)} \|u\|_x.
\]
We get
\[
\|F[u](x) - F[\tilde{u}](x)\| \leq \sum_{i=1}^{m} \bar{k}_i(x)p_i(x)[1 + \lambda \mu_i(x)] \int_{H_i(x)} \|\tilde{u} - u\|_{\alpha_i(s)}(ds)p_i + \bar{l}(x)[1 + \lambda \rho(x)]\|u - \tilde{u}\|_{\beta(x)}.
\]
Note that for \( \tau \in B \) and \( s \in G \) we have
\[
\|u(\tilde{\alpha}_i(s) + \tau) - \tilde{u}(\tilde{\alpha}_i(s) + \tau)\| \leq \frac{1}{\tilde{z}(\tilde{\alpha}_i(s) + \tau)} \|u(\tilde{\alpha}_i(s) + \tau) - \tilde{u}(\tilde{\alpha}_i(s) + \tau)\| \tilde{z}(\tilde{\alpha}_i(s) + \tau)
\]
\[
\leq \|u - \tilde{u}\|_* \tilde{z}(\tilde{\alpha}_i(s)).
\]
Analogously we have a such estimation for \( \|u(\tilde{\beta}(x) + \tau) - \tilde{u}(\tilde{\beta}(x) + \tau)\| \). Therefore \( \|u - \tilde{u}\|_{\alpha_i(s)} \leq \|u - \tilde{u}\|_* \tilde{z}(\tilde{\alpha}_i(s)) \) and \( \|u - \tilde{u}\|_{\beta(x)} \leq \|u - \tilde{u}\|_* \tilde{z}(\tilde{\beta}(x)) \). Now we get
\[
\|F[u](x) - F[\tilde{u}](x)\| \leq \left\{ \sum_{i=1}^{m} \bar{k}_i(x)p_i(x)[1 + \lambda \mu_i(x)] \int_{H_i(x)} \tilde{z}(\tilde{\alpha}_i(s))ds + \bar{l}(x)[1 + \lambda \rho(x)]\tilde{z}(\tilde{\beta}(x)) \right\}\|u - \tilde{u}\|_*
\]
\[
\leq (\tilde{z}(x) - \hat{h}(x))\|u - \tilde{u}\|_*,
\]
where \( x \in G \), and finally
\[
\|F[u](x) - F[\tilde{u}](x)\|_* \leq \left( 1 - \inf \frac{\hat{h}(x)}{\tilde{z}(x)} \right)\|u - \tilde{u}\|_*.
\]
Thus by the Banach fixed point theorem the problem (1), (2) has a unique solution in the class \( D([-r, a], E, \lambda]) \), where \( \lambda \) is defined in Lemma 2.2. The main theorem is proved. 

4. Some effective conditions

Now we give some examples of effective conditions for Assumptions (iii)–(v) of \( H_2 \) to be satisfied (see [18]).
Example 1. Suppose that there exist
(a) $\bar{K}_i, \bar{l}, \bar{\zeta}_j \in \mathbb{R}_+, i = 1, \ldots, m, j = 1, \ldots, n$, such that

(i) $l(x) \leq \bar{l}, K_i(x) \leq \bar{K}_i, i = 1, \ldots, m$
(ii) $\zeta_j(x) \leq \bar{\zeta}_j, \zeta \leq 1, j = 1, \ldots, n$
(iii) $\bar{l} \prod_{s \in \bar{\Gamma}} \bar{\zeta}_s < 1$ for $i = 1, \ldots, m$
(iv) $\sum_{N=0}^{+\infty} \bar{\l}_N(\bar{\zeta}_N x_1, \ldots, \bar{\zeta}_N x_n) < +\infty$;
(b) $\bar{\gamma}(i)_{k_j} \in \mathbb{R}_+$, such that $\phi_{k_j} \leq \bar{\gamma}(i)_{k_j} x_{k_j}$, and $\bar{\gamma}(i)_{k_j} \leq 1$, where $k_j \in \Gamma_i$.

Then the conditions (iii)–(v) of Assumption $H_2$ are satisfied.

Example 2. Suppose that
(a) there exist $\bar{l}$ and $\bar{K}_j \in \mathbb{R}_+, j = 1, \ldots, n$, such that $l(x) \leq \bar{l}, K_j(x) \leq \sum_{i=1}^n \bar{K}_i x_i, i = 1, \ldots, m$;
(b) conditions (ii), (iv) of Assumption (a) and Assumption (b) of Example 1 are satisfied.

Then the conditions (iii)–(v) of Assumption $H_2$ are fulfilled.

Example 3. Suppose that
(a) $G = [0, \bar{a}]$, $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$, $\bar{a}_j > 0, j = 1, \ldots, n$;
(b) condition (ii) of Assumption (a), and Assumption (b) of Example 1 are satisfied;
(c) there exist $\bar{l} = (\bar{l}_1, \ldots, \bar{l}_n)$, $\bar{K} = (\bar{K}_1, \ldots, \bar{K}_m)$, such that $\bar{l}_j, \bar{K}_i \in \mathbb{R}_+$, $j = 1, \ldots, n, i = 1, \ldots, m$, and $l(x) \leq \sum_{j=1}^n \bar{l}_j x_j, K_i(x) \leq \bar{K}_i$, and the condition $\sum_{j=1}^n \bar{l}_j \zeta_s \bar{a}_j < 1$ holds.

Then the conditions (iii)–(v) of Assumption $H_2$ are fulfilled.

Example 4. Suppose that
(a) conditions (ii), (iv) of Assumption (a) of Example 1 are satisfied;
(b) there exist $\bar{l}$, $\bar{K} = (\bar{K}_1, \ldots, \bar{K}_n)$, such that $\bar{l}, \bar{K}_i \in \mathbb{R}_+, i = 1, \ldots, n$, and $l(x) \leq \bar{l}, K_j(x) \leq \sum_{i=1}^n \bar{K}_i x_i, j = 1, \ldots, m$, and $\bar{l}_s (\prod_{s \in \bar{\Gamma}} \bar{\zeta}_s)^2 \leq 1$,
(i) $i = 1, \ldots, n, j = 1, \ldots, m$,
(c) $\phi_{k_j}(x) \leq (\bar{\gamma}_{k_j} x_{k_1}^2, \ldots, \bar{\gamma}_{k_j} x_{k_p}^2), k_j \in \bar{\Gamma}_i, \bar{\gamma}_{k_j} \in \mathbb{R}_+$.

Then the conditions (iii)–(v) of Assumption $H_2$ are fulfilled.
Example 5. Suppose that
(a) there exist $G = [0, \bar{a}], \bar{a} = (\bar{a}_1, \ldots, \bar{a}_n), 0 < \bar{a}_i \leq 1, i = 1, \ldots, n$, such that
\[ \prod_{s \in \bar{\Gamma}_j} \bar{a}_s^2 < 1, j = 1, \ldots, m; \]
(b) $\zeta(x) = (\zeta_1(x), \ldots, \zeta_n(x)) \leq (x_1^2, \ldots, x_n^2);
(c) condition (i) of Assumption (a), and Assumption (b) of Example 1 are fulfilled, and the condition
\[ \sum_{N=0}^{+\infty} \bar{h}(x_1^{2N}, \ldots, x_n^{2N}) < +\infty \]
holds.
Then the conditions (iii)–(v) of Assumption $H_2$ are satisfied.

Example 6. Suppose that
(a) there exist $\bar{H}, \bar{P} \in \mathbb{R}_+$, such that $\tilde{h}(x) = \bar{h}(x_1, \ldots, x_n) \leq \bar{H}(\prod_{i=1}^n x_i)^\bar{P};$
(b) conditions (i), (ii) of Assumption (a) and Assumption (b) of Example 1 are fulfilled, and
\[ \left( \prod_{s \in \bar{\Gamma}_j} \bar{\zeta}_s \right)^\nu \leq 1, \] where $\nu = \min[1, \bar{P}]$.
Then the conditions (iii)–(v) of Assumption $H_2$ are satisfied.

References


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