On a Class of Second-Order Differential Inclusions on the Positive Half-Line

Gheorghe Moroşanu

Abstract. Consider in a real Hilbert space $H$ the differential equation (inclusion) (E):
\[ p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t) \text{ a.e. in } (0, \infty), \]
with the condition (B): $u(0) = x \in \overline{D(A)}$, where $A: D(A) \subset H \to H$ is a (possibly set-valued) maximal monotone operator whose range contains 0; $p, q \in L^\infty(0, \infty)$, such that ess inf $p > 0$, $\frac{q}{p}$ is differentiable a.e., and ess inf $\left[ \left( \frac{q}{p} \right)^2 + 2\left( \frac{q}{p} \right)' \right] > 0$. We prove existence of a unique (weak or strong) solution $u$ to (E), (B), satisfying $u(0) = x \in \overline{D(A)}$, showing in particular the behavior of $u$ as $t \to \infty$.

Keywords. Strong solution, weak solution, existence, uniqueness, asymptotic behavior

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1. Introduction

Let $H$ be a real Hilbert space with the inner product $(\cdot, \cdot)$ and the induced norm $\|x\| = (x, x)^{1/2}$. Consider the following second-order, non-homogeneous, differential equation (inclusion)
\[ p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t) \text{ for a.a. } t \in \mathbb{R}_+ := [0, \infty), \]
with the condition
\[ u(0) = x \in \overline{D(A)}, \]
where
(H1) $A: D(A) \subset H \to H$ is a (possibly set-valued) maximal monotone operator, such that $[0, 0] \in \text{Graph}(A)$;
(H2) $p, q \in L^\infty(\mathbb{R}_+) := L^\infty(\mathbb{R}_+; \mathbb{R})$, such that ess inf $p > 0$, $\frac{q}{p}$ is differentiable a.e., and ess inf $\left[ \left( \frac{q}{p} \right)^2 + 2\left( \frac{q}{p} \right)' \right] > 0$;

G. Moroşanu: Department of Mathematics, Central European University, Nador u. 9, 1051 Budapest, Hungary; morosanug@ceu.hu
and $f$ is a given $H$-valued function whose (required) properties will be specified later. In fact, one can assume the more general condition that the range $R(A)$ of $A$ contains $0$. Indeed, this case reduces to $[0, 0] \in \text{Graph}(\tilde{A})$, where $\tilde{A}$ is obtained from $A$ by shifting its domain.

For information on monotone operators we refer the reader to [5, 7, 12].

V. Barbu [3, 4] (see also [5, Chapter V]) established the existence of a unique bounded solution to (E), (B) in the particular case $p \equiv 1$, $q \equiv 0$ and $f \equiv 0$. Subsequently the existence and uniqueness of bounded solutions in the homogeneous case ($f \equiv 0$) has been further investigated by H. Brezis [6], N. Pavel [14], L. Véron [17, 18], and by E. I. Poffald and S. Reich [15, 16] when $A$ is an $m$-accretive operator in a Banach space. The non-homogeneous case has received less attention from this point of view. Bruck [8] proved that if (E), (B), with $p \equiv 1$, $q \equiv 0$ and $f \in L^2_{\text{loc}}(0, \infty; H)$, has a bounded solution $u \in C([0, \infty); H) \cap W^{2,2}_{\text{loc}}((0, \infty); H)$ for some $x \in \overline{D}(A)$, then (E), (B) has a unique bounded solution $u \in C([0, \infty); H) \cap W^{2,2}_{\text{loc}}((0, \infty); H)$ for every $x \in \overline{D}(A)$. This result has been extended to the case when $A$ is an $m$-accretive operator in a Banach space by Poffald and Reich [15, 16]. Note that in these papers the existence for some $x \in \overline{D}(A)$ was hypothesized to derive existence for all $x \in \overline{D}(A)$. Recall also that Bruck [9] established the existence of a bounded solution on $\mathbb{R}$ of equation (E) (implying that all solutions of (E) are bounded on $[0, \infty)$), in the case $p \equiv 1$, $q \equiv 0$ and $f \in L^\infty(\mathbb{R}; H)$, under the restrictive condition that $A$ is coercive. We also mention the relatively recent article by Apreutesei [2] addressing the case of smooth coefficients $p, q$, with $p(t) \geq p_0 > 0$, $q(t) \geq q_0 > 0$, and $x \in D(A)$.

In a recent paper [13] we established the existence of a unique bounded solution of equation (E), subject to (B), for all $x \in \overline{D}(A)$, under the same mild conditions (H1) and (H2) above, with one exception: instead of the condition on $\frac{q}{p}$ specified above, we assumed there $q^+ \in L^1(0, \infty; H)$. Our present alternative condition on $\frac{q}{p}$ ensures the existence of a unique (weak or strong) solution to (E), (B) satisfying $a^\frac{1}{2}u \in L^\infty(0, \infty; H)$ and $t^\frac{1}{2}a^\frac{3}{2}u' \in L^2(0, \infty; H)$, where $a(t) = \exp(\int_0^t \frac{q}{p} \, d\tau)$. So, in addition to existence and uniqueness, we get information about the asymptotic behavior of $u$ as $t \to \infty$. If in particular $q(t) \geq q_0 > 0$, then $\|u(t)\|$ decays exponentially to zero as $t \to \infty$.

The new framework requires separate analysis. However, some steps in our proofs are similar to those developed in [13]. In such cases, the reader will be referred to that paper.

It is worth pointing out that this paper covers in particular the case $q(t) < 0$ which allows using our existence theory to approximate the solutions of some parabolic and hyperbolic problems by the method of artificial viscosity, introduced by J. L. Lions [11]. See [13] for details. Note that the case $q \equiv 0$ was covered in [10, 13].
2. Results

Let us first recall the concepts of strong and weak solution for equation (E) (respectively, equation (E) plus condition (B)). These concepts have been introduced in [10, 13].

For an interval \( J \subset \mathbb{R} \), open or not, denote by \( L^p_{\text{loc}}(J; H) \) (resp. \( W^{k,p}_{\text{loc}}(J; H) \)) the space of all \( H \)-valued functions defined on \( J \), whose restrictions to compact intervals \([a, b] \subset J\) belong to \( L^p(a, b; H) \) (respectively, to \( W^{k,p}(a, b; H) \)).

**Definition 2.1.** Let \( f \in L^2_{\text{loc}}([0, \infty); H) \) and let \( x \in D(A) \). A \( H \)-valued function \( u = u(t) \) is said to be a strong solution of equation (E) (respectively, of equation (E) plus condition (B)) if \( u \in C([0, \infty); H) \cap W^{2,2}_{\text{loc}}((0, \infty); H) \) and \( u(t) \) satisfies equation (E) for a.a. \( t > 0 \) (and, in addition, \( u(0) = x \), respectively).

Denote \( Y = L^1(0, \infty; H; t\sqrt{a(t)} dt) \), where \( a(t) = \exp\left(\int_0^t q(\tau)p(\tau) d\tau d\tau\right) \). Obviously, \( Y \) is real Banach space with respect to the norm

\[
\|f\|_Y = \int_0^\infty \|f(t)\|t\sqrt{a(t)} dt.
\]

If \( f \in Y \) we cannot expect in general existence of strong solutions for (E), so we need the following definition of a weaker concept:

**Definition 2.2.** Let \( f \in Y \) and let \( x \in D(A) \). A \( H \)-valued function \( u = u(t) \) is said to be a weak solution of equation (E) (respectively, of equation (E) plus condition (B)) if there exist sequences \( u_n \in C([0, \infty); H) \cap W^{2,2}_{\text{loc}}((0, \infty); H) \) and \( f_n \in Y \cap L^2_{\text{loc}}([0, \infty); H) \), such that:

(i) \( f_n \) converges to \( f \) in \( Y \);
(ii) \( u_n(t) \) satisfies equation (E) with \( f = f_n \) for a.a. \( t > 0 \) and all \( n \in \mathbb{N} \);
(iii) \( u_n \) converges uniformly to \( u \) on any compact interval \([0, T]\) (and, in addition, \( u(0) = x \), respectively).

Note that the couple (E), (B) is an incomplete problem. We need an additional condition to obtain a complete problem. In this paper we consider the following condition

\[
\sup_{t \geq 0} a(t)\|u(t)\|^2 < \infty. \tag{C}
\]

Obviously, if \( \frac{q}{p} \in L^1(\mathbb{R}_+) \) (which is equivalent to \( q \in L^1(\mathbb{R}_+) \) if \( p \in L^\infty(\mathbb{R}_+) \) and ess inf \( p > 0 \), then (C) becomes \( \sup_{t \geq 0} \|u(t)\| < \infty \).

Before stating the first main result of the paper, let us recall two lemmas from [13]:

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Lemma 2.3. Let $A$ satisfy (H1), $p, q \in L^\infty(0, T)$, with $\text{ess inf } p > 0$, and let $f \in L^2(0, T; H)$, where $T$ is a given positive number. Then, for all $x, y \in D(A)$, there exists a unique $u = u(t) \in W^{2,2}(0, T; H)$ satisfying

$$p(t)u''(t) + q(t)u'(t) \in Au(t) + f(t) \quad \text{for a.a. } t \in (0, T),\tag{1}$$

$$u(0) = x, \quad u(T) = y. \tag{2}$$

Lemma 2.4. Assume that $A$ satisfies (H1), $p, q \in L^\infty(0, T)$, with $\text{ess inf } p > 0$, $f \in L^2(0, T; H)$, and $x, y \in D(A)$. For $\lambda > 0$ denote by $A_\lambda$ the Yosida approximation of $A$ and by $u_\lambda$ the unique solution of

$$p(t)u''_\lambda(t) + q(t)u'_\lambda(t) = A_\lambda u_\lambda(t) + f(t) \quad \text{for a.a. } t \in (0, T),$$

$$u_\lambda(0) = x, \quad u_\lambda(T) = y$$

(which exists by Lemma 2.3). Then, $u_\lambda \to u$ in $C([0, T]; H)$ as $\lambda \to 0^+$, where $u$ is the solution of problem (1), (2). Moreover, $u'_\lambda \to u'$ in $C([0, T]; H)$ and $u''_\lambda \to u''$ weakly in $L^2(0, T; H)$, as $\lambda \to 0^+$.

Theorem 2.5. Assume (H1) and (H2) hold. If $x \in \overline{D(A)}$ and $f \in Y \cap L^2_{\text{loc}}([0, \infty); H)$, then there exists a unique strong solution $u$ of (E), (B), (C), such that $t^2u'' \in L^2(\mathbb{R}_+; H)$ and $t^2u'' \in L^2_{\text{loc}}([0, \infty); H)$. If in addition $x \in D(A)$, then $u \in W^{2,2}(0, \infty; H)$.

Proof. Assume in a first stage that $x \in D(A)$ (and $f \in Y \cap L^2_{\text{loc}}([0, \infty); H)$, as hypothesized). For each $\lambda > 0$ and $n \in \mathbb{N}$, denote by $u_{n\lambda}, u_n$ the solutions of the following problems

$$pu''_{n\lambda} + qu'_{n\lambda} = A_\lambda u_{n\lambda} + f \quad \text{a.e. in } (0, n),\tag{3}$$

$$u_{n\lambda}(0) = x, \quad u_{n\lambda}(n) = 0, \tag{4}$$

and

$$pu''_n + qu'_n = Au_n + f \quad \text{a.e. in } (0, n),\tag{5}$$

$$u_n(0) = x, \quad u_n(n) = 0. \tag{6}$$

Lemma 2.3 ensures the existence and uniqueness of $u_{n\lambda}, u_n \in W^{2,2}(0, n; H)$. By Lemma 2.4, $u_{n\lambda} \to u_n, u'_{n\lambda} \to u'_n$ in $C([0, n]; H)$, as $\lambda \to 0^+$, and $u''_{n\lambda} \to u''_n$ weakly in $L^2(0, n; H)$, as $\lambda \to 0^+$. Note that equations (3), (5) can be equivalently expressed as follows (see [1])

$$(au'_{n\lambda})' = b(A_\lambda u_{n\lambda} + f) \quad \text{a.e. in } (0, n),\tag{7}$$

and, respectively,

$$(au'_n)' \in b(Au_n + f) \quad \text{a.e. in } (0, n), \tag{8}$$
where

\[ b(t) = \frac{a(t)}{p(t)}. \]

Recall that

\[ a(t) = \exp \left( \int_0^t \frac{q}{p} \, d\tau \right). \]

We have for a.a. \( t \in (0, n) \)

\[
\frac{d^2}{dt^2} \left[ a \| u_n \|^2 \right] = \frac{d}{dt} \left[ a \frac{q}{p} \| u_n \|^2 + 2(au'_n, u_n) \right]
\]

\[
= a \left( \left( \frac{q}{p} \right)^2 + \left( \frac{q}{p} \right)' \right) \| u_n \|^2 + 2a \frac{q}{p} (u'_n, u_n) + 2a \| u'_n \|^2 + 2\left( (au'_n)' , u_n \right)
\]

\[
\geq a \left( \left( \frac{q}{p} \right)^2 + \left( \frac{q}{p} \right)' \right) \| u_n \|^2 + 2 \frac{q}{p} (u'_n, u_n) + 2\| u'_n \|^2 \right) - 2b \| u_n \| \cdot \| f \|. \quad (9)
\]

The last inequality follows from (8) and the monotonicity of \( A \). Taking into account the condition on \( \frac{q}{p} \) (see (H2)) we derive from (9)

\[
\frac{d^2}{dt^2} \left[ a \| u_n \|^2 \right] \geq -2b \| f \| \cdot \| u_n \|. \quad (10)
\]

Integration of (10) over \([\tau, n]\) leads to

\[
\frac{d}{d\tau} \left( a(\tau) \| u_n(\tau) \|^2 \right) \leq 2 \int_\tau^n \frac{q}{p} \| f \| \cdot \| u_n \| \, ds.
\]

A new integration, this time over \([0, t]\), yields

\[
a(t) \| u_n(t) \|^2 \leq \| x \|^2 + 2 \int_0^t \frac{q}{p} \| f \| \cdot \| u_n \| \, ds
\]

\[
\leq \| x \|^2 + 2 \int_0^t \frac{q}{p} \| f \| \cdot \| u_n \| \, ds
\]

\[
= \| x \|^2 + 2 \int_0^t \tau b \| f \| \cdot \| u_n \| \, d\tau, \quad 0 \leq t \leq n. \quad (11)
\]

Denoting \( M_n = \sup_{0 \leq t \leq n} \sqrt{a(t)} \| u_n(t) \| \), from (11) we derive

\[
M_n^2 \leq \| x \|^2 + 2M_n \int_0^n \frac{\sqrt{a}}{p} \| f \| \, d\tau \leq \| x \|^2 + 2 \frac{M_n}{p_0} \| f \|_Y,
\]

where \( p_0 = \text{ess inf} \, p \). Therefore,

\[
M_n \leq \frac{1}{p_0} \| f \|_Y + \sqrt{\frac{1}{p_0^2} \| f \|_Y^2 + \| x \|^2} =: E = E(x, f).
\]

Thus,

\[
\sup_{0 \leq t \leq n} a(t) \| u_n(t) \|^2 \leq E^2. \quad (12)
\]

Similarly,

\[
\sup_{0 \leq t \leq n} a(t) \| u_{n\lambda}(t) \|^2 \leq E^2.
\]
Now, let $0 < R < m < n$, with $m, n \in \mathbb{N}$. Denote
\[ g(t) = a(t)\|u_n(t) - u_m(t)\|^2, \quad 0 \leq t \leq m. \]

We have
\[
g'(t) = \frac{a}{p} \|u_n - u_m\|^2 + 2(a(u_n' - u_m'), u_n - u_m),
\]
\[
g''(t) = \frac{a}{p^2} \|u_n - u_m\|^2 + a \left( \frac{q}{p} \right) \|u_n - u_m\|^2 + 2a \left( \frac{q}{p} \right)'(u_n' - u_m', u_n - u_m)
+ 2 \left( (a(u_n' - u_m'))', u_n - u_m \right) + 2\|u_n' - u_m'\|^2.
\]

Therefore,
\[
g''(t) \geq a \left( \left[ \frac{q}{p^2} + \left( \frac{q}{p} \right) \right] \|u_n - u_m\|^2 + 2\left( \frac{q}{p} \right)'(u_n' - u_m', u_n - u_m) + 2\|u_n' - u_m'\|^2 \right). \tag{13}
\]

Denoting $\alpha := \text{ess inf} \left\{ \left( \frac{q}{p} \right)^2 + 2\left( \frac{q}{p} \right)' \right\} > 0$, and observing that
\[
\left( \frac{q}{p} \right)^2 + \left( \frac{q}{p} \right)' \geq \frac{1}{2} \left( \frac{q}{p} \right)^2 + \frac{\alpha}{2},
\]
from (13) we derive
\[
g''(t) \geq \frac{a}{2} \left( \left[ \frac{q}{p^2} + \alpha \right] \|u_n - u_m\|^2 + 4\left( \frac{q}{p} \right)'(u_n' - u_m', u_n - u_m) + 4\|u_n' - u_m'\|^2 \right)
\[ \geq \beta a\|u_n' - u_m'\|^2, \tag{14} \]
for a.a. $t \in (0, m)$, where $\beta$ is a small positive number. We multiply (14) by $(m - t)$ and then integrate the resulting inequality over $[0, m]$:
\[
\beta \int_0^m (m - t)a\|u_n' - u_m'\|^2 dt \leq (m - t)g'(t)\bigg|_0^m + \int_0^m g'(t) dt
\]
\[
= g(m)
\]
\[
= a(m)\|u_n(m)\|^2
\]
\[
\leq E^2.
\]

We have used (12). It follows that $\beta(m - R) \int_0^R a\|u_n' - u_m'\|^2 dt \leq E^2$, which shows that $(u_n')$ is a Cauchy (hence convergent) sequence in $L^2(0, R; H)$. Therefore, since $u_n(t) - u_m(t) = \int_0^t (u_n - u_m)'(s) ds$, $u_n$ converges in $C([0, R]; H)$ to some $u \in C([0, R]; H)$, and so $u_n' \to u'$ in $L^2(0, R; H)$. In particular, $u(0) = x$. Obviously, since $R > 0$ was arbitrarily chosen, $u$ can be extended to $[0, \infty)$, such that $u \in C([0, \infty); H) \cap W^{1, 2}_{loc}([0, \infty); H)$, and $u$ satisfies (cf. (12))
\[
\sup_{t \geq 0} a(t)\|u(t)\|^2 \leq E^2 < \infty. \tag{15}
\]
By arguments similar to those used in [13], we deduce that \( u''_n \) is bounded in \( L^2(0, \frac{t}{2}; H) \), hence weakly convergent to \( u'' \) in this space, and finally that \( u \) is a strong solution of equation (E).

Now, assume that \( x \in \overline{D(A)} \) and \( f \in Y \cap L^2_{loc}([0, \infty); H) \). Let \( x_k \in D(A) \), \( \|x_k - x\| \to 0 \). Denote by \( u_k \) the strong solution of equation (E) satisfying \( u_k(0) = x_k \), and \( \sqrt{a}\|u_k\| \in L^\infty(R_+) \). Existence of \( u_k \) is ensured by the first part of the proof. In fact, according to (15),

\[
\sup_{t \geq 0} \sqrt{a(t)}\|u_k(t)\| \leq E(x_k, f) \leq E_0 < \infty. \tag{16}
\]

Denote by \( u_{kn}, u_{kn\lambda} \) the corresponding approximations of \( u_k \) and \( u_{kn} \) (as defined above, see problems (5), (6) and (3), (4)). We see that for a.a. \( t \in (0, n) \)

\[
\frac{1}{2} \frac{d}{dt} \left( a(t) \|u_k - u_jn\|^2 \right) \geq a\|u_k' - u_jn'\|^2,
\]

so the function \( t \to a(t) \frac{d}{dt} \|u_k(t) - u_jn(t)\|^2 \) is nondecreasing on \([0, n]\). Since it is equal to zero at \( t = n \), it follows that it is non-positive in \([0, n]\). Then the function \( t \to \|u_k(t) - u_jn(t)\| \) is nonincreasing on \([0, n]\). In particular,

\[
\|u_{kn}(t) - u_{jn}(t)\| \leq \|x_k - x_j\| \quad \forall t \in [0, n].
\]

Therefore, according to the first part of the proof, we have

\[
\|u_k(t) - u_j(t)\| \leq \|x_k - x_j\| \quad \forall t \geq 0.
\]

This shows that there exists a function \( u \in C([0, \infty); H) \) such that \( u_k \) converges to \( u \) in \( C([0, R]; H) \) for all \( R \in (0, \infty) \), so in particular \( u(0) = x \). According to (16), we also have \( \sqrt{a}\|u\| \in L^\infty(R_+) \). Now, set

\[
h(t) = a(t)\|u_{kn\lambda}(t)\|^2, \quad 0 \leq t \leq n.
\]

We have

\[
h'(t) = a \left( \frac{q}{p} \right) \|u_{kn\lambda}\|^2 + 2(a u_{kn\lambda}', u_{kn\lambda}),
\]

\[
h''(t) = a \left( \left( \frac{q}{p} \right)^2 + \left( \frac{q}{p} \right) \right) \cdot \|u_{kn\lambda}\|^2 + 2a \frac{q}{p} (u_{kn\lambda}', u_{kn\lambda}) + 2(a u_{kn\lambda}', u_{kn\lambda}) + 2a \|u_{kn\lambda}'\|^2.
\]

Therefore,

\[
h''(t) \geq a \left( \left( \frac{q}{p} \right)^2 + \left( \frac{q}{p} \right) \right) \|u_{kn\lambda}\|^2 + 2a \frac{q}{p} (u_{kn\lambda}', u_{kn\lambda}) + 2\|u_{kn\lambda}'\|^2 - 2b\|f\| \cdot \|u_{kn\lambda}\|,
\]

\[
\geq \beta a \|u_{kn\lambda}'\|^2 - 2b \frac{E_0}{p_0} \sqrt{a}\|f\| \tag{17}
\]
Multiply (17) by $t$ and integrate the resulting inequality over $[0, n]$ to obtain

$$
\beta \int_0^n t a \| u'_{kn}\|^2 dt \leq 2 \frac{E_0}{p_0} \int_0^n t \sqrt{a} \| f \| dt + \int_0^n th''(t) dt
\leq 2 \frac{E_0}{p_0} \| f \|_Y + \int_0^n h'(t) dt
\leq 2 \frac{E_0}{p_0} \| f \|_Y + \| x_k \|^2
\leq K_0 < \infty.
$$

(18)

According to Lemma 2.4, it follows by (18) that

$$
\beta \int_0^n t a \| u'_{kn}\|^2 dt \leq K_0.
$$

(19)

By the first part of the proof, we also have

$$
\beta \int_0^\infty t a \| u'_{kn}\|^2 dt \leq K_0.
$$

(20)

In fact, $\sqrt{tau'} \in L^2(\mathbb{R}_+; H)$ and $\sqrt{tau'_k} \to \sqrt{tau'}$ in $L^2(\mathbb{R}_+; H)$. Indeed, denoting

$$
r(t) = a(t) \| u_{kn}(t) - u_{jn}(t) \|^2, \quad 0 \leq t \leq n,
$$

we derive by a computation similar to that we have used above for $g(t)$

$$
r''(t) \geq \beta a(t) \| u'_{kn}(t) - u'_{jn}(t) \|^2 \quad \text{for a.a. } t \in (0, n),
$$

which implies $\beta \int_0^n t a \| u'_{kn} - u'_{jn}\|^2 dt \leq tr'(0) - \int_0^n r'(t) dt = \| x_k - x_j \|^2$. Hence,

$$
\beta \int_0^\infty t a \| u'_{kn} - u'_{jn}\|^2 dt \leq \| x_k - x_j \|^2,
$$

which confirms our assertion above.

Next, using the sequence $(u_{kn\lambda})$ (and in particular (18)), we can show by a procedure similar to that used in [13] that $t^\frac{3}{2} u'' \in L^2_{\text{loc}}([0, \infty); H)$ and that $u$ is a strong solution of equation (E). Uniqueness of $u$ follows as in [13], so the proof of the theorem is complete.

\[ \square \]

**Theorem 2.6.** Assume (H1) and (H2) hold. Then, for each $x \in \overline{D(A)}$ and $f \in Y$, there exists a unique weak solution $u$ of (E), (B), (C), and $\sqrt{tau'} \in L^2(\mathbb{R}_+; H)$.

**Proof.** Let $x \in \overline{D(A)}$ and let $f_1, f_2 \in Y \cap L^2_{\text{loc}}([0, \infty); H)$. Denote by $u(t, x, f_i)$, $i = 1, 2$, the corresponding strong solutions given by Theorem 2.5, and by $u_n(t, x, f_i)$ their approximations ($i = 1, 2, \ldots, n \in \mathbb{N}$), as defined above (see
(5), (6)). Recall that (by the uniqueness property) every strong solution is obtained by the limiting procedure developed in the proof of Theorem 2.5. By a computation involving (H2), similar to that performed above for \(g(t)\), we derive the inequality

\[
\frac{d^2}{dt^2} [a(t)\|u_n(t, x, f_1) - u_n(t, x, f_2)\|^2] \\
\geq -2b(t)\|f_1(t) - f_2(t)\| \cdot \|u_n(t, x, f_1) - u_n(t, x, f_2)\|. \tag{21}
\]

Successive integrations of (21), over \([\tau, n]\) and then over \([0, t]\), lead to

\[
a(t)\|u_n(t, x, f_1) - u_n(t, x, f_2)\|^2 \\
\leq 2 \int_0^n d\tau \int_\tau^n b(s)\|f_1(s) - f_2(s)\| \cdot \|u_n(s, x, f_1) - u_n(s, x, f_2)\| \, ds \\
= 2 \int_0^n \tau b(\tau)\|f_1(\tau) - f_2(\tau)\| \cdot \|u_n(\tau, x, f_1) - u_n(\tau, x, f_2)\| \, d\tau. \tag{22}
\]

Obviously, (22) implies \(\sqrt{a(t)}\|u_n(t, x, f_1) - u_n(t, x, f_2)\| \leq \frac{2}{p_0}\|f_1 - f_2\|_Y\), for \(0 \leq t \leq n\), and hence

\[
\sqrt{a(t)}\|u(t, x, f_1) - u(t, x, f_2)\| \leq \frac{2}{p_0}\|f_1 - f_2\|_Y \quad \forall t \geq 0. \tag{23}
\]

From inequality (23) we can derive the existence of a unique weak solution \(u(t; x, f)\) for each \((x, f) \in D(A) \times Y\). Indeed, \(f\) can be approximated (with respect to the norm of \(Y\)) by a sequence \((f_k)\) of smooth functions with compact support \(\subset (0, \infty)\), so it is enough to take in (23) \(f_1 := f_k\) and \(f_2 := f_j\). So, there exists uniquely \(u(\cdot; x, f) \in C([0, \infty); H)\) the uniform limit on compact intervals of \(u(\cdot; x, f_k)\) as \(k \to \infty\).

Note that (19) holds true for \(u_n'(t; x, f_k)\) with another constant \(K_0\) (since \(E(x, f_k)\) is also bounded), so (20) also holds true for \(u'(t; x, f_k)\). Therefore, \(\sqrt{a} u' \in L^2(\mathbb{R}_+; H)\) (as the weak limit in \(L^2(\mathbb{R}_+; H)\) of the sequence \((\sqrt{a} u_k')\)). This completes the proof of the theorem.

\[\square\]

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**References**


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