Vector-Valued Integration in $BK$-Spaces

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Abstract. Questions of convergence in $BK$-spaces, i.e. Banach spaces of complex-valued sequences $x = (x_k)_{k \in \mathbb{Z}}$ with continuity of all functionals $x \to x_k$ ($k \in \mathbb{Z}$) will be studied by methods of Fourier analysis. An elegant treatment is possible if the Cesàro sections of a $BK$-space element $x$ can be represented by vector-valued Riemann integrals. This was done by Goes [2] following the example of Katznelson [5: pp. 10 - 12]. The purpose of this paper is to make precise the conditions in [2] concerning Riemann integration and to demonstrate relations between $BK$-spaces which are generated by a given $BK$-space.

Keywords: $BK$-spaces, Riemann integration, Cesàro-sectional (weak) convergence and boundedness

AMS subject classification: Primary 46 B 45, 46 A 45, 28 B 05, secondary 40 C 05, 42 A 24

1. Introduction

This paper is motivated by a letter of Boettcher to Goes (Beispiel eines translations-invarianten $BK$-Raumes, der nicht die Eigenschaft $\sigma B$ hat) from May 31, 1990, in which the space $E = L^2(T) \oplus M^d(T)$ (cf. Example 3.7) is considered as example of a translation-invariant $BK$-space which fails to have the so-called property $\sigma B$. Choosing some element $x = \delta_0$ Boettcher proves that the sequence $(\sigma_n x)_{n \in \mathbb{N}_0}$ with $\sigma_n x \in L^2(T) \subseteq E$ for all $n \in \mathbb{N}_0$ is not bounded.

This shows that in general Proposition 4.1/(i) in Goes [2] is not valid. Actually the $BK$-valued Riemann integral used in the proof of Proposition 4.1/(i) may not exist. It is evident that the function $xe$ (cf. Definition 2.7) is not Riemann integrable because $\sigma_n x \notin M^d(T)$ for all $n \in \mathbb{N}_0$. Thus Riemann integrability of $xe$ is sufficient for $x$ to have property $\sigma B$ (hence [2: Proposition 4.1/(i)]) is valid in this case (cf. Theorem 3.5)). However it is not necessary, as $u \in E_3$ in Example 3.7 shows.

From this point of view the advantages and the limitations in the representation of $\sigma_n x$ as vector-valued integral shall be demonstrated. Especially some properties of Riemann and Bochner integration in translation-invariant $BK$-spaces are considered. Beyond this significant relations between certain $BK$-subspaces of the linear space $\Omega$ of all complex-valued sequences are presented.

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2. Notations, definitions and preliminary remarks

2.1 Convergence and boundedness in BK-spaces. Let \( \mathbb{R} \) and \( \mathbb{C} \) be the set of real and complex numbers, respectively, let \( \mathbb{N} \) and \( \mathbb{N}_0 \) be the set of positive and non-negative integers, respectively, and \( \mathbb{Z} \) the set of integers. Furthermore let \( \Omega \) be the linear space of complex-valued sequences on \( \mathbb{Z} \), i.e.

\[
\Omega = \left\{ (x_k)_{k \in \mathbb{Z}} \mid x_k \in \mathbb{C} \text{ for all } k \in \mathbb{Z} \right\}.
\]

For \( k \in \mathbb{Z} \) let \( \delta^k \) the Kronecker symbol and define \( \sigma_n : \Omega \to \Omega \) \((n \in \mathbb{N}_0)\) by

\[
\sigma_n : x \mapsto \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right)x_k \delta^k \quad (x \in \Omega).
\]

The function \( \sigma_n x \) is called \( n \)-th Cesàro section of order one of \( x \).

**Definition 2.1** (BK-space). \((E; \| \cdot \|_E)\) with \( E \subseteq \Omega \) is called BK-space if

1. \( E \) endowed with the norm \( \| \cdot \|_E \) is a Banach space
2. \( Pr_k : E \to \mathbb{C} \) with \( x \mapsto x_k \) is continuous for all \( k \in \mathbb{Z} \).

**Definition 2.2** (Convergence and boundedness). Let \( E \) be a BK-space, \( x \in \Omega \) and \( \sigma_n x \in E \) for all \( n \in \mathbb{N}_0 \). Then \( x \) has the

1. property \( aK \) of Cesàro-sectional convergence if \((\sigma_n x)_{n \in \mathbb{N}_0}\) is a Cauchy sequence in \( E \), i.e. \( \lim_{n \to \infty} \sigma_n x = x \in E \);
2. property \( oB \) of Cesàro-sectional boundedness if \( \sup_{n \in \mathbb{N}_0} \| \sigma_n x \|_E < \infty \);
3. property \( SaK \) of weak Cesàro-sectional convergence if \( x \in E \) and \( \lim_{n \to \infty} \phi(\sigma_n x) = \phi(x) \) for all \( \phi \in E' \).

**Remark 2.3.** Let \( P \) be one of the properties \( aK, oB \) or \( SaK \). Then the space \( E_P = \{ x \in \Omega \mid x \text{ has the property } P \} \) endowed with the norm \( \| x \|_{E_P} = \sup_{n \in \mathbb{N}_0} \| \sigma_n x \|_E \) is a BK-space with \( \| x \|_E \leq \sup_{n \in \mathbb{N}_0} \| \sigma_n x \|_E \) for all \( x \in E_{aK} \), and thus for all \( x \in E_{aK} \) (cf. Yosida [6: Theorem 2/p. 120]) as well known.

**Definition 2.4.** A closed subspace \( G \) of \( E \) has sectional density (Abschnittsdichte AD) if the set \( \Phi = \{ x \in \Omega \mid \{ k \in \mathbb{Z} | x_k \neq 0 \} \text{ finite} \} \) is dense in \( G \), i.e. \( G = \overline{\Phi \cap G} \). In particular let \( E_{AD} \) be defined by \( E_{AD} = \overline{\Phi \cap \Omega} \).

According to Zeller [7: Sätze 2.2, 3.3 and 3.4] we have the following lemma.

**Lemma 2.5.** Let \((E; \| \cdot \|_E)\) be a BK-space. Then the following assertions are true.

1. \( E \) has property \( aK \) if and only if \( E = E_{AD} \) and \( E \subseteq E_{oB} \).
2. \( E \) has property \( SaK \) if and only if \( E \) has property \( aK \).

**Remarks 2.6.** Obviously we have the relation \( E_{aK} \subseteq E_{SaK} \subseteq E_{AD} \), and if \( E_{AD} \subseteq E_{oB} \), then even \( E_{aK} = E_{SaK} = E_{AD} \). If \( E_{SaK} \) is closed with respect to \( \| \cdot \|_E \), then \( E_{SaK} = E_{aK} \).
Definition 2.7 (Invariance and continuity of translation). Let $T = \mathbb{R}/2\pi\mathbb{Z}$, where $\mathbb{R}$ is the additive group of real numbers. Then

$$e(t) = (e^{ikt})_{k \in \mathbb{Z}} \quad \text{for all } t \in T.$$ 

A BK-space $E$ is called

1. **translation invariant** if $xe(t) = (xke^{ikt})_{k \in \mathbb{Z}} \in E$ and $\|xe(t)\|_E = \|x\|_E$ for all $x \in E$ and $t \in T$

and it is called

2. **homogeneous BK-space** if in addition the translation is continuous, i.e. the convergence $\lim_{t \to t_0} \|xe(t) - xe(t_0)\|_E = 0$ for all $x \in E$ and $t_0 \in T$ is true.

A translation invariant BK-space $E$ has weakly continuous translation if

$$\lim_{t \to t_0} \phi(xe(t) - xe(t_0)) = 0 \quad \text{for all } x \in E, \phi \in E' \text{ and } t_0 \in T.$$

Remark 2.8. The continuity of $xe$ for $x \in E$ in a particular point $t_0 \in T$ implies trivially the continuity of $xe$ on $T$.

2.2 Vector-valued integration. In this subsection we refer to Gordon [3], a survey article, where essential criteria of Riemann integration are stated.

The Riemann integral and some of its properties. Let $[a, b]$ be a real finite interval and $X$ a Banach space. Furthermore let a partition $\mathcal{P}$ of $[a, b]$ be given with

$$\mathcal{P} = \left\{ t_i \mid 0 \leq i \leq N; a = t_0 < t_1 < \ldots < t_N = b \right\}$$

and

$$|\mathcal{P}| = \max \left\{ t_i - t_{i-1} : 1 \leq i \leq N \right\}$$

its norm. If $\tilde{\mathcal{P}}_1 \subseteq \tilde{\mathcal{P}}_2$, then $\tilde{\mathcal{P}}_2$ is called a refinement of $\tilde{\mathcal{P}}_1$.

If we choose $s_i \in [t_{i-1}, t_i]$ for all $1 \leq i \leq N$, we obtain from $\tilde{\mathcal{P}}$ a tagged partition $\mathcal{P}$ of $[a, b]$, i.e.

$$\mathcal{P} = \left\{ (s_i, [t_{i-1}, t_i]) \mid 1 \leq i \leq N; a = t_0 < t_1 < \ldots < t_N = b; s_i \in [t_{i-1}, t_i] \right\}.$$

For $f : [a, b] \to X$ we call

$$f(\mathcal{P}) = \sum_{i=1}^{N} f(s_i)(t_i - t_{i-1})$$

a Riemann sum of $f$.

Definition 2.9 (Riemann integral). The function $f : [a, b] \to X$ is called Riemann integrable ($R$-integrable) if

$$\exists \varepsilon \in X : \forall \varepsilon > 0 : \exists \delta > 0 : \forall \mathcal{P} \text{ tagged with } |\mathcal{P}| < \delta : \|f(\mathcal{P}) - z\|_X < \varepsilon$$

and $z = (R) - \int_{a}^{b} f(t) \, dt$ is called the Riemann integral of $f$.

Evidently an $R$-integrable function $f$ must be bounded (cf. [3: p. 924]). The proofs of the following two theorems are obvious and omitted.
Theorem 2.10 (Cauchy criteria). Let a function \( f : [a, b] \to X \) be given. Then the following assertions are pairwise equivalent.

1. \( f \) is \( R \)-integrable on \([a, b]\).

2. \( \forall \varepsilon > 0 : \exists \delta > 0 : \forall P_1, P_2 \text{ tagged with } |P_1|, |P_2| < \delta : \|f(P_1) - f(P_2)\|_X < \varepsilon \).

3. \( \forall \varepsilon > 0 : \exists \widehat{P}_\varepsilon : \forall P_1, P_2 \text{ refinements of } \widehat{P}_\varepsilon, P_1, P_2 \text{ tagged} : \|f(P_1) - f(P_2)\|_X < \varepsilon \).

Theorem 2.11. Let the function \( f : [a, b] \to X \) be \( R \)-integrable on \([a, b]\). Then we have:

1. \( f \) is \( R \)-integrable on every subinterval of \([a, b]\).

2. If \( \|f(t)\|_X \leq M \) on \([a, b]\), then \( \| \int_a^b f(t) \, dt \|_X \leq M(b - a) \).

3. If \( h : [a, b] \to X \) is continuous, then \( h \) is \( R \)-integrable.

4. If \( Y \) is a Banach space and \( T : X \to Y \) a continuous linear operator, then

\[
\int_a^b T(f(t)) \, dt = T \left( \int_a^b f(t) \, dt \right).
\]

5. If \( g : [a; b] \to X \) is \( R \)-integrable on \([a, b]\), then \( f + g \) is \( R \)-integrable on \([a, b]\) and

\[
\int_a^b (f + g)(t) \, dt = \int_a^b f(t) \, dt + \int_a^b g(t) \, dt.
\]

Theorem 2.12. Let \( f : [a, b] \to X \) be an \( R \)-integrable and \( g : [a, b] \to C \) a continuous function. Then the product function \( gf : [a, b] \to X \) is \( R \)-integrable.

Proof. Let \( \|f(t)\|_X \leq M \) on \([a, b]\). We consider a sequence of step functions \((g_n)_{n \in \mathbb{N}}\) with \( g_n : [a, b] \to \mathbb{C} \) converging uniformly on \([a, b]\) to \( g \). Each function \( g_n \) can be represented as

\[
g_n = \sum_{k=1}^{m_n} \alpha_k^{(n)} \chi_{[\tau_{k-1}, \tau_k)} + \alpha_m^{(n)} \chi_{(b)}
\]

where \( a = \tau_0 < \tau_1 < \ldots < \tau_{m_n} = b \) and \( \alpha_k^{(n)} \in \mathbb{C} \) for all \( 1 \leq k \leq m_n \). Obviously by the assertions 1 and 5 of Theorem 2.11 \( g_n f \) is \( R \)-integrable for all \( n \in \mathbb{N} \), and we get

\[
\int_a^b g_n f(t) \, dt = \sum_{k=1}^{m_n} \alpha_k^{(n)} \int_{\tau_{k-1}}^{\tau_k} f(t) \, dt.
\]

We choose \( n \in \mathbb{N} \) such that

\[
\sup_{t \in [a, b]} |g_n(t) - g(t)| < \frac{\varepsilon}{3M(b - a)}.
\]

Let \( \widehat{P}_\varepsilon \) be a partition such that for two arbitrarily chosen refinements \( \widehat{P}_1 \) and \( \widehat{P}_2 \) of \( \widehat{P}_\varepsilon \) with \( P_1 \) and \( P_2 \) tagged

\[
\|g_n f(P_1) - g_n f(P_2)\|_X < \frac{\varepsilon}{3}.
\]
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(c.f. Theorem 2.10/3.). With
\[ \|g f(P_1) - g f(P_2)\|_X \leq \|g f(P_1) - g_n f(P_1)\|_X + \|g_n f(P_1) - g_n f(P_2)\|_X + \|g_n f(P_2) - g f(P_2)\|_X \]
we obtain
\[ \|g f(P_1) - g_n f(P_1)\|_X < \frac{\varepsilon}{3M(b-a)} M(b-a) = \frac{\varepsilon}{3} \]
and analogously
\[ \|g f(P_2) - g_n f(P_2)\|_X < \frac{\varepsilon}{3}. \]
Thus $\|g f(P_1) - g f(P_2)\|_X < \varepsilon$ for all tagged partitions $P_1$ and $P_2$ with refinements $\tilde{P}_1$ and $\tilde{P}_2$ of $P$, i.e. $\int_a^b g f(t) \, dt$ exists.

The Bochner integral and some of its properties. Beside the Riemann integral we will have a look at the Bochner integral. Later we shall see consequences of these two possibilities of vector-valued integration for $BK$-spaces.

Definition 2.13 (Bochner integral). Let $(\Sigma; \mathcal{A}; \mu)$ be a measure space. A function $f : \Sigma \rightarrow X$ is called simple if there exist $E_1, \ldots, E_n \in \mathcal{A}$ and $x_1, \ldots, x_n \in X$ such that $f = \sum_{i=1}^n x_i \chi_{E_i}$. For a simple function $f$ the Bochner integral is defined by
\[ (B) \int_{\Sigma} f \, d\mu = \sum_{i=1}^n x_i \mu(E_i). \]
A function $f : \Sigma \rightarrow X$ is called $\mu$-measurable if $f$ is $\mu$-almost everywhere the limit of a sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions, i.e. if
\[ \lim_{n \rightarrow \infty} \|f_n - f\|_X = 0 \quad \text{a.e.} \]
A $\mu$-measurable function $f : \Sigma \rightarrow X$ is called Bochner integrable if there exists a sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions such that for the sequence of Lebesgue integrals $(\int_{\Sigma} \|f_n - f\|_X \, d\mu)_{n \in \mathbb{N}}$ we have
\[ \lim_{n \rightarrow \infty} \int_{\Sigma} \|f_n - f\|_X \, d\mu = 0. \]
Then the Bochner integral is defined by
\[ (B) \int_{\Sigma} f \, d\mu = \lim_{n \rightarrow \infty} (B) \int_{\Sigma} f_n \, d\mu. \]
In the following we consider only Lebesgue measure spaces $(\Sigma; \mathcal{L}; \lambda)$. The next theorem is due to Diestel and Uhl [1: Theorem 9/p. 49].

Theorem 2.14. Let the function $f$ be Bochner integrable on $[a,b]$ with respect to the Lebesgue measure $\lambda$. Then
\[ \lim_{h \rightarrow 0} \frac{1}{h} \int_a^{s+h} \|f(t) - f(s)\|_X \, d\lambda(t) = 0 \]
for almost all $s \in [a,b]$.

Now we obtain
Lemma 2.15. Let $h > 0$ and $f : [a; b] \to X$ be a Bochner integrable function. Then
\[
\lim_{h \to 0} \frac{1}{h} \int \lambda \left( \left\{ t \in [s; s + h] \, \mid \, \| f(t) - f(s) \|_X < \varepsilon \right\} \right) = 1
\]
for almost all $s \in [a, b]$ and all $\varepsilon > 0$.

Proof. For $\varepsilon > 0$ set
\[
a_\varepsilon(s) = \lambda \left( \left\{ t \in [s; s + h] \, \mid \, \| f(t) - f(s) \|_X < \varepsilon \right\} \right)
\]
\[
b_\varepsilon(s) = \lambda \left( \left\{ t \in [s; s + h] \, \mid \, \| f(t) - f(s) \|_X \geq \varepsilon \right\} \right).
\]
Then $a_\varepsilon(s) + b_\varepsilon(s) = h$. Hence by Theorem 2.14
\[
\frac{1}{h} \int a_\varepsilon(s) \leq \frac{1}{h} \int_{s}^{s+h} \| f(t) - f(s) \|_X d\lambda(t) \to 0 \quad \text{as} \quad h \to 0
\]
for almost all $s \in [a, b]$. Thus $\lim_{h \to 0} \frac{1}{h} a_\varepsilon(s) = 1$.

3. Properties of BK-valued integrals

In the following we consider BK-subspaces of $\Omega$ which are induced by a translation-invariant BK-space $E$ and certain properties.

Definition 3.1. For a translation invariant BK-space $E$ we define
\[
E_c = \{ x \in E \mid xe \text{ is continuous on } T \}
\]
\[
E_\lambda = \{ x \in E \mid xe \text{ is } \lambda\text{-measurable on } T \}
\]
\[
E_{Ri} = \{ x \in E \mid (R) \cdot \frac{1}{2\pi} \int_0^{2\pi} xe(t) dt \text{ exists} \}.
\]

Remarks 3.2. $E_c$ and $E_{Ri}$ are with respect to $\| \cdot \|_E$ translation-invariant BK-spaces. Obviously $E_c \subseteq E_{Ri}$ because every continuous function is $R$-integrable. Furthermore $E_c$ is a homogeneous BK-space, and $xe$ is Bochner integrable for all $x \in E_\lambda$ (cf. [1: Theorem 2/p. 45]).

It will be our aim to demonstrate relations between the spaces $E_P$ for $P$ equal one of the properties $\sigma K, \sigma B, S\sigma K, AD, c, Ri$ or $\lambda$ and to compare these relations with results in [2].

Theorem 3.3. The function $xe$ is continuous if and only if $xe$ is $\lambda$-measurable, i.e. $E_c = E_\lambda$.

Proof. Evidently the continuity of the function $xe$ implies its measurability. Lemma 2.15 and the translation invariance of $E$ imply
\[
\lim_{h \to 0} \frac{1}{h} \lambda \left( \left\{ \tau \in [t; t + h] \, \mid \, \| xe(\tau) - xe(t) \|_E < \varepsilon \right\} \right) = 1
\]
for all $t \in T$ and all $\varepsilon > 0$. Supposing that $xe$ is not continuous we get

$$\exists \delta > 0 : \exists (t_n)_{n \in \mathbb{N}} : t_n \downarrow t_0 : \forall t_n : \| xe(t_n) - xe(t_0) \|_E \geq \delta$$

(1)

for $t_0 \in (0, 2\pi)$. Let $\epsilon = \frac{\delta}{16}$ and

$$A_k(h) = \left\{ r \in [t_k, t_k + h] : \| xe(r) - xe(t_k) \|_E < \frac{\delta}{16} \right\}$$

for all $k \in \mathbb{N}_0$. Choose $h$ such that $\lambda(A_0(h)) \geq \frac{9}{10} h$. Then by the translation invariance we also have $\lambda(A_n(h)) \geq \frac{9}{10} h$ for all $n \in \mathbb{N}$. Let now be $n_0 \in \mathbb{N}$ such that $|t_n - t_0| < \frac{1}{10} h$ for all $n \geq n_0$. Then $A_n(h) \cap A_0(h) \neq \emptyset$ for $n \geq n_0$ because otherwise

$$\frac{11}{10} h > \lambda([t_0, t_n + h]) \geq \lambda(A_n(h) \cup A_0(h)) = \lambda(A_n(h)) + \lambda(A_0(h)) \geq \frac{18}{10} h.$$ 

With $\tau_n \in A_n(h) \cap A_0(h)$ we obtain

$$\| xe(t_n) - xe(t_0) \|_E \leq \| xe(t_n) - xe(\tau_n) \|_E + \| xe(\tau_n) - xe(t_0) \|_E < \frac{\delta}{8}$$

for all $n \geq n_0$. This is a contradiction to (1). Thus $xe$ is continuous in $t_0$ and therefore on $T \Box$

**Theorem 3.4.** Let $(K_n)_{n \in \mathbb{N}_0}$ with

$$K_n(t) = \sum_{k=-n}^{n} \left( 1 - \frac{|k|}{n+1} \right) e^{ikt}$$

be the Fejér kernel. Then

$$\sigma_n x = \frac{1}{2\pi} \int_{0}^{2\pi} K_n(t) xe(-t) \, dt$$

for all $n \in \mathbb{N}_0$ and all $x \in E_{R_i}$.

**Proof.** Since $\frac{1}{2\pi} \int_{0}^{2\pi} xe(-t) \, dt$ exists for all $x \in E_{R_i}$, we obtain by Theorem 2.12 for all $n \in \mathbb{N}_0$ the existence of the integrals $\frac{1}{2\pi} \int_{0}^{2\pi} K_n(t) xe(-t) \, dt \in E$. Using assertion 4 of Theorem 2.11 we have for all $k \in \mathbb{Z}$

$$Pr_k \left( \frac{1}{2\pi} \int_{0}^{2\pi} K_n(t) xe(-t) \, dt \right) = \frac{1}{2\pi} \int_{0}^{2\pi} Pr_k(K_n(t) xe(-t)) \, dt$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} K_n(t) xe^{-ikt} \, dt$$

$$= \left\{ \begin{array}{ll}
(1 - \frac{|k|}{n+1}) x_k & \text{for } |k| \leq n \\
0 & \text{for } |k| > n.
\end{array} \right.$$ 

Thus $\frac{1}{2\pi} \int_{0}^{2\pi} K_n(t) xe(-t) \, dt = \sigma_n x \Box$
According to [2] we obtain the following Theorems 3.5 and 3.6.

**Theorem 3.5.** Let $E$ be a translation invariant $BK$-space. Then

1. $E_{Ri} \subseteq E_{\sigma B}$
2. $E_{\sigma K} = E_{AD} = E_e = E_{\sigma K}$.

**Proof.** We prove the two assertions in the following way.

1. Let $x \in E_{Ri}$. Then for all $n \in \mathbb{N}_0$

$$\|\sigma_n x\|_E = \left\| \frac{1}{2\pi} \int_0^{2\pi} K_n(t)x e^{-i t} dt \right\|_E \leq \frac{1}{2\pi} \int_0^{2\pi} \|K_n(t)\|_E dt = \|x\|_E.$$ 

Thus $\sup_{n \in \mathbb{N}_0} \|\sigma_n x\|_E \leq \|x\|_E < \infty$, i.e. $E_{Ri} \subseteq E_{\sigma B}$.

2. We prove $E_{\sigma K} \subseteq E_{AD} \subseteq E_e \subseteq E_{\sigma K} \subseteq E_{\sigma K}$. Let $x \in E_{\sigma K}$. Then by [6: Theorem 2/p. 120]

$$\forall \varepsilon > 0 : \exists \alpha^{(n)}_i \geq 0 \ (0 \leq i \leq n) : \sum_{i=0}^{n} \alpha^{(n)}_i = 1 \text{ and } \left\| \sum_{i=0}^{n} \alpha^{(n)}_i x - z \right\|_E < \varepsilon,$$

i.e. $x \in E_{AD}$. Hence $E_{\sigma K} \subseteq E_{AD}$. From $x \in E_{AD}$ one obtains that for all $\varepsilon > 0$ there exists an $z_x \in \Phi \cap E$ with $\|x - z_x\|_E < \frac{\varepsilon}{3}$. Obviously $z_x \in E_e$ and therefore

$$\forall t_0 \in T : \exists U_\delta(t_0) : \forall t \in U_\delta(t_0) : \|z_x e(t) - z_x e(t_0)\|_E < \frac{\varepsilon}{3}.$$ 

From this we get

$$\|z_x e(t) - z_x e(t_0)\|_E \leq \|z_x e(t) - x e(t)\|_E \leq \|z_x e(t) - z_x e(t_0)\|_E + \|z_x e(t_0) - x e(t_0)\|_E < \varepsilon,$$

for all $t \in U_\delta(t_0)$, i.e. $x \in E_e$. Trivially $x \in E_{Ri}$ and $\|\sigma_n x - x\|_E \to 0$ (cf. [2: p. 246]), i.e. $x \in E_{\sigma K}$. Consequently $|\phi(\sigma_n x) - \phi(x)| \to 0$ for all $\phi \in E'$, i.e. $x \in E_{\sigma K}$.

**Theorem 3.6.** Let $E_{wc}$ be the space of those elements $x \in E$ for which $xe$ has weakly continuous translation. Then $E_{wc} = E_{\sigma K} = E_e$. Thus in $E$ weakly continuous translation is equivalent to continuous translation.

**Proof.** We have $E_{\sigma K} = E_e$ and $E_e \subseteq E_{wc}$. We only have to prove $E_{wc} \subseteq E_e$. Let $x \in E_{wc}$. Then $xe$ is weakly continuous and therefore weakly measurable. The range of $xe$, i.e. $\{xe(t) | t \in T\}$, is a subset of the closure of

$$\left\{ \sum_{k=0}^{n} \beta_k xe(t_k) \mid n \in \mathbb{N}_0 \text{ and } \beta_k \in \mathbb{Q}, t_k \in \mathbb{Q} \cap T \text{ for all } 0 \leq k \leq n \right\},$$

(cf. [6: Theorem 2/p. 120]), where $\mathbb{Q}$ denotes the set of all rational numbers. Therefore $\{xe(t) | t \in T\}$ is a separable set. Thus the conditions of the Pettis theorem [6: p. 131] are fulfilled, and we obtain $x \in E_{wc} = E_e$ (cf. Theorem 3.3), i.e. $E_{wc} = E_e = E_{\sigma K}$.
Boettcher and Goes noticed that Theorem 3.6 can also be interpreted as an application of the theorem in [6: p. 233]. Furthermore we have to remark that in the proof of Proposition 4.3 in [2] there is not paid attention to the fact that the additional condition $E \subseteq E_{R_i}$ is used.

The following example illustrates inclusion relations between the considered $BK$-spaces.

**Example 3.7.** First we define the following spaces: Let $L^p(T)$ $(1 \leq p < \infty)$ be the space of all complex-valued Lebesgue measurable functions on $T$ with $\int_0^{2\pi} |f|^p d\lambda < \infty$ and $\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f|^p d\lambda\right)^{1/p}$ (cf. Katznelson [5: p. 14]) and $L^\infty(T)$ the subspace of $L^1(T)$ of all essentially bounded functions endowed with the norm $\|f\|_\infty = \text{ess sup}_{t \in T} |f(t)|$. Furthermore let $M^d(T)$ be the space of all purely discontinuous (Borel) measures on $T$ (cf. Hewitt and Ross [4: Definition 19.13/p. 269] and [5: p. 37]).

Now we consider the associated translation invariant $BK$-spaces $\hat{L}^p(T), \hat{L}^\infty(T)$ and $\hat{M}^d(T)$ of sequences of Fourier(-Stieltjes) coefficients, $l^\infty = \{u \in \Omega | \sup_{k \in \mathbb{Z}} |u_k| < \infty\}$ and $c_0 = \{v \in l^\infty | \lim_{|k| \to \infty} |v_k| = 0\}$. Then the translation invariant $BK$-space $(E; \|\cdot\|_E)$ shall be constructed by

$$E = E_1 \oplus E_2 \oplus E_3 \oplus E_4$$

and endowed with the norm $\|\cdot\|_E = \sum_{k=1}^4 \|\cdot\|_{E_k}$. Let

$$E_1 = \{x \in \hat{L}^\infty(T) \mid x_{2k} = 0 \text{ for all } k \in \mathbb{Z} \setminus \{0\}\}.$$

Then

$$\forall x \in E_1 : \exists f \in L^\infty(T) : \forall k \in \mathbb{Z} : x_k = \hat{f}(k).$$

Let be $\|x\|_{E_1} = \|f\|_\infty$. Then $E_1$ endowed with this norm is a $BK$-space. With

$$E_2 = \{y \in \hat{M}^d(T) \mid y_{2k} = 0 \text{ for all } k \in \mathbb{Z} \setminus \{0\}\}$$

one obtains

$$\forall y \in E_2 : \exists \mu \in M^d(T) : \forall k \in \mathbb{Z} : y_k = \hat{\mu}(k).$$

Correspondingly let be $\|y\|_{E_3} = \text{var}(\mu)$ (total variation of $\mu$). According to [4: Theorem 19.20/p. 273] we have $M^d(T) \cap L^1(T) = \{0\}$, and with $L^\infty(T) \subseteq L^1(T)$ we have $M^d(T) \cap L^\infty(T) = \{0\}$ respectively $E_1 \cap E_2 = \{0\}$. Furthermore let $E_3$ and $E_4$ be defined by

$$E_3 = \{u \in l^\infty \mid u_{2k+1} = u_{4k} = 0 \text{ for all } k \in \mathbb{Z}\}$$

and

$$E_4 = \{v \in c_0 \mid v_0 = v_{2k+1} = v_{4k+2} = 0 \text{ for all } k \in \mathbb{Z}\},$$

and let $E_3$ and $E_4$ be endowed with the usual norm $\|w\|_{E_3} = \|w\|_\infty = \sup_{k \in \mathbb{Z}} |w_k|$ and $\|w\|_{E_4} = \|w\|_\infty = \sup_{k \in \mathbb{Z}} |w_k|$, respectively.
As direct sum of BK-spaces \( E \) is evidently a BK-space. First we prove that in general \( E_c \) is a proper subset of \( E_{R_1} \). Choose \( f \in L^\infty(T) \) with

\[
f(t) = \begin{cases} 
0 & \text{for } t \in [0, \pi) \\
1 & \text{for } t \in [\pi, 2\pi). 
\end{cases}
\]

Then \( x \in E_1 \) with

\[
x_k = \hat{f}(k) = \begin{cases} 
0 & \text{for } 0 \neq k \text{ even} \\
\frac{1}{2} & \text{for } k = 0 \\
\frac{i}{\pi k} & \text{for } k \text{ odd} 
\end{cases}
\]

and \( x(\tau) = \hat{f}_\tau \) (where \( f_\tau(t) = f(t + \tau) \) for all \( t \in T \)) such that

\[
\|x(\tau_1) - x(\tau_2)\|_E = \|f_{\tau_1} - f_{\tau_2}\|_\infty = 1
\]

for all \( \tau_1 \) and \( \tau_2 \) with \( \tau_1 \neq \tau_2 \). Thus \( x \not\in E_c \). Now we prove the existence of

\[
\frac{1}{2\pi} \int_0^{2\pi} x(t) \, dt.
\]

For \( \varepsilon > 0 \) let

\[
\tilde{P}_\varepsilon = \left\{ (t_j^{(\varepsilon)}) \left| 0 = t_0^{(\varepsilon)} < t_1^{(\varepsilon)} < \ldots < t_N^{(\varepsilon)} = 2\pi \right. \right\}
\]

be a partition with \( |\tilde{P}_\varepsilon| < \frac{\varepsilon \pi}{4} \), and let \( \tilde{P}_1 \) and \( \tilde{P}_2 \) be two refinements of \( \tilde{P}_\varepsilon \). Then

\[
\tilde{P}_k = \bigcup_{j=1}^{N_k} \tilde{P}_{kj} \quad (k \in \{1, 2\}) \quad \text{with}
\]

\[
\tilde{P}_{kj} = \left\{ (t_j^{(kj)}) \left| t_j^{(kj)} = t_0^{(kj)} < t_1^{(kj)} < \ldots < t_{N_k}^{(kj)} = t_N^{(kj)} \right. \right\} \quad (1 \leq j \leq N_k).
\]

For corresponding tagged partitions \( P_1, P_2 \) and \( P_\varepsilon \) we get

\[
\left\| \frac{1}{2\pi} x(\tau(P_k)) - \frac{1}{2\pi} x(\tau(P_\varepsilon)) \right\|_{E_1}
\]

\[
= \frac{1}{2\pi} \left\| \sum_{j=1}^{N_k} \sum_{i=1}^{N_{kj}} \left( x(\tau(t_i^{(kj)})) - x(\tau(t_j^{(\varepsilon)})) \right) \left( t_i^{(kj)} - t_j^{(\varepsilon)} \right) \right\|_{E_1}
\]

\[
= \frac{1}{2\pi} \left\| \sum_{j=1}^{N_k} \sum_{i=1}^{N_{kj}} \left( f(t_i^{(kj)}) - f(t_j^{(\varepsilon)}) \right) \left( t_i^{(kj)} - t_{i-1}^{(kj)} \right) \right\|_{\infty}
\]

\[
\leq \frac{1}{2\pi} \left\| \sum_{j=1}^{N_k} \left( \chi_{[\pi - t_j^{(\varepsilon)}, \pi - t_{j-1}^{(\varepsilon)}]} + \chi_{[2\pi - t_j^{(\varepsilon)}, 2\pi - t_{j-1}^{(\varepsilon)}]} \right) \left( t_j^{(\varepsilon)} - t_{j-1}^{(\varepsilon)} \right) \right\|_{\infty}
\]

\[
\leq \frac{1}{2\pi} \frac{4|\tilde{P}_\varepsilon|}{\varepsilon}
\]

\[
< \frac{\varepsilon}{2}.
\]
Thus we have
\[ \left\| \frac{1}{2\pi} x e(\mathcal{P}_1) - \frac{1}{2\pi} x e(\mathcal{P}_2) \right\|_{E_1} \]
\[ \leq \left\| \frac{1}{2\pi} x e(\mathcal{P}_1) - \frac{1}{2\pi} x e(\mathcal{P}_\varepsilon) \right\|_{E_1} + \left\| \frac{1}{2\pi} x e(\mathcal{P}_2) - \frac{1}{2\pi} x e(\mathcal{P}_\varepsilon) \right\|_{E_1} \]
\[ < \varepsilon. \]

Therefore \( \frac{1}{2\pi} x e \) is R-integrable by assertion 3 in Theorem 2.10 (cf. also [3: Example 12/p. 930]).

Furthermore there exist elements of \( E \) which do not have \( \sigma B \). These elements cannot be R-integrable. Let \( \mu \in M^d(\mathcal{T}) \) be such that \( \mu \) is the difference of the two Dirac measures \( \delta_0 \) and \( \delta_\pi \), i.e. \( \mu = \delta_0 - \delta_\pi \). Then \( y \in E_2 \) with
\[ y_k = \mu(k) = \begin{cases} 0 & \text{for } k \text{ even} \\ \frac{1}{\pi} & \text{for } k \text{ odd.} \end{cases} \]

We have \( \sigma_n y \in E_1 \) for all \( n \in \mathbb{N}_0 \), but \( (\|\sigma_n y\|_{E_1})_{n \in \mathbb{N}_0} \) is not bounded. (This is a modification of Boettcher's example.)

We know that \( E_{R_1} \subseteq E_{\sigma B} \). In general this inclusion is proper. Let \( u \in E_3 \) be such that
\[ u_k = \begin{cases} 1 & \text{for } k = 4p + 2 \ (p \in \mathbb{Z}) \\ 0 & \text{otherwise.} \end{cases} \]

Evidently \( \|u\|_E = \|u\|_{E_3} = 1 \) and \( \sup_{n \in \mathbb{N}_0} \|\sigma_n u\|_E = 1 \), i.e. \( u \in E_{\sigma B} \). The function \( u e \) is not R-integrable, otherwise \( u_0 = 0 \) would imply
\[ \frac{1}{2\pi} \int_0^{2\pi} u e(t) \, dt = \frac{1}{2\pi} \int_0^{2\pi} u e(-t) \, dt = \sigma_0 u = 0 \ (\in E) \]

or written as a limit
\[ \lim_{k \to \infty} \left\| \frac{1}{2\pi} \sum_{j=1}^{k} u e \left( \frac{2\pi j}{k} \right) \frac{2\pi}{k} \right\|_E = 0. \]

But for any \( k = 4p + 2 \) one obtains
\[ \left\| \frac{1}{2\pi} \sum_{j=1}^{k} u e \left( \frac{2\pi j}{k} \right) \frac{2\pi}{k} \right\|_E \geq \left\| \frac{1}{2\pi} \sum_{j=1}^{k} u_k e^{2\pi i j \frac{2\pi}{k}} \frac{2\pi}{k} \right\|_E = 1. \]

The function \( u e \) cannot be R-integrable, i.e. \( E_{R_1} \) is a proper subset of \( E_{\sigma B} \). Finally it is well known that there exist elements in \( E_{\sigma B} \) which are not in \( E \). Let \( w = (w_k)_{k \in \mathbb{Z}} \) be such that
\[ w_k = \begin{cases} 1 & \text{for } k = 4p \ (p \neq 0) \\ 0 & \text{otherwise.} \end{cases} \]
Obviously \( w \notin E \), \( \sigma_n w \in E_4 \) for all \( n \in \mathbb{N}_0 \) and \( \sup_{n \in \mathbb{N}_0} \| \sigma_n w \|_E < \infty \).

The following chart shows the relations between the spaces, which are generated by a translation-invariant \( BK \)-space \( E \).

\[ \begin{array}{c}
\Omega \\
E \\
E_{eB} \\
E_{\text{Ri}} \\
E_{S\sigma K} = E_{\sigma K} = E_{AD} \\
= E_c = E_{mr} = E_\lambda \\
\end{array} \]

References


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