Abstract: For remainder functionals (e.g., approximation or quadrature errors), estimates by the moduli of smoothness are obtained. As a by-product, the constants in the estimate of the $\mathcal{K}$-functional by the moduli of smoothness are improved.

Keywords: Error estimates, modulus of continuity, modulus of smoothness, $\mathcal{K}$-functional

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1. Introduction

Let $R : C[a,b] \to \mathbb{R}$ be a bounded linear functional, and let $R[\mathcal{P}_{r-1}] = 0$, where $\mathcal{P}_{r-1}$ are the polynomials of degree less $r$ (the setting in the following sections will be somewhat more general). The standard estimates for $R[f]$ are

$$|R[f]| \leq \|R\| \|f^{(j)}\|_{\infty} \quad \text{for } f \in C^j[a,b] \text{ and } 1 \leq j \leq r,$$

where

$$\|R\|_j = \sup \left\{ \frac{|R[f]|}{\|f^{(j)}\|} : f \in C^j[a,b], \|f^{(j)}\|_{\infty} \neq 0 \right\}.$$

But if additional information on $f$ is available, e.g. $f^{(j)} \in \text{Lip } \alpha$, this information cannot be used by estimates of the type (1.1). Therefore, it is of interest to have estimates by the moduli of continuity, i.e., estimates of the form

$$|R[f]| \leq c(t)\omega_*(f,t).$$

It is well known that such estimates exist, but, in general, no estimates for the constants $c(t)$ are available (see, e.g., Esser [3] and Ivanov [4]). The aim of this paper is to obtain

c(t) are available such estimates in the non-periodic case, and to improve them in the periodic case. Section 2 deals with functionals on $C^s_{2\pi}$ and $L^s_{p,2\pi}$, Section 3 with functionals on $C^s[a,b]$, and Section 4 with the $K$-functional.

2. Functionals on $C^s_{2\pi}$ and $L^s_{p,2\pi}$

Let $C^s_{2\pi}$ be the class of $s$-times continuously differentiable, real-valued functions with period $2\pi$, and $L^s_{p,2\pi}$ the class of $2\pi$-periodic, real-valued functions with absolutely continuous $(s-1)$-th and $p$-integrable $s$-th derivative, with

$$
\|f\|_p^p = \int_0^{2\pi} |f(x)|^p \, dx \quad (1 \leq p < \infty) \quad \text{and} \quad \|f\|_\infty = \sup_{0 \leq x \leq 2\pi} |f(x)|.
$$

For convenience, we denote these classes by $X^s_p$, i.e., $X^s_\infty = C^s_{2\pi}$ and $X^s_p = L^s_{p,2\pi}$ for $1 \leq p < \infty$. We consider functionals $R : X^s_p \to \mathbb{R}$ satisfying

$$
|R[f + g]| \leq |R[f]| + |R[g]|
$$

and

$$
\|R\|_{i,p} = \sup \left\{ \frac{|R[f]|}{\|f^{(i)}\|_p} : f \in X^i_p, \|f^{(i)}\|_p \neq 0 \right\} < \infty
$$

for $i = s, \ldots, r$, for some $r > s \geq 0$. We therefore have

$$
|R[f]| \leq \|R\|_{i,p} \|f^{(i)}\|_p \quad \text{for} \quad f \in X^i_p, \quad i = s, \ldots, r.
$$

Further let

$$
\omega_j(f,t)_p = \sup_{|h| \leq t} \|\Delta^j_h f\|_p, \quad \text{where} \quad \Delta^j_h f(x) = \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} f(x + ih),
$$

be the $j$-th modulus of smoothness (for the properties of $\omega_j$, see Schumaker [12: Chap. 2.8]). To simplify notation, we introduce the following abbreviations. Let

$$
[x] = \min\{i \in \mathbb{Z} : i \geq x\}, \quad \beta_r = \begin{pmatrix} r \\ \lfloor r/2 \rfloor \end{pmatrix}
$$

and

$$
\rho_r = \int_0^{[r]} [x]^r N_r \left( x + \frac{r}{2} \right) \, dx = \sum_{i=1}^{[r]} i^r N_{r+1} \left( i + \frac{r}{2} \right) \leq \frac{1}{2} \left[ \frac{r}{2} \right]^r
$$

($\rho_1 = 1/2$, $\rho_2 = 1/2$, $\rho_3 = 31/48$, $\rho_4 = 9/8$). Here, $N_r$ is the B-spline of degree $r - 1$ for the knots 0, 1, \ldots, $r$, satisfying

$$
\int_0^r N_r(x) \, dx = 1.
$$
Further, there holds
\[
\int_{t-1}^{t} N_r(x) \, dx = N_{r+1}(t) \tag{2.6}
\]
(Schoenberg [11: p. 12]; this has been used to obtain the second formula for \(\rho_r\) given in (2.4)), and
\[
\int_{0}^{r} F^{(r)}(x + \alpha v) N_r(v) \, dv = \alpha^{-r} \Delta_\alpha F(x) \quad \text{for } F \in X_r^n \tag{2.7}
\]
and \(\alpha > 0\) (Schumaker [12: p. 54]). The following lemma is well known (see, e.g., DeVore [2]), but we have modified the proof such that better constants are obtained.

**Lemma 2.1.** Let \(r \geq 1\) and \(1 \leq p \leq \infty\). Then for all \(f \in X_p^0\) and all \(t > 0\), there exists a function \(g \in X_p^r\) such that

a) \(\|f - g\|_p \leq 2 \beta_{r-1} \rho_r \omega_r(f, t)_p\) and \(\|g^{(r)}\|_p \leq \beta_{r-1}(2^r - \beta_r) t^{-r} \omega_r(f, t)_p\),

or

b) \(\|f - g\|_p \leq \beta_{r-1} \omega_r(f, t)_p\) and \(\|g^{(r)}\|_p \leq \left[\frac{1}{2} \right]^r \delta_1 t^{-r} \omega_r(f, t)_p\), where \(\delta_1 = \delta_2 = 1\) and \(\delta_r = 2\) for \(r \geq 3\).

**Proof.** Let \(\alpha_r = \lfloor r/2 \rfloor\), and let \(g\) be defined by

\[
g(x) = \beta_{r-1} \int_{-\lfloor r/2 \rfloor}^{\lfloor r/2 \rfloor} (\beta_r f(x) + (-1)^{r-\alpha_r-1} \Delta_{\alpha_r} f(x - \alpha_r vt)) N_r(v + \frac{r}{2}) \, dv
\]

\[
= \beta_{r-1} \sum_{i=0}^{r} \left(\frac{r}{i}\right) (-1)^{i-\alpha_r-1} ((i - \alpha_r)t)^{-r} \Delta_{(i-\alpha_r)t} F \left( x + \frac{r}{2} (\alpha_r - i)t \right),
\]

by (2.7), where \(F^{(r)} = f\).

a) Minkowski's inequality and the periodicity of \(f\) yield

\[
\|f - g\|_p \leq \beta_{r-1} \int_{-\lfloor r/2 \rfloor}^{\lfloor r/2 \rfloor} \|\Delta_{\alpha_r} f\|_p N_r(v + \frac{r}{2}) \, dv
\]

\[
\leq \beta_{r-1} \int_{-\lfloor r/2 \rfloor}^{\lfloor r/2 \rfloor} \omega_r(f, \|vt\|)_p N_r(v + \frac{r}{2}) \, dv
\]

\[
\leq \beta_{r-1} \int_{-\lfloor r/2 \rfloor}^{+r/2} \omega_r(f, t)_p N_r(v + \frac{r}{2}) \, dv
\]

\[
= 2 \beta_{r-1} \rho_r \omega_r(f, t)_p,
\]

while, by the triangle inequality,

\[
\|g^{(r)}\|_p \leq \beta_{r-1} \sum_{i=0}^{r} \left(\frac{r}{i}\right) |i - \alpha_r|^{-r} t^{-r} \|\Delta_{(i-\alpha_r)t} F^{(r)}\|_p
\]
\[
\beta_t^{-1} r^{-r} \sum_{i=0, i \neq \alpha_r}^{r} \left( \frac{r}{i} \right) |i - \alpha_r|^r \omega_r(f, |i - \alpha_r|^r) p \\
\leq \beta_t^{-1} r^{-r} \sum_{i=0}^{r} \left( \frac{r}{i} \right) \omega_r(f, t) p \\
= \beta_t^{-1} r^{-r} (2^r - \beta_r) \omega_r(f, t) p.
\]

b) From (2.9), we also get
\[
\|g^{(r)}\|_p \leq \beta_t^{-1} r^{-r} \sum_{i=0, i \neq \alpha_r}^{r} \left( \frac{r}{i} \right) |i - \alpha_r|^r \omega_r(f, \alpha_r t) p \leq \delta_t r^{-r} \omega_r(f, \alpha_r t) p,
\]
since \(\beta_t^{-1} r^{-r} \sum_{i=0, i \neq \alpha_r}^{r} \left( \frac{r}{i} \right) |i - \alpha_r|^r \leq \delta_r\), by some elementary calculations, while (2.8) together with (2.5) gives
\[
\|f - g\|_p \leq \beta_t^{-1} \omega_r \left( f, \frac{r}{2} t \right) p \leq \beta_t^{-1} \omega_r(f, \alpha_r t) p.
\]
Replacing \(u = \alpha_r t\) proves part b) \(\blacksquare\)

From Lemma 2.1, we get the following estimates for \(\|R[f]\|\) in terms of the moduli of smoothness.

**Theorem 2.1.** Let \(R : X^*_p \to \mathbb{R}\) be a functional satisfying (2.1) and (2.3), and let \(t > 0\) and \(1 \leq j \leq r - s\). Then
\[
|R[f]| \leq \beta_t^{-1} \omega_j \left( 2 \rho_j \|R\|_{x,p} + (2^j - \beta_j) t^{-j} \|R\|_{j+s,p} \right) \omega_j(f^{(s)}, t) p \quad \text{for } f \in X^*_p.
\]

**Proof.** Let \(F = f^{(s)} \in X^*_p\). By Lemma 2.1, there exists \(G \in X^*_j\) such that
\[
\|F - G\|_p \leq c_4 \omega_j(F, t) p \quad \text{and} \quad \|G^{(j)}\|_p \leq c_2 \omega_j(F, t) p. \tag{2.10}
\]
Now choose \(g \in X^{j+s}_p\) with \(g^{(s)} = G\). Then
\[
|R[f]| \leq |R[f - g]| + |R[g]| \\
\leq \|R\|_{x,p} \|f^{(s)} - g^{(s)}\|_p + \|R\|_{j+s,p} \|g^{(j+s)}\|_p \\
= \|R\|_{x,p} \|F - G\|_p + \|R\|_{j+s,p} \|G^{(j)}\|_p.
\]
Inserting the estimates from (2.10), together with the constants of Lemma 2.1a) (with \(r\) replaced by \(j\)) completes the proof \(\blacksquare\)

**Example 2.1.** Let \(E_n[f] = \inf_{g \in X_n} \|f - g\|_\infty\) be the error in the approximation of \(f \in C_{2\pi}\) by trigonometric polynomials of degree lesser or equal \(n\). By the Theorem of Favard-Achieser-Krein,
\[
\|E_n\|_{r, \infty} = K_r(n + 1)^{-r} \quad \text{where} \quad K_r = \frac{4}{\pi} \sum_{\nu = 0}^{\infty} \left( \frac{(-1)^\nu}{2\nu + 1} \right)^{r+1} \leq \frac{\pi}{2}.
\]
is Favard's constant. Choosing \( t = 2/(n + 1) \) in Theorem 2.1, we obtain

\[
E_n[f] \leq \frac{\pi c_j}{(n + 1)^s} \omega_j \left( f^{(s)}, \frac{2}{n + 1} \right) \infty \quad \text{for } f \in C^2_{2n},
\]

\( s \geq 0 \) and \( j \geq 1 \); where

\[
c_j = \frac{1}{\beta_j} \left( \rho_j + \frac{1}{2} \right) - \frac{1}{2j + 1}
\]

(2.11)

\( c_1 = 3/4, c_2 = 3/8, c_3 = 23/72 \) and \( c_4 = 23/96 \), but \( c_j \to \infty \) for \( j \to \infty \). The asymptotic behaviour of \( c_r \) is probably be given by

\[
c_r \sim \frac{\sqrt{\pi r}}{2} \left( \frac{r}{4\sqrt{3e}} \right)^r \quad \text{for } r \to \infty,
\]

but we have not been able to prove this.

It can be shown that Theorem 2.1 remains true if \( \| R \|_{i,p} \) is replaced by

\[
\| R \|_{i,p} = \sup \left\{ \frac{\| R[f] \|}{\| f^{(i)} \|_p} : f \in X^i_p, \| f^{(i)} \|_p \neq 0, \int_0^{2\pi} f^{(i)}(x) \, dx = 0 \right\}
\]

(obviously, \( \| R \|_{i,p} \leq \| R \|_{i,p} \), but with equality for \( i \geq 1 \) because of the periodicity of the functions considered). E.g., for quadrature errors

\[
R[f] = \int_a^b f(x) \, dx - \sum_{i=1}^n a_i f(x_i)
\]

there holds

\[
\| R \|_{0,\infty} = b - a + \sum_{i=1}^n i = 1^n |a_i|, \quad \text{but} \quad \| R \|_{0,\infty} = \sum_{i=1}^n |a_i|
\]

(for an example and some other details omitted here, see [6]).

3. Functionals on \( C^s[a,b] \)

Let \( C^s[a,b] \) be the class of \( s \)-times continuously differentiable, real-valued functions on \( [a,b] \), and \( \| f \| = \sup |f(x)| \). We consider functionals \( R : C^s[a,b] \to IR \) satisfying (2.1), and assume that, for \( f \in C^s[a,b] \) and \( i = s, \ldots, r \),

\[
|R[f]| \leq \| R \|_{i} \| f^{(i)} \|
\]

(3.1)

holds, where \( \| R \| \), is defined analogous to (2.2), and \( \| R \|_{i} < \infty \). Further, let

\[
\omega_j(f,t) = \sup \{ |\Delta^j_h f(x)| : |h| \leq t \text{ and } x, x + jh \in [a,b] \}
\]

denote the \( j \)-th modulus of continuity of \( f \). The proof of an analogue of Lemma 2.1 is more complicated, since \( f \) is not defined outside \( [a,b] \). A standard method to overcome this difficulty is to extend \( f \) in a suitable way (see DeVore [2]), but then one has to know the constants related to this extension. For the case of the sup-norm considered in this
section, this difficulties can be avoided by a modification of the step size of the differences involved in the proof (this does not work for the $L_p$-norms, $1 \leq p < \infty$, which is the reason why we do not treat this case here). This modification is a useful tool to obtain explicit constants, when Steklov functions are applied to non-periodic functions. It was derived by the author some years ago for a first version of this paper, but afterwards, I discovered that it had already been used by Sendov in [9]; inserting $t = (b-a)/r^2$ in part b) of the following lemma, gives the estimate of Sendov. Let

$$\tau_r = \int_0^r \left| x \right|^r N_r(x) \, dx = \sum_{i=1}^r i^r N_{r+1}(i) \leq r^r \quad \text{(3.2)}$$

($\tau_1 = 1, \tau_2 = 5/2, \tau_3 = 10, \tau_4 = 331/6$; the second formula for $r$ follows from (2.6)).

**Lemma 3.1.** Let $f \in C[a,b], \ r \geq 1$ and $t \in (0,(b-a)/r^2]$. Then there exists a function $g \in C^r[a,b]$ such that

a) $\|f - g\| \leq \tau_r \omega_r(f,t)$ and $\|g^{(r)}\| \leq (2^r - 1)t^{-r} \omega_r(f,t)$, or

b) $\|f - g\| \leq \omega_r(f,rt)$ and $\|g^{(r)}\| \leq (r + 1)t^{-r} \omega_r(f,rt)$.

**Proof.** Let

$$g(x) = \int_0^r \left( f(x) + (-1)^{r+1} \Delta_u f(x) \right) N_r(u) \, du \quad \text{with} \quad u = vt - \frac{x-a}{b-a}rt.$$ 

Here, $u$ and the restriction for $t$ stated in the lemma have been chosen such that $x + itu \in [a,b]$ always (i.e., for $x \in [a,b]$ and $i = 0, \ldots, r$).

a) We obtain

$$|f(x) - g(x)| \leq \int_0^r \omega_r\left(f, t\left|u - \frac{x-a}{b-a}r^2\right|\right) N_r(v) \, dv. \quad \text{(3.3)}$$

For simplicity, let $a = 0$ and $b = 1$, and let

$$\phi(x) = \omega_r(f,tx) \quad \text{and} \quad \psi(x) = \int_0^r \phi(|v-rx|) N_r(v) \, dv.$$ 

From the monotonicity of $\omega_r(f,\cdot)$, it follows that $\phi$ is an increasing function, and the symmetry of $N_r$ yields $\psi(1-x) = \psi(x)$. Using the symmetry properties of $N_r$ and $\psi$, and the monotonicity of $\phi$ and of $N_r$ on $[0,r/2]$, we obtain for $x \in [0,1/2]$

$$\psi(0) - \psi(x) = \int_0^x (\phi(v) - \phi(rx-v)) N_r(v) \, dv + \int_x^r (\phi(v) - \phi(v-rx)) N_r(v) \, dv$$

$$\geq \int_0^{r/2} (\phi(v) - \phi(rx-v)) N_r(v) \, dv$$

$$= \int_0^{r/2} (\phi(rx-v) - \phi(v))(N_r(rx-v) - N_r(v)) \, dv$$

$$\geq 0.$$
Because of the symmetry, the same holds for \( x \in [1/2, 1] \), so that

\[
\|f - g\| \leq \psi(0) = \int_0^1 \omega_r(f, tv) N_r(v) dv \leq \int_0^1 [v]^{\tau} N_r(v) dv \omega_r(f, t) = \tau \omega_r(f, t).
\]

For the \( r \)-th derivative of \( g \), we obtain, using (2.7),

\[
|g^{(r)}(x)| = \left| \sum_{i=1}^{r} \binom{r}{i} (-1)^{i-1} (it)^{-r} \left( 1 - \frac{irt}{b - a} \right)^r \Delta_i f \left( x - \frac{x - a}{b - a} \right) \right| \\
\leq \sum_{i=1}^{r} \binom{r}{i} (it)^{-r} \omega_r(f, it) \\
\leq \sum_{i=1}^{r} \binom{r}{i} t^{-r} \omega_r(f, t) \\
= (2^r - 1) t^{-r} \omega_r(f, t).
\]

(3.4)

b) From (3.4), we also get

\[
|g^{(r)}(x)| \leq \sum_{i=1}^{r} \binom{r}{i} t^{-r} \omega_r(f, rt) \leq (r + 1) t^{-r} \omega_r(f, rt).
\]

Further, \( \|f - g\| \leq \omega_r(f, rt) \), by (3.3) and (2.5)

In the same way as Theorem 2.1, we obtain the following one.

**Theorem 3.1.** Let \( R : C^s[a, b] \rightarrow \mathbb{R} \) be a functional satisfying (2.1) and (3.1), and let \( 1 \leq j \leq r - s \) and \( t \in (0, (b - a)/j^2) \). Then

\[
|R[f]| \leq \left( \tau_j \|R\|_s + (2^j - 1) t^{-j} \|R\|_{j+s} \right) \omega_j(f^{(s)}, t) \quad \text{for } f \in C^s[a, b].
\]

The estimate of Theorem 3.1 makes use of \( \|R\|_s \) and \( \|R\|_{j+s} \), which, however, may not be known. This difficulty can be overcome if \( \|R\|_m \) and \( \|R\|_{r} \), are known for some \( m < s \) and some \( r > j + s \), by using the estimate

\[
\|R\|_r \leq K_{r-m} \|R\|_{m}^{(r-i)/(r-m)} \left( \frac{\|R\|_{i}}{K_{r-m}} \right)^{(i-m)/(r-m)}
\]

(3.5)

which holds for \( 0 \leq m \leq i \leq r \) (see Ligun [7] and Köhler [5]; the \( K_i \) are again Favard's constants).

**Example 3.1.** Let \( R_n^G[f] = \int_0^1 f(x) dx - Q_n^G[f] \) be the error of the \( n \)-point Gauss-Legendre quadrature formula. It is well known that

\[
\|R_n^G\|_0 = 4 \quad \text{and} \quad \|R_n^G\|_{2n} = \frac{2^{2n+1} \pi^4}{(2n + 1)(2n)!^3}.
\]
Using Stirling's and Wallis' formula, it can be shown that

\[ t_{2n} := \left( \frac{\| R_n^G \|_{2n}}{\| R_n^G \|_0 K_{2n}} \right)^{1/(2n)} \leq \frac{e}{4n}. \]

Using (3.5) to estimate \( \| R_n^G \|_s \) and \( \| R_n^G \|_{j+s} \) by \( \| R_n^G \|_0 \) and \( \| R_n^G \|_{2n} \), and applying Theorem 3.1 with \( r = 2n \) and \( t = t_{2n} \), yields

\[ |R_n^G[f]| \leq 4 \left( \frac{e}{4n} \right)^s \left( \tau_j K_s + (2j - 1)K_{j+s} \right) \omega_j \left( f^{(s)}, \frac{e}{4n} \right) \quad \text{for } f \in C^s[a,b], \]

\[ 0 \leq s < j + s \leq 2n \text{ and } n \geq ej^2/8. \]

Finally, let us shortly consider compound functionals, and state an estimate given by Sendov and Popov \[10: \text{p. 49}\] for the \( \tau \)-moduli, in the framework of this paper. Let the function \( R : C^s[0,1] \rightarrow \mathbb{R} \) satisfy (2.1) and (3.1), and let the \( N \)-compound functional \( R_N : C^s[a,b] \rightarrow \mathbb{R} \) be defined by

\[ R_N[f] = \frac{b - a}{N} \sum_{i=0}^{N-1} R \left( f(a + (i + \cdot) \frac{b - a}{N}) \right) \quad \text{for } f \in C^s[a,b]. \]

**Theorem 3.2.** Let \( 1 \leq j \leq r - s \). Then

\[ |R_N[f]| \leq 6 \left( \frac{b - a}{N} \right)^{r+1} \| R \|_s \omega_j \left( f^{(s)}, \frac{b - a}{N(j + 1)} \right) \quad \text{for } f \in C^s[a,b]. \]

Especially, this can be applied to compound quadrature rules. Better (and partly sharp) estimates by \( \omega_r(f, \cdot) \) have been obtained by Büttgenbach, Lüttgens and Nessel \[1\] for the compound midpoint and the first four compound Newton-Cotes rules, using representations for the error by differences of order \( r \).

### 4. Estimates for the \( K \)-functional

**a)** Let us first consider the \( K \)-functional of Peetre in the periodic case, i.e.,

\[ K_r(f,u)_p = \inf_{g \in X^0_p} \left( \|f - g\|_p + u\|g^{(r)}\|_p \right) \quad \text{for } f \in X^0_p. \]

Specializing to \( g \) from Lemma 2.1a), and choosing \( t = 2u \), yields the following theorem.

**Theorem 4.1.** Let \( 1 \leq p \leq \infty, r \geq 1 \) and \( u > 0 \). Then

\[ K_r(f,u^*)_p \leq 2c_r \omega_r(f,2u)_p \quad \text{for } f \in X^0_p, \]

with \( c_r \) as in (2.11).
For \( r = 1, 2 \), this coincides with the estimate given by Maligranda [8]. The unmodified proof of Lemma 2.1, as given by DeVore [2], yields \( K_r(f, u^r) \leq (r^r + 1 - 2^{-r})\omega_r(f, 2u) \) (see also Maligranda [8]).

b) For the \( K \)-functional in the non-periodic case,

\[
K_r(f, u) = \inf_{g \in C^r[a, b]} (\|f - g\| + u\|g^{(r)}\|) \quad \text{for} \quad f \in C[a, b],
\]

choose, e.g., \( g \) from Lemma 3.1a) with \( t = 2u \).

**Theorem 4.2.** Let \( r > 1 \) and \( 0 < u \leq (b - a)/(2r^2) \). Then

\[
K_r(f, u^r) \leq (r^r + 1 - 2^{-r})\omega_r(f, 2u) \quad \text{for} \quad f \in C[a, b].
\]

It is well known that \( K_r \) can be estimated by \( \omega_r \), but, as far as we know, no explicit constants have been given for \( r > 2 \).

**References**


